Math 225 (Q1) Solution to Homework Assignment 2.

1.

(a) The characteristic equation of A is $0 = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc)$. By quadratic formula, the roots are: $\lambda_\pm = \frac{1}{2}[(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}] = \frac{1}{2}[a + d \pm \sqrt{D}]$.

(b) For $D > 0$, $\lambda_+ > \lambda_-$ and A has two distinct eigenvalues.

(c) For $D = 0$, $\lambda_+ = \lambda_-$ and A has an eigenvalue with algebraic multiplicity 2.

(d) For $D < 0$, $\sqrt{D}$ is not a real number and so A has no real eigenvalues.

2.

(a) The characteristic equation is $0 = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 & -1 \\ -1 & -2-\lambda & 1 \\ 3 & 9 & -\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ 9 & -\lambda \end{vmatrix} - 4\begin{vmatrix} -1 & 1 \\ 3 & -\lambda \end{vmatrix} + (-1)\begin{vmatrix} -1 & -2-\lambda \\ 3 & 9 \end{vmatrix} = -\lambda^3 + \lambda^2 + 8\lambda - 12.$

(b) Since $-\lambda^3 + \lambda^2 + 8\lambda - 12$ can be factorized as $-(\lambda+3)(\lambda-2)^2$, therefore the eigenvalues are: $\lambda_1 = -3$ (algebraic multiplicity 1) and $\lambda_2 = 2$ (algebraic multiplicity 2).

(c) Consider the eigenvalue $\lambda = 2$, its eigenspace is the solution space of the equation $(A - 2I)\underline{x} = \underline{0}$, which is just $\mathrm{Nul}(A - 2I)$. Now $A - 2I = \begin{pmatrix} 1 & 4 & -1 \\ -1 & -4 & 1 \\ 3 & 9 & -2 \end{pmatrix}$ has reduced row echelon $\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}$. The equations are: $x_1 + \frac{1}{3}x_3 = 0$ and $x_2 - \frac{1}{3}x_3 = 0$ so that $\underline{x} = x_3 \begin{pmatrix} -1/3 \\ 1/3 \\ 1 \end{pmatrix}$ Thus, $\left\{ \begin{pmatrix} -1/3 \\ 1/3 \\ 1 \end{pmatrix} \right\}$ is a basis of the eigenspace of A corresponing to the eigenvalue λ and $\lambda = 2$ has geometric multiplicity 1. Similarly, we find that the eigenvalue $\lambda = -3$ has geometric multiplicity 1 and $\left\{ \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} \right\}$ is a basis of the corresponding eigenspace.

(d) Since the geometric multiplicity of the eigenvalue 2 is not equal to its algebraic multiplicity, A is not diagonalizable.

3.

(a) When we expand the determinant for the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$, we will get a term $(a_{11} - \lambda)\cdots(a_{nn} - \lambda)$. Thus the highest order term in $p_A(\lambda)$ is $(-\lambda)\cdots(-\lambda) = (-1)^n\lambda^n$. On the other hand, since $\lambda_1, \cdots, \lambda_n$ are roots of $p_A(\lambda)$, the linear terms $\lambda - \lambda_1, \cdots, \lambda - \lambda_n$ are factors of $p_A(\lambda)$. Since the highest order term of $(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$ is λ^n, $p_A(\lambda) = c(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$, where c is a constant. By comparing the highest order term on both sides, we see that $c = (-1)^n$. Thus, $p_A(\lambda) = (-1)^n(\lambda-\lambda_1)\cdots(\lambda-\lambda_n) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$.

(b) Let $\lambda = 0$ in the identity $\det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$, we get $\det(A) = \lambda_1 \cdots \lambda_n$.

(b) By part (a)

$$\det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$$

holds for all λ. If we let $\lambda = 0$ in this, we get,

$$\det(A) = \det(A - 0I) = (\lambda_1 - 0)\cdots(\lambda_n - 0) = \lambda_1 \cdots \lambda_n.$$

4.

(a) $p_A(\lambda) = \begin{vmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{vmatrix} = (7-\lambda)(-1-\lambda) - (4)(-3) = \lambda^2 - 6\lambda + 5.$

(b) Since $p_A(\lambda) = (\lambda - 1)(\lambda - 5)$, $\lambda = 1, 5$ are the eigenvalues of A. To find the eigenvector(s) corresponding to the eigenvalue $\lambda = 1$, we solve $(A - (1)I)\underline{x} = \underline{0}$, which is, $\begin{pmatrix} 6 & 4 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The augmented coefficient matrix is $\begin{pmatrix} 6 & 4 & | & 0 \\ -3 & -2 & | & 0 \end{pmatrix}$ and its reduced row echelon form is $\begin{pmatrix} 1 & 2/3 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$ Thus, $\underline{x} = \begin{pmatrix} -\frac{2}{3}x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$. Similarly, we find the $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 5$.

(c) Let $D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ (a diagonal matrix) and $P = \begin{pmatrix} -\frac{2}{3} & -2 \\ 1 & 1 \end{pmatrix}$ (an invertible matrix). Then $P^{-1}AP = D$.

(d) $A = PDP^{-1}$ and

$$A^{10} = PD^{10}P^{-1} = \begin{pmatrix} -\frac{2}{3} & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1^{10} & 0 \\ 0 & 5^{10} \end{pmatrix}\begin{pmatrix} 3/4 & 3/2 \\ -3/4 & -1/2 \end{pmatrix} = \begin{pmatrix} 14648437 & 9765624 \\ -7324218 & -4882811 \end{pmatrix}.$$

(e) Let $C = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{pmatrix}$. Then $C^2 = D$. Now define $B = PCP^{-1}$. Then

$$B^2 = (PCP^{-1})(PCP^{-1}) = PCP^{-1}PCP^{-1} = PCCP^{-1} = PC^2P^{-1} = PDP^{-1} = A.$$

Finally,

$$B = \begin{pmatrix} -\frac{2}{3} & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} 3/4 & 3/2 \\ -3/4 & -1/2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} + \frac{3\sqrt{5}}{2} & -1 + \sqrt{5} \\ -\frac{3}{4} - \frac{3\sqrt{5}}{4} & \frac{3}{2} - \frac{\sqrt{5}}{2} \end{pmatrix}.$$

5.

(a) Since $p_{A^T}(\lambda) = \det(A^T - \lambda I) = \det(A^T - (\lambda I)^T) = \det((A - \lambda I)^T) = \det(A - \lambda I) = p_A(\lambda)$, the matrices A^T and A have the same characteristic polynomial. Since the eigenvalues are the roots of the characteristic polynomial, A^T and A have the same eigenvalues.

(b) Recall that μ is an eigenvalue if and only if $\lambda = \mu$ satisfies the characteristic equation $\det(A - \lambda I) = 0$. Now let $\mu = 0$ and we see that 0 is an eigenvalue if and only if (in short, iff) $\det(A) = \det(A - 0I) = 0$.