Math 225 (Q1) Homework Assignment 1.

1. Let $\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ be two vectors in $\mathbf{R}^n$. Recall, the dot product of $\underline{u}$ and $\underline{v}$ is defined as

$$\underline{u} \cdot \underline{v} = u_1 v_1 + \cdots + u_n v_n.$$

and the length (also called magnitude or norm) of $\underline{u}$ is defined as

$$||\underline{u}|| = \sqrt{\underline{u} \cdot \underline{u}}.$$

(a) Show that

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v},$$

where $\underline{u}^T = (u_1, \cdots, u_n)$ is the transpose of $\underline{u}$.

(b) Show that $||\underline{u}|| = 0$ if and only if $\underline{u} = \underline{0}$, that is, the vector $\underline{u}$ has zero length if and only if $\underline{u}$ is the zero vector.

2. Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a $m \times n$ matrix and let $B = (b_{jk})_{1 \leq j \leq n, 1 \leq k \leq p}$ be a $n \times p$ matrix. Recall that the product C of the matrices A and B is a $m \times p$ matrix, $C = (c_{ik})_{1 \leq i \leq m, 1 \leq k \leq p}$, where the (i, k)–th entry of C is given by

$$c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + \cdots + a_{in} b_{nk}.$$

Note that c_{ik} is the "dot product" of the i–th row of A with the k–th column of B. Also recall that the transpose of A is the matrix A^T which is a $n \times m$ matrix whose i–th row is the i-th column of A. That is, if $D = A^T$ and $D = (d_{rs})_{1 \leq r \leq n, 1 \leq s \leq m}$, then

$$d_{rs} = a_{sr}, \qquad \text{for all} \quad 1 \leq r \leq n, \ 1 \leq s \leq m.$$

Show that $(AB)^T = B^T A^T$.

3. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation, that is, T is a mapping taking vectors in $\mathbf{R}^n$ to vectors in $\mathbf{R}^m$ and T satisfies the properties

$$T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$$

$$T(c\underline{u}) = cT(\underline{u})$$

for all $\underline{u}, \underline{v} \in \mathbf{R}^n$ and $c \in \mathbf{R}$. Recall, the kernel of T is the subset of $\mathbf{R}^n$ defined by

$$\mathrm{Ker}(T) = \{\underline{u} \in \mathbf{R}^n \ : \ T(\underline{u}) = \underline{0}\}.$$

and the range of T is the subset of $\mathbf{R}^m$ defined by

$$\mathrm{Ran}(T) = \{\underline{v} \in \mathbf{R}^m \ : \ \underline{v} = T(\underline{u}) \quad \text{for some} \quad \underline{u} \in \mathbf{R}^n\}.$$

(a) Show that $\mathrm{Ker}(T)$ is a subspace of $\mathbf{R}^n$.

(b) Show that $\mathrm{Ran}(T)$ is a subspace of $\mathbf{R}^m$.

4. Let A be a $m \times n$ matrix. Recall that the null space of A is the set

$$\mathrm{Nul}(A) = \{\underline{x} \in \mathbf{R}^n \ : \ A\underline{x} = \underline{0}\}.$$

Show that $\mathrm{Nul}(A^T A) = \mathrm{Nul}(A)$.

5. Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ be any three vectors in $\mathbf{R}^n$. Define the vectors $\underline{p} = \underline{u} - \underline{v}$, $\underline{q} = \underline{v} - \underline{w}$ and $\underline{r} = \underline{w} - \underline{u}$. Show that $\underline{p}$, $\underline{q}$ and $\underline{r}$ are linearly dependent by expressing one of the vectors as a linear combination of the other two vectors.