Review of Math 118

Definition: Let f be bounded on $[a,b]$ and $P = \{x_0, x_1, \ldots, x_n\}$, where

$$a = x_0 < x_1 < \ldots < x_n = b.$$

$$\mathcal{L}(P,f) = \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f \cdot (x_i - x_{i-1})$$
$$\mathcal{U}(P,f) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f \cdot (x_i - x_{i-1})$$

$$\underline{\int_a^b} f = \sup\{\mathcal{L}(P,f) : \forall P \text{ of } [a,b]\}$$
$$\overline{\int_a^b} f = \inf\{\mathcal{U}(P,f) : \forall P \text{ of } [a,b]\}$$

If

$$\underline{\int_a^b} f = \overline{\int_a^b} f = \alpha \in \mathbb{R},$$

we say

$$\int_a^b f \ \exists = \alpha.$$

Integrable $\iff$ $\exists \{P_n\}_{n=1}^{\infty} \ni \lim_{n\to\infty} \mathcal{L}(P_n, f) = \lim_{n\to\infty} \mathcal{U}(P_n, f)$
Can Integrate Piecewise
Linearity & Positivity of Integral Operator
Bounds on Integrand $\Rightarrow$ Bounds on Integral
Integrals on $[a,b]$ are Continuous Functions of a and b.
Continuous $\Rightarrow$ Integrable on $[a,b]$
Monotonic $\Rightarrow$ Integrable on $[a,b]$
Darboux: $\int_a^b f = \alpha \iff$ all Riemann sums converge to α whenever $x_i - x_{i-1} \to 0$

MVT for Integrals (Continuous Functions Take on their Average Value)
FTC:

$$\int_a^b F' = F(b) - F(a)$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))b'(x) - f(a(x))a'(x)$$

$$x^a = e^{a\log x}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int \sec x\,dx = \log(\sec x + \tan x) + C$$

1. Change of Variables

$$\int_a^b f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du$$

- $\int \sin^n x \cos^m x\,dx$ odd power $\Rightarrow$ Let $u =$ other trig function

2. Integration by Parts

$$\int_a^b fg' = [fg]_a^b - \int_a^b f'g$$

3. Partial Fractions

(i)

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}, \quad \deg R < \deg Q$$

(ii) Factor

$$Q(x) = (x-\alpha)^n \ldots (x^2+\gamma x+\lambda)^m \ldots, \text{where } \gamma^2 - 4\lambda < 0$$

(iii) Expand

$$\begin{aligned}\frac{R(x)}{Q(x)} &= \frac{A_1}{(x-\alpha)} + \frac{A_2}{(x-\alpha)^2} + \ldots + \frac{A_n}{(x-\alpha)^n} \\ &+ \ldots \\ &+ \frac{\Gamma_1 x + \Lambda_1}{(x^2+\gamma x+\lambda)} + \frac{\Gamma_2 x + \Lambda_2}{(x^2+\gamma x+\lambda)^2} + \ldots + \frac{\Gamma_m x + \Lambda_m}{(x^2+\gamma x+\lambda)^m} \\ &+ \ldots\end{aligned}$$

Area

$$A = \int_a^b |f(x) - g(x)|\, dx$$

Volumes by Slices

$$V = \int A(x)\, dx = \pi \int \left({r_{\text{out}}}^2 - {r_{\text{in}}}^2\right) \qquad \text{(Slice axis of revolution)}$$

Volume by Shells

$$V = 2\pi \int rh\, dw \qquad \text{(Slice other axis)}$$

Work

$$W = \int \rho g \underbrace{D(x)}_{\text{distance}} \underbrace{A(x)\, dx}_{\text{volume}}$$

Pappus Theorem Area (Volume)=Length (Area) × Distance Travelled by Centroid

$$\bar{x} = \frac{1}{L} \int x\, ds,$$
$$\bar{y} = \frac{1}{L} \int y\, ds,$$

where $L = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{r'^2 + r^2}\, d\theta$.

$$\bar{x} = \frac{1}{A} \int x\, dA = \frac{1}{A} \int x\, |f(x) - g(x)|\, dx,$$
$$\bar{y} = \frac{1}{A} \int y\, dA = \frac{1}{2A} \int \left[f^2(x) - g^2(x)\right] dx \qquad \text{if } f(x) \geq g(x),$$

where $A = \int |f(x) - g(x)|\, dx = \frac{1}{2} \int r^2\, d\theta$.

The surface area generated by rotating a curve about an axis is

$$2\pi \int_a^b r\, ds,$$

where r measures the (perpendicular) distance to the axis of revolution.

Comparison Test $0 \leq f \leq g$:

(i)

$$\int g \in \mathcal{C} \Rightarrow \int f \in \mathcal{C},$$

(ii)

$$\int f \in \mathcal{D} \Rightarrow \int g \in \mathcal{D}.$$