Review of Math 101

$$\int_a^b F' = F(b) - F(a)$$

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))b'(x) - f(a(x))a'(x)$$

$$x^a = e^{a\log x}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int \sec x\,dx = \log(\sec x + \tan x) + C$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

1. Change of Variables

$$\int_a^b f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du$$

2. Integration by Parts

$$\int_a^b fg' = [fg]_a^b - \int_a^b f'g$$

3. Partial Fractions

(i)

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}, \quad \deg R < \deg Q$$

(ii) Factor

$$Q(x) = (x-a)^n \dots (x^2 + \gamma x + \lambda)^m \dots, \text{where } \gamma^2 - 4\lambda < 0$$

(iii) Expand

$$\frac{R(x)}{Q(x)} = \left[\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}\right] + \dots$$
$$+ \left[\frac{\Gamma_1 x + \Lambda_1}{x^2 + \gamma x + \lambda} + \dots + \frac{\Gamma_m x + \Lambda_m}{(x^2 + \gamma x + \lambda)^m}\right] + \dots$$

Expression	x Domain	Substitution	θ or t Domain	Identity
$\sqrt{a^2 - x^2}$	$[-a, a]$	$x = a\sin\theta$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$(-\infty, \infty)$	$x = a\tan\theta$ $x = a\sinh t$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $(-\infty, \infty)$	$1 + \tan^2\theta = \sec^2\theta$ $1 + \sinh^2 t = \cosh^2 t$
$\sqrt{x^2 - a^2}$	$(-\infty, -a] \cup [a, \infty]$	$x = a\sec\theta$ $x = a\cosh t$	$\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ $[0, \infty)$	$\sec^2\theta - 1 = \tan^2\theta$ $\cosh^2 t - 1 = \sinh^2 t$

Table 1: Useful trigonometric substitutions

$\int \sin^n x \cos^m x \, dx$

odd power: let $u =$ other trig function
even power: reduce with

$$2\sin x\cos x = \sin 2x, \qquad \sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Area

$$A = \int_a^b |f(x) - g(x)|\, dx$$

Volumes by Slices

$$V = \int A(x)\, dx = \pi \int \left({r_{\text{out}}}^2 - {r_{\text{in}}}^2\right) \qquad \text{Slice axis of revolution}$$

Volume by Shells

$$V = 2\pi \int rh\, dw \qquad \text{Slice other axis}$$

Comparison Test $0 \le f \le g$

$$\int_a^\infty g \text{ converges } \Rightarrow \int_a^\infty f \text{ converges}$$

$$\int_a^\infty f \text{ diverges } \Rightarrow \int_a^\infty g \text{ diverges}$$

Convergent	Divergent
$\int_1^\infty \frac{1}{x^p}\, dx \quad (p > 1)$	$\int_1^\infty \frac{1}{x^p}\, dx \quad (p \le 1)$
$\int_0^\infty e^{-\alpha x}\, dx \quad (\alpha > 0)$	$\int_0^\infty e^{-\alpha x}\, dx \quad (\alpha \le 0)$
$\int_{0^+}^1 \frac{1}{x^p}\, dx \quad (p < 1)$	$\int_{0^+}^1 \frac{1}{x^p}\, dx \quad (p \ge 1)$
$\int_{0^+}^1 \log x\, dx$	$\int_0^{\pi/2^-} \tan x\, dx$

Table 2: Useful integrals for Comparison Test

Linear Differential Equations

$$\frac{dy}{dx} + P(x)y = Q(x),$$

Multiply both sides by $I = e^{\int P(x)\, dx}$:

$$\frac{d}{dx}(Iy) = I\frac{dy}{dx} + IPy = IQ$$

Geometric Series

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \qquad |r|< 1$$

p**-Series**

$$\sum_{k=0}^{\infty} \frac{1}{k^p} \text{ converges } \iff p > 1$$

Harmonic Series

$$\sum_{k=0}^{\infty} \frac{1}{k} \text{ diverges}$$

Divergence Test

$$\sum_{k=1}^{\infty} a_k \text{ converges } \Rightarrow \lim_{n\to\infty} a_n = 0$$

Integral Test

If f is integrable on any closed interval and decreasing and non-negative on $[1,\infty)$,

$$\sum_{k=1}^{\infty} f(k) \text{ converges } \iff \int_1^{\infty} f \text{ converges}$$

Remainder Estimate

Let f be integrable on any closed interval and decreasing and non-negative on $[1,\infty)$. Then the *remainder* $\sum_{k=n+1}^{\infty} f(k)$ of $\sum_{k=1}^{\infty} f(k)$ on truncating the series after n terms satisfies

$$\int_{n+1}^{\infty} f \leq \sum_{k=n+1}^{\infty} f(k) \leq \int_n^{\infty} f$$

Comparison Test

If $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$ then

$$\sum_{k=1}^{\infty} b_k \text{ converges } \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$\sum_{k=1}^{\infty} a_k \text{ diverges } \Rightarrow \sum_{k=1}^{\infty} b_k \text{ diverges}$$

Limit Comparison Test

Suppose $a_k \geq 0$ and $b_k > 0$ for all $k \in \mathbb{N}$ and $L = \lim_{k\to\infty} \frac{a_k}{b_k}$.

$0 < L < \infty$:

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \sum_{k=1}^{\infty} b_k \text{ converges}$$

$L = 0$:

$$\sum_{k=1}^{\infty} b_k \text{ converges} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

Leibniz Alternating Series Test

The alternating series $\sum_{k=1}^{\infty}(-1)^k a_k$ is convergent if the sequence $\{a_k\}_{k=1}^{\infty}$ decreases monotonically to 0

Alternating Series Remainder Estimate

If $\{a_k\}_{k=1}^{\infty}$ is a monotonically decreasing sequence that converges to 0,

$$\left|\sum_{k=n+1}^{\infty}(-1)^k a_k\right| \leq a_{n+1}$$

Absolute Convergence

An absolutely convergent series is convergent

Conditional Convergence

Convergence without absolute convergence (e.g. alternating harmonic series)

Limit Ratio Test Suppose $a_k > 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$

$$r < 1 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$r > 1 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ diverges}$$

$$r = 1 \Rightarrow ?$$

Limit Root Test Suppose $a_k \geq 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \sqrt[k]{a_k} = r$

$$r < 1 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges}$$

$$r > 1 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ diverges}$$

$$r = 1 \Rightarrow ?$$

Power Series

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

Radius of Convergence

$$\sum_{k=0}^{\infty} c_k x^k \begin{cases} \text{converges absolutely} & \text{if } |x| < R \\ ? & \text{if } |x| = R \\ \text{diverges} & \text{if } |x| > R \end{cases}$$

Taylor Expansion

$$f(b) = \sum_{k=0}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a) + \underbrace{\frac{(b-a)^n}{n!} f^{(n)}(c)}_{R_n}$$

for some $c \in (a, b)$

Taylor's Inequality

If $|f^{(n)}|$ is an increasing function on $[a, b]$, then $|R_n| \leq |f^{(n)}(b)||b-a|^n/n!$
If $|f^{(n)}|$ is an decreasing function on $[a, b]$, then $|R_n| \leq |f^{(n)}(a)||b-a|^n/n!$

Binomial Series

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k, \qquad |x| < 1$$

where

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \\ \dfrac{n(n-1)\dots(n-k+1)}{k!} & \text{if } k \geq 1 \end{cases}$$

Polar Coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{1}$$

Area in Polar Coordinates

$$A = \frac{1}{2}\int r^2(\theta)\,d\theta$$

First Derivative of Parametrized Curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Second Derivative of Parametrized Curve

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{dt}{dx}\cdot\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

Cylindrical Coordinates

$$\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned}$$

where $\varphi \in [0, 2\pi]$

Spherical Polar Coordinates

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}$$

where $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$

Unit Tangent Vector

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$$

$$\boldsymbol{T}'\cdot\boldsymbol{T} = 0$$

Parametric Equation of a Line

$$\boldsymbol{r}(t) = (1-t)\boldsymbol{p} + t\boldsymbol{q}, \qquad t \in \mathbb{R}$$

Point-Direction Equation of a Line

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$$

Symmetric Equation of a Line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Point-Direction Equation of a Plane

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Parametric Equation of a Plane Spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{u} + s\boldsymbol{v}, \qquad t, s \in \mathbb{R}$$

Distance of (X, Y, Z) from a Plane $ax + by + cz = d$

$$D = \left| \frac{(X, Y, Z) \cdot (a, b, c) - d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Ellipsoid
$x^2 + y^2 + z^2 = 1$

Hyperboloid of One Sheet
$x^2 + y^2 - z^2 = 1$

Hyperboloid of Two Sheets
$-x^2 - y^2 + z^2 = 1$

Elliptic Paraboloid
$z = x^2 + y^2$

Hyperbolic Paraboloid
$z = x^2 - y^2$

Arc length

$$L = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{r'^2 + r^2}\, d\theta = \int |\boldsymbol{r}'(t)|\; dt$$

Surface area of curve rotated about axis

$$2\pi \int_a^b r\, ds$$

$r =$ (perpendicular) distance to the axis of revolution

Curvature

$$\kappa = \left| \frac{d\boldsymbol{T}}{ds} \right| = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3}$$