Math 225 (Q1) Solution to Homework Assignment 3

1.

(a) Since $A\underline{v_1} = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix} = \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix} = (1)\underline{v_1}$, therefore $\underline{v_1}$ is an eigenvector of A corresponding to the eigenvalue 1. The characteristic equation of A is

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{pmatrix} 0.6-\lambda & 0.3 \\ 0.4 & 0.7-\lambda \end{pmatrix} = (0.6-\lambda)(0.7-\lambda) - (0.3)(0.4) \\ &= \lambda^2 - 1.3\lambda + 0.3 = (\lambda - 1)(\lambda - 0.3), \end{aligned}$$

so that the eigenvalues of A are: $\lambda = 1, 0.3$. To find an eigenvector of A corresponding to the eigenvalue 0.3, we solve the equation: $(A - 0.3I)\underline{x} = \underline{0}$. Since $A - 0.3I = \begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \end{pmatrix}$ which can be row reduced to $\begin{pmatrix} \boxed{1} & 1 \\ 0 & 0 \end{pmatrix}$, x_2 is a free variable and $x_1 + x_2 = 0$. Thus, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue 0.3.

(b) We write

$$\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = x_0 = \underline{v_1} + c\underline{v_2} = \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix} + c\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Clearly, $c = 3/7 - 1/2 = -1/14$.

(c)

$$\underline{x_1} = A\underline{x_0} = A(\underline{v_1} + c\underline{v_2}) = A\underline{v_1} + cA\underline{v_2} = \underline{v_1} + c(0.3)\underline{v_2}.$$

$$\underline{x_2} = A\underline{x_1} = A(\underline{v_1} + c(0.3)\underline{v_2}) = A\underline{v_1} + c(0.3)A\underline{v_2} = \underline{v_1} + c(0.3)^2\underline{v_2}.$$

In general, $\underline{x_k} = \underline{v_1} + c(0.3)^k\underline{v_2}$. As k gets larger and larger, $(0.3)^k$ becomes very small. Thus, $\underline{x_k}$ tends to $\underline{v_1}$ as k tends to infinity.

2.

(a) $z\overline{z} = |z|^2 = 2^2 + (-3)^2 = 4 + 9 = 13.$

(b) $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}.$

(c) $zw = (2 - 3i)(3 + 4i) = 6 + 8i - 9i - 12i^2 = (6 + 12) + (8 - 9)i = 18 - i.$

(d) $\frac{z}{w} = \frac{2-3i}{3+4i} = \frac{2-3i}{3+4i}\frac{3-4i}{3-4i} = \frac{(2-3i)(3-4i)}{(3+4i)(3-4i)} = \frac{6-8i-9i+12i^2}{3^2+4^2} = \frac{-6-17i}{25} = -\frac{6}{25} - \frac{17}{25}i.$

3.

(a) The characteristic equation for A is

$$0 = \det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ -8 & 4-\lambda \end{vmatrix} = -\lambda(4-\lambda) - (1)(-8) = \lambda^2 - 4\lambda + 8.$$

Using quadratic formula to solve this quadratic equation, we get

$$\lambda = \frac{1}{2}[4 \pm \sqrt{(-4)^2 - 4(1)(8)}] = \frac{1}{2}[4 \pm \sqrt{-16}] = \frac{1}{2}[4 \pm \sqrt{16}\sqrt{-1}] = 2 \pm 2i.$$

Thus, the eigenvalues of A are: $\lambda_1 = 2+2i$ and $\lambda_2 = 2-2i = \overline{\lambda_1}$. The eigenspace of A corresponding to the eigenvalue λ_1 is the solution space of the system of equations $(A - \lambda_1 I)\underline{z} = \underline{0}$, which is

$$\begin{pmatrix} 0-(2+2i) & 1 \\ -8 & 4-(2+2i) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\text{or,} \quad \begin{pmatrix} -2-2i & 1 \\ -8 & 2-2i \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The augmented coefficient matrix $\begin{pmatrix} -2-2i & 1 & | & 0 \\ -8 & 2-2i & | & 0 \end{pmatrix}$ has the reduced row echelon form $\begin{pmatrix} 1 & -1/4+(1/4)i & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$. Thus, $z_1 + (-\frac{1}{4} + \frac{1}{4}i)z_2 = 0$ so that $z_1 = (\frac{1}{4} - \frac{1}{4}i)z_2$ and solution vector looks like $\underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_2 \begin{pmatrix} \frac{1}{4} - \frac{1}{4}i \\ 1 \end{pmatrix}$. Thus, $\left\{ \begin{pmatrix} \frac{1}{4} - \frac{1}{4}i \\ 1 \end{pmatrix} \right\}$ is a basis of the eigenspace of A corresponding to the eigenvalue λ_1. By taking complex conjugate, $\left\{ \begin{pmatrix} \frac{1}{4} + \frac{1}{4}i \\ 1 \end{pmatrix} \right\}$ is a basis of the eigenspace of A corresponding to the eigenvalue $\lambda_2 = \overline{\lambda_1}$.

(b) Let $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2+2i & 0 \\ 0 & 2-2i \end{pmatrix}$ and $P = \begin{pmatrix} \frac{1}{4} - \frac{1}{4}i & \frac{1}{4} + \frac{1}{4}i \\ 1 & 1 \end{pmatrix}$. Then $P^{-1}AP = D$.

(c) By (b), corresponding to the eigenvalue $a - bi = 2 - 2i$, the eigenvector is $\underline{v} = \begin{pmatrix} \frac{1}{4} + \frac{1}{4}i \\ 1 \end{pmatrix}$. Let $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$ and $Q = (\,\mathrm{Re}(\underline{v}) \quad \mathrm{Im}(\underline{v})\,) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ 1 & 0 \end{pmatrix}$. Then $Q^{-1}AQ = C$.

4. $\overline{q} = \overline{\overline{\underline{x}}^T A \underline{x}}$ follows from taking complex conjugate on both sides of $q = \overline{\underline{x}}^T A \underline{x}$. $\overline{\overline{\underline{x}}^T A \underline{x}} = \underline{x}^T\ \overline{A \underline{x}}$ comes from the facts: $\overline{BC} = \overline{B}\,\overline{C}$, $\overline{B^T} = \overline{B}^T$ and $\overline{\overline{B}} = B$. $\underline{x}^T\ \overline{A \underline{x}} =$

$\underline{x}^T A\overline{\underline{x}}$ is due to $\overline{BC} = \overline{B}\,\overline{C}$ and that $\overline{A} = A$, since A is a real matrix. $\underline{x}^T A\overline{\underline{x}} = (\underline{x}^T A\overline{\underline{x}})^T$ follows from the fact that $\underline{x}^T A\overline{\underline{x}}$ is a number (that is, 1×1 matrix) and for a 1×1 matrix B, $B^T = B$. $(\underline{x}^T A\overline{\underline{x}})^T = \overline{\underline{x}}^T A^T \underline{x}$ follows from $(BCD)^T = D^T C^T B^T$ and $(B^T)^T = B$. $\overline{\underline{x}}^T A^T \underline{x} = \overline{\underline{x}}^T A\underline{x} = q$ follows from $A^T = A$, since A is a symmetric matrix. Once we have shown that $\overline{q} = q$, the complex number q must in fact be real, since only a real number can be the same as its complex conjugate.

5.

$$||\underline{u}+\underline{v}||^2 + ||\underline{u}-\underline{v}||^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) + (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v})$$

$$= [\underline{u}\cdot\underline{u} + \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{u} + \underline{v}\cdot\underline{v}] + [\underline{u}\cdot\underline{u} - \underline{u}\cdot\underline{v} - \underline{v}\cdot\underline{u} + \underline{v}\cdot\underline{v}] = 2\underline{u}\cdot\underline{u} + 2\underline{v}\cdot\underline{v}$$

$$= 2||\underline{u}||^2 + 2||\underline{v}||^2.$$