Math 225 (Q1) Solution to Homework Assignment 10

1.

(a) Let $B = A^TA = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix}$. Then the eigenvalues of B are: $\lambda_1 = 25$, $\lambda_2 = 9$, $\lambda_3 = 0$ with corresponding unit eigenvectors: $\underline{v_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} -\frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{pmatrix}$, $\underline{v_3} = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$. The singular values of A are: $\sigma_1 = 5$, $\sigma_2 = 3$, $\sigma_3 = 0$. Since there are two positive singular values, the rank of A is 2.

(b) Let $V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & -\frac{4}{3\sqrt{2}} & \frac{1}{3} \end{pmatrix}$. Let $\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$. Let $\underline{u_1} = \frac{1}{\sigma_1}A\underline{v_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $\underline{u_2} = \frac{1}{\sigma_2}A\underline{v_2} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. $A = U\Sigma V^T$ is a singular value decomposition of A.

(c) A basis of Col(A) is $\{A\underline{v_1}, A\underline{v_2}\}$ or we can use $\{\underline{u_1}, \underline{u_2}\} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$. A basis of Nul$(A)$ is $\{\underline{v_3}\} = \left\{ \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \right\}$.

2. Let A be a $n \times n$ matrix and let $A = U\Sigma V^T$ be a singular value decomposition of A. Then $\det(A) = \det(U)\det(\Sigma)\det(V^T)$. Since U is an orthogonal matrix, $U^TU = I$. This implies

$$1 = \det(I) = \det(U^T)\det(U) = \det(U)\det(U) = \det(U)^2,$$

so that $\det(U) = \pm 1$. Similarly, $\det(V^T) = \det(V) = \pm 1$. Thus, $\det(A) = \pm\det(\Sigma)$. Finally, Σ is a diagonal matrix with the singular values of A on its diagonal, therefore $\det(\Sigma)$ is the product of the singular values of A. Hence, $|\det(A)|$ is the product of the singular values, $\sigma_1, \cdots, \sigma_n$ of A, since $\sigma_i \geq 0$.

3. Let $S = \{\underline{s_1}, \cdots, \underline{s_n}\}$. Then S is a linear independent set. In order to show S is a basis of V, it remains to show Span$(S) = V$. Let $\underline{v} \in V$. we want to show that $\underline{v}$ is a linear combination of the vectors in S. If $\underline{v} \in S$, then $v = \underline{s_i}$ for some i and

$$\underline{v} = 0\underline{s_1} + \cdots + 1\underline{s_i} + \cdots + 0\underline{s_n}.$$

That is, $\underline{v}$ is a linear combination of vectors in S. On the other hand, if $\underline{v} \notin S$, then $\underline{s_1}, \cdots, \underline{s_n}, \underline{v}$ must be linearly dependent, since S is a maximal linearly independent set. This means, there exist scalars $c_1, \cdots, c_n$ and c (not all of them are zero) such that

$$c_1\underline{s_1} + \cdots + c_n\underline{s_n} + c\underline{v} = \underline{0}.$$

The number c cannot be zero because if $c = 0$, then

$$c_1\underline{s_1} + \cdots + c_n\underline{s_n} = \underline{0}.$$

By the linear independence of S, $c_1 = \cdots = c_n = 0$ which contradicts the assumption that not all of the $c_1, \cdots, c_n$ and c are zero. Now since $c \neq 0$, we have

$$\underline{v} = (-\frac{c_1}{c})\underline{s_1} + \cdots + (-\frac{c_1}{c})\underline{s_n}$$

so that $\underline{v}$ is indeed a linear combination of $\underline{s_1}, \cdots, \underline{s_n}$, as desired.

4. Note: We could have consider A^T, which is a 2×3 matrix. Use the method in Question 1 to find a SVD for A^T as $A^T = U_1\Sigma_1V_1^T$ and finally take transpose again to get a SVD for A as $A = V_1\Sigma_1^TU_1^T = U\Sigma V^T$.

(a) A SVD for $A = U\Sigma V^T$ can be found as follows. Let $B = A^TA = \begin{pmatrix} 20 & -10 \\ -10 & 5 \end{pmatrix}$. The eigenvalues of B are: $\lambda_1 = 25$ and $\lambda_2 = 0$ with corresponding eigenvectors $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$. After we normalize these eigenvectors, we get, $\underline{v_1} = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ and $\underline{v_2} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$. Notice that $\underline{v_1}$ and $\underline{v_2}$ are orthogonal because they are eigenvectors corresponding to distinct eigenvalues of B. The set $\{\underline{v_1}.\underline{v_2}\}$ is an orthonormal basis of $\mathbf{R}^2$. Let $V = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$. The singular values of A are: $\sigma_1 = \sqrt{\lambda_1} = 5$ and $\sigma_2 = \sqrt{\lambda_2} = 0$. The matrix Σ has the same size as A and so $\Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\underline{u_1} = \frac{1}{\sigma_1}A\underline{v_1}$. Then $\underline{u_1} = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$. We have to find two more vectors, $\underline{u_2}$ and $\underline{u_3}$ in $\underline{u_1}^\perp$ so that $\underline{u_1}$, $\underline{u_2}$ and $\underline{u_3}$ form an orthonormal basis of $\mathbf{R}^3$. Let $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \underline{u_1}^\perp$. Then $(-\frac{2}{\sqrt{5}})x_1 + (-\frac{1}{\sqrt{5}})x_2 + (0)x_3 = 0$

so that $2x_1 + x_2 = 0$. Clearly, x_2 and x_3 are free variables and

$$\underline{x} = \begin{pmatrix} -\frac{1}{2}x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis of $\underline{u_1}^{\perp}$. Notice that these two basis vectors are already orthogonal and so there is no need to apply the Gram-Schmidt process to them. Normalize, we get $\underline{u_2} = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix}$ and $\underline{u_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Thus, $U = \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. A singular value decompostion of A is

$$A = \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^T.$$

(b)

$$A^+ = V_1 \Sigma_1^{-1} U_1^T = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \frac{1}{5} \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{25} & \frac{2}{25} & 0 \\ -\frac{2}{25} & -\frac{1}{25} & 0 \end{pmatrix}.$$

5. Let $S = \{\underline{s_1}, \cdots, \underline{s_n}\}$. Then $\text{Span}(S) = V$. It remains to show S is a linear independent set. Suppose not. Then one of the vectors in S will be a linear combination of the other vectors in S. Let's say $\underline{s_1}$ is a linear combination of $\underline{s_2}, \cdots, \underline{s_n}$. Then

$$V = \text{Span}\{\underline{s_1}, \underline{s_2}, \cdots, \underline{s_n}\} = \text{Span}\{\underline{s_2}, \cdots, \underline{s_n}\}$$

which contradicts the assumption that S is a minimal spanning set. This contradiction shows that S has to be linear independent.