Math 100/101:
Calculus I/II

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Contents

Preface

These notes were developed for a first-year engineering mathematics course on differential and integral calculus at the University of Alberta. The author would like to thank Andy Hammerlindl and Tom Prince for coauthoring the high-level graphics language `Asymptote` (freely available at `http://asymptote.sourceforge.net`) that was used to draw the mathematical figures in this text. The code to lift TEX characters to three dimensions and embed them as surfaces in `PDF` files was developed with in collaboration with Orest Shardt.

Chapter 0

Real Numbers

Mathematics deals with different types of numbers:

$\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of natural (counting) numbers;

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of integers;

$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\}$, the set of rational numbers (fractions);

$\mathbb{R}$, the set of all real numbers.

Notice that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Remark: The decimal expansion of a rational number ends in a repeating pattern of digits:

$$1/2 = 0.5000\ldots = 0.5\bar{0}$$

$$1/3 = 0.3333\ldots = 0.\bar{3}$$

$$2/7 = 0.285714285714\ldots = 0.2\overline{8571}4$$

Remark: The real numbers are those numbers like $\sqrt{2} = 1.414213562373\ldots$ and $\pi = 3.1415926535897\ldots$ that do not end in a repeating pattern and thus cannot be represented as a ratio of two integers.

0.1 Open and Closed Intervals

Let $a, b \in \mathbb{R}$ and $a < b$. There are 4 types of (finite) intervals:

$$\begin{aligned}
[a, b] &= \{x : a \leq x \leq b\}, \leftarrow \textit{closed} \text{ (contains both endpoints)} \\
(a, b) &= \{x : a < x < b\}, \leftarrow \quad \textit{open} \text{ (excludes both endpoints)} \\
[a, b) &= \{x : a \leq x < b\}, \\
(a, b] &= \{x : a < x \leq b\}.
\end{aligned}$$

It is convenient to also define:

$$\begin{aligned}
&(-\infty, \infty) = \mathbb{R}, \\
&[a, \infty) = \{x : x \geq a\}, \\
&(a, \infty) = \{x : x > a\}, \\
&(-\infty, a] = \{x : x \leq a\}, \\
&(-\infty, a) = \{x : x < a\}.
\end{aligned}$$

0.2 Inequalities

- $a < b \Rightarrow a + c < b + c$

- $a < b$ and $c < d \Rightarrow a + c < b + d$

- $a < b$ and $c > 0 \Rightarrow ac < bc$

- $a < b$ and $c < 0 \Rightarrow ac > bc$

- $0 < a < b \Rightarrow 1/a > 1/b$

- To determine the set of x values for which $x^2 - 5x + 6 < 0$, we factor $x^2 - 5x + 6 = (x-2)(x-3)$ and consider the following table:

Interval	$x-2$	$x-3$	$(x-2)(x-3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

We thus see that $x^2 - 5x + 6 = (x-2)(x-3) < 0$ if and only if $2 < x < 3$, in other words, when $x \in (2, 3)$.

0.3 Absolute Value

The fact that for any nonzero real number either $x > 0$ or $-x > 0$ makes it convenient to define an *absolute value* function:

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0, \\ -x & \text{if} \quad x < 0. \end{cases}$$

Properties: Let x and y be any real numbers.

(A1) $|x| \geq 0$.

(A2) $|x| = 0 \iff x = 0$.

(A3) $|-x| = |x|$.

(A4) $|xy| = |x|\,|y|$.

(A5) If $a \geq 0$, then

$$|x| \leq a \iff -a \leq x \leq a.$$

Proof: First note the equivalence

$$\begin{aligned} |x| \leq a \quad &\iff \quad 0 \leq x \leq a \quad \text{or} \quad 0 < -x \leq a \\ &\iff \quad -a \leq x \leq a. \end{aligned}$$

(A6) $-|x| \leq x \leq |x|$.

Proof: Apply (A5) with $a = |x|$.

(A7)

$$|x + y| \leq |x| + |y|. \qquad \textit{(Triangle Inequality)}$$

Proof:

$$\textbf{(A6)} \Rightarrow \begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases}$$

$$\begin{aligned} \Rightarrow -(|x| + |y|) \leq x + y \leq |x| + |y| = a \\ \textbf{(A5)} \Rightarrow |x + y| \leq |x| + |y|. \end{aligned}$$

Remark: On letting $y \to -y$, we can use **(A3)** to rewrite the Triangle Inequality as

$$|x - y| \leq |x| + |y|.$$

- If $|u-1|< 0.1$ and $|v-1| < 0.2$ then

$$\begin{aligned}|u-v| &= |(u-1)-(v-1)| \leq |u-1|+|v-1| \\ &< 0.1+0.2 = 0.3.\end{aligned}$$

Remark: For all real x, $\sqrt{x^2} = |x| \geq 0$. By definition, $\sqrt{x}$ and $x^{1/2}$ denote the **non-negative** square root of x. For example, $\sqrt{4} = 2$, not ± 2.

Remark: Note the following equivalences:

- $|x| = a \iff x = \pm a$;
- $|x| < a \iff -a < x < a$;
- $|x| > a \iff x < -a$ or $x > a$.

Problem 0.1: Find the set of x such that $|x-1| \leq x$.

To handle the absolute value, we must break the problem into cases:

We know that the argument $x-1$ of the absolute value is non-negative when $x \geq 1$. In this case, the inequality reduces to

$$x - 1 \leq x,$$

which always holds. The solutions set for this case is thus $[1, \infty)$.

In the remaining case, where $x < 1$, the inequality becomes

$$-x + 1 \leq x,$$

which means that $1 \leq 2x$, or equivalently, $x \geq 1/2$. But this is still under the restriction that $x < 1$ so our solution set for this case is the interval $[1/2, 1)$.

The complete solution to the original inequality $|x-1| \leq x$ is the union of our two solutions sets, namely $[1, \infty) \cup [1/2, 1) = [1/2, \infty)$.

Chapter 1

Functions

1.1 Examples of Functions

Definition: A *function* f is a rule that associates a real number y to each real number x in some subset $\mathcal{D}$ of $\mathbb{R}$. The set $\mathcal{D}$ is called the *domain* of f.

Definition: The *range* $f(\mathcal{D})$ of f is the set $\{f(x) : x \in \mathcal{D}\}$.

- $f(x) = x^2$ on domain $\mathcal{D} = [0, 2)$:

$$f(\mathcal{D}) = \{x^2 : x \in [0, 2)\} = [0, 4).$$

An equivalent definition is:

Definition: A *function* is a collection of pairs of numbers (x, y) such that if (x, y_1) and (x, y_2) are in the collection, then $y_1 = y_2$. That is,

$$x_1 = x_2 \Rightarrow f(x_1) = f(x_2).$$

This can be restated as the *vertical line test*: an set of ordered pairs (x, y) is a function if every vertical line intersects their graph at most once.

Definition: If a function f has domain A and range B, we write $f : A \to B$.

Definition: *Constant functions* are functions of the form $f(x) = c$, where c is a constant.

- $f(x) = 1$ is a constant function.

Definition: *Polynomials* are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0.$$

When $a_n \neq 0$, we say that the *degree* of f is n and write $\deg f = n$. While a nonzero constant function has degree 0, it turns out to be convenient to define the degree of the zero function $f(x) = 0$ to be $-\infty$.

- $f(x) = x^2 + 1$ and $f(x) = 3x^2 - 1$ are polynomials of degree 2.

Note that a polynomial $f(x)$ with only even-degree terms (all the odd-degree coefficients are zero) satisfies the property $f(-x) = f(x)$, while a polynomial $f(x)$ with only odd-degree terms satisfies $f(-x) = -f(x)$. We generalize this notion with the following definition.

Definition: A function f is said to be *even* if $f(-x) = f(x)$ for every x in the domain of f.

Definition: A function f is said to be *odd* if $f(-x) = -f(x)$ for every x in the domain of f.

- The functions x, x^3, and $\sin x$ are odd.

- The functions 1, x^2, and $\cos x$ are even.

- The functions $x + 1$, $\log x$, e^x are neither even nor odd.

Problem 1.1: Show that an odd function f with domain $\mathbb{R}$ satisfies $f(0) = 0$.

Definition: *Rational functions* are functions of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials. They are defined on the set of all x for which $Q(x) \neq 0$.

- $\dfrac{1}{x}$ and $\dfrac{x^3 + 3x^2 + 1}{x^2 + 1}$ are both rational functions.

Composition Once we have defined a few elementary functions, we can create new functions by combining them together using $+$, $-$, $\cdot$, $\div$, or by introducing the *composition* operator $\circ$.

Definition: If $f : A \to B$ and $g : B \to C$ then we define $g \circ f : A \to C$ to be the function that takes $x \in A$ to $g(f(x)) \in C$.

- $$\begin{array}{ll} f(x) = x^2 + 1 & f : \mathbb{R} \to [1, \infty), \\ g(x) = 2\sqrt{x} & g : [1, \infty) \to [2, \infty), \\ g(f(x)) = 2\sqrt{x^2 + 1} & g \circ f : \mathbb{R} \to [2, \infty). \end{array}$$

  Note however that $f(g(x)) = 4x + 1$, so that $f \circ g : [0, \infty) \to [1, \infty)$.

- $$\begin{array}{ll} f(x) = x^2 + 1 & f : \mathbb{R} \to [1, \infty), \\ g(x) = \frac{1}{x} & g : [1, \infty) \to (0, 1], \\ g(f(x)) = \frac{1}{x^2+1} & g \circ f : \mathbb{R} \to (0, 1]. \end{array}$$

  One can also build new functions from old ones using *cases*, or *piecewise* definitions:

- $$f(x) = \begin{cases} 0 & x < 0, \\ \frac{1}{2} & x = 0, \\ 1 & x > 0. \end{cases}$$

- $$f(x) = |x| = \begin{cases} x & x \geq 0, \\ -x & x < 0. \end{cases}$$

  Cases can sometimes introduce jumps in a function.

Problem 1.2: Graph the function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ 2 - x & \text{if } 1 < x \leq 2. \end{cases}$$

Definition: A function is said to be *increasing* (*decreasing*) on an interval I if

$$x, y \in I, x \leq y \Rightarrow f(x) \leq f(y) \quad (f(x) \geq f(y))$$

and *strictly increasing* (*strictly decreasing*) if

$$x, y \in I, x < y \Rightarrow f(x) < f(y) \quad (f(x) > f(y)).$$

Note that a strictly increasing function is increasing.

1.2 Transformations of Functions

Transformations can be used to obtain new functions from old ones:

- To obtain the graph of $y = f(x) + c$, shift the graph of $f(x)$ a distance c upwards.

- To obtain the graph of $y = f(x - c)$, shift the graph of $f(x)$ a distance c rightwards.

- To obtain the graph of $y = cf(x)$, stretch the graph of $f(x)$ vertically by the factor $c > 0$.

- To obtain the graph of $y = f(x/c)$, stretch the graph of $f(x)$ horizontally by the factor $c > 0$.

- To obtain the graph of $y = -f(x)$, reflect the graph of $f(x)$ about the x axis.

- To obtain the graph of $y = f(-x)$, reflect the graph of $f(x)$ about the y axis.

Problem 1.3: Sketch the graphs of $|x - 1|$ and x on the same axes and use your graph to verify the results of Prob. 0.1.

1.3 Trigonometric Functions

Trigonometric functions are functions relating the shape of a right-angle triangle to one of its other angles.

Definition: If we label one of the non-right angles by θ, the length of the hypotenuse by hyp, and the lengths of the sides opposite and adjacent to x by opp and adj, respectively, then

$$\sin\theta = \frac{\text{opp}}{\text{hyp}},$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}},$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}.$$

Note here that since θ is one of the nonright angles of a right-angle triangle, these definitions apply only when $0 < \theta < 90°$. Note also that $\tan\theta = \sin\theta/\cos\theta$. Sometimes it is convenient to work with the reciprocals of these functions:

$$\csc\theta = \frac{1}{\sin\theta},$$

$$\sec\theta = \frac{1}{\cos\theta},$$

$$\cot\theta = \frac{1}{\tan\theta}.$$

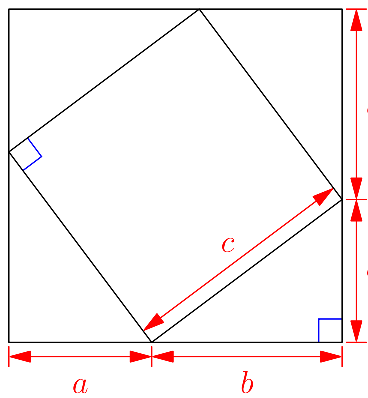


Figure 1.1: Pythagoras' Theorem

Pythagoras' Theorem states that the square of the length c of the hypotenuse of a right-angle triangle equals the sum of the squares of the lengths a and b of the other two sides. A simple geometric proof of this important result is illustrated in Figure 1.1. Four identical copies of the triangle, each with area $ab/2$, are placed around a square of side c, so as to form a larger square with side $a+b$. The area c^2 of the inner square is then just the area $(a+b)^2 = a^2 + 2ab + b^2$ of the large square minus the total area $2ab$ of the four triangles. That is, $c^2 = a^2 + b^2$.

Remark: If we scale a right-angle triangle with angle θ and $90^\circ - \theta$, so that the hypotenuse $c = 1$, the length of the sides opposite and adjacent to the angle θ are $\sin\theta$ and $\cos\theta$, respectively. Pythagoras' Theorem then leads to the following important identity:

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1. \tag{1.1}$$

Other useful identities result from dividing both sides of this equation either by $\sin^2\theta$:

$$1 + \cot^2\theta = \csc^2\theta,$$

or by $\cos^2\theta$:

$$\tan^2\theta + 1 = \sec^2\theta.$$

Note that Eq. (1.1) implies both that $|\sin\theta| \leq 1$ and $|\cos\theta| \leq 1$.

Definition: We define the number π to be the area of a *unit circle* (a circle with radius 1).

Definition: Instead of using degrees, in our development of calculus it will be more convenient to measure angles in terms of the area of the sector they subtend on the unit circle. Specifically, we define an angle measured in *radians* to be twice[1] the area of the sector that it subtends, as shown in Figure 1.2. For example, our definition of π says that a full unit circle ($360°$) has area π; the corresponding angle in radians would then be 2π. Thus, we can convert between radians and degrees with the formula

$$\pi \text{ radians} = 180°.$$

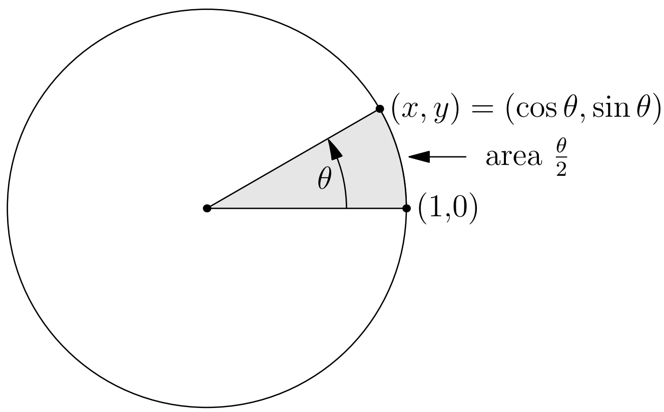


Figure 1.2: The unit circle

The coordinates x and y of a point P on the unit circle are related to θ as follows:

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x,$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y.$$

Complementary Angle Identities:

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right),$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

[1]The reason for introducing the factor of two in this definition is to make the angle x expressed in radians equal to the length of the arc it subtends on the unit circle, as we will see later using integral calculus, once we have developed the notion of the length of an arc. For example, the circumference of a full circle of unit radius will be found to be precisely 2π.

Supplementary Angle Identities:

$$\sin(\pi - \theta) = \sin\theta,$$

$$\cos(\pi - \theta) = -\cos\theta.$$

Symmetries:

$$\sin(-\theta) = -\sin\theta,$$
$$\cos(-\theta) = \cos\theta,$$
$$\sin(\theta + 2\pi) = \sin\theta,$$
$$\cos(\theta + 2\pi) = \cos\theta.$$

Problem 1.4: We thus see that $\sin\theta$ is an odd periodic function of θ and $\cos\theta$ is an even periodic function of θ, both with period 2π. Use these facts to prove that $\tan\theta$ is an odd periodic function of θ with period π.

Special Values:

$$\begin{aligned}
\sin(0) &= \cos\left(\frac{\pi}{2}\right) = 0,\\
\sin\left(\frac{\pi}{2}\right) &= \cos(0) = 1,\\
\sin\left(\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},\\
\sin\left(\frac{\pi}{6}\right) &= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\\
\sin\left(\frac{\pi}{3}\right) &= \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.
\end{aligned}$$

Addition Formulae:

Claim:
$$\cos(A - B) = \cos A\cos B + \sin A\sin B.$$

Proof: Consider the points $P = (\cos A, \sin A)$, $Q = (\cos B, \sin B)$, and $R = (1, 0)$ on the unit circle, as illustrated in Fig. 1.3. We can use Pythagoras' Theorem to obtain a formula for the length (squared) of a chord subtended by an angle:

$$\overline{QR}^2 = (1 - \cos B)^2 + \sin^2 B = 1 - 2\cos B + \cos^2 B + \sin^2 B = 2 - 2\cos B.$$

For example, since the angle subtended by $\overline{PQ}$ is $A - B$,

$$\overline{PQ}^2 = 2 - 2\cos(A - B).$$

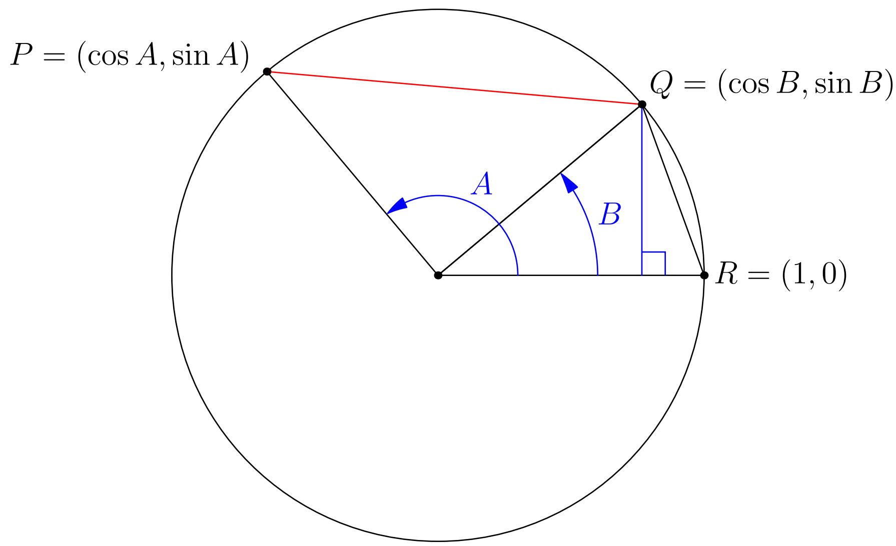


Figure 1.3: The unit circle with points $P = (\cos A, \sin A)$, $Q = (\cos B, \sin B)$, and $R = (1, 0)$

Alternatively, we could compute $\overline{PQ}^2$ directly:

$$\begin{aligned}\overline{PQ}^2 &= (\cos A - \cos B)^2 + (\sin A - \sin B)^2 \\ &= \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B \\ &= 2 - 2(\cos A\cos B + \sin A\sin B).\end{aligned}$$

On comparing these two results, we conclude that

$$\cos(A - B) = \cos A\cos B + \sin A\sin B.$$

The claim thus holds.

Remark: Other trigonometric addition formulae follow easily from the above result:

$$\begin{aligned}\cos(A + B) &= \cos(A - (-B)) \\ &= \cos A\cos(-B) + \sin A\sin(-B) \\ &= \cos A\cos B - \sin A\sin B.\end{aligned}$$

$$\begin{aligned}\sin(A + B) &= \cos\left[\frac{\pi}{2} - (A + B)\right] \\ &= \cos\left[\left(\frac{\pi}{2} - A\right) - B\right] \\ &= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B \\ &= \sin A\cos B + \cos A\sin B.\end{aligned}$$

$$\begin{aligned}\sin(A-B) &= \sin(A-(-B)) \\ &= \sin A\cos(-B)+\cos A\sin(-B) \\ &= \sin A\cos B-\cos A\sin B.\end{aligned}$$

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B} \\ &= \frac{(\sin A\cos B+\cos A\sin B)\cdot\frac{1}{\cos A\cos B}}{(\cos A\cos B-\sin A\sin B)\cdot\frac{1}{\cos A\cos B}} \\ &= \frac{\tan A+\tan B}{1-\tan A\tan B}, \qquad \text{provided } A,\ B,\ A+B \text{ are not odd multiples of } \tfrac{\pi}{2}.\end{aligned}$$

Double-Angle Formulae:

$$\begin{aligned}\sin 2A &= \sin(A+A) \\ &= \sin A\cos A+\sin A\cos A \\ &= 2\sin A\cos A.\end{aligned}$$

$$\begin{aligned}\cos 2A &= \cos(A+A) \\ &= \cos A\cos A-\sin A\sin A \\ &= \cos^2 A-\sin^2 A \\ &= \cos^2 A-(1-\cos^2 A) \\ &= 2\cos^2 A-1 \\ &= (1-\sin^2 A)-\sin^2 A \\ &= 1-2\sin^2 A.\end{aligned}$$

Also, if A is not an odd multiple of $\pi/4$ or $\pi/2$,

$$\begin{aligned}\tan 2A &= \tan(A+A) \\ &= \frac{\tan A+\tan A}{1-\tan A\tan A} \\ &= \frac{2\tan A}{1-\tan^2 A}.\end{aligned}$$

Inequalities: We have already seen that $|\sin x|\le 1$ and $|\cos x|\le 1$. Our development of trigonometric calculus will rely on the following additional key result:

$$\sin x\le x\le \tan x \quad \text{for all } x\in\left[0,\frac{\pi}{2}\right).$$

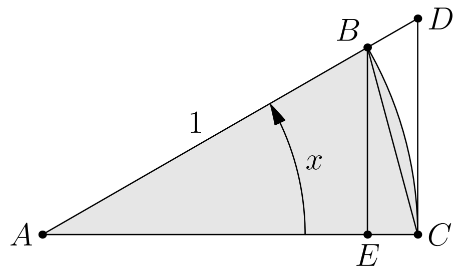
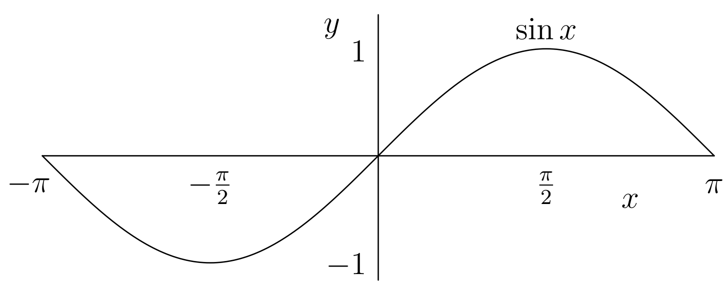


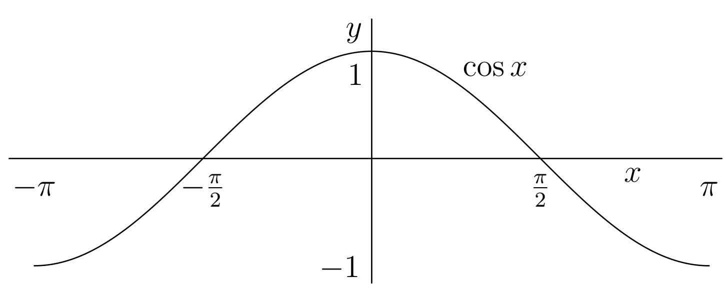
Figure 1.4: Geometric proof of $\sin x \leq x \leq \tan x$

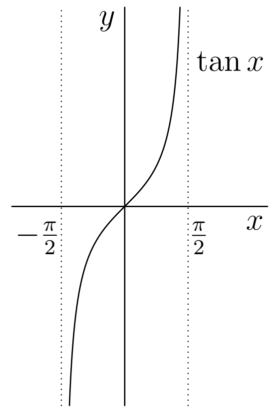
We establish this result geometrically, referring to the arc of unit radius in Fig 1.4. The shaded area of the sector ABC subtended by the angle x (measured in radians) is $x/2$. Since $BE = \sin x$ and $DC = \tan x$, we deduce

$$
\begin{aligned}
&\text{Area}_{\triangle_{ABC}} \leq \text{Area}_{\text{Sector}_{ABC}} \leq \text{Area}_{\triangle_{ADC}} \\
\Rightarrow\ &\frac{1}{2}(1)\sin x \leq \frac{x}{2} \leq \frac{1}{2}(1)\tan x \\
&\Rightarrow \sin x \leq x \leq \tan x \quad \text{for all } x \in \left[0, \frac{\pi}{2}\right).
\end{aligned}
$$

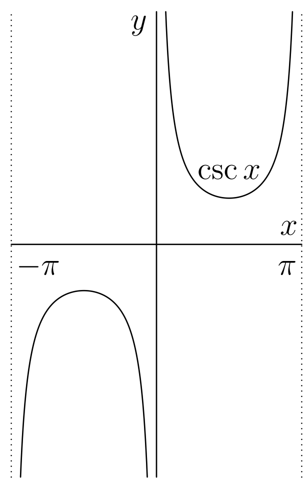
Problem 1.5: Verify that the graphs of the functions $y = \sin x$, $y = \cos x$, and $y = \tan x$ are periodic extensions of the illustrated graphs.

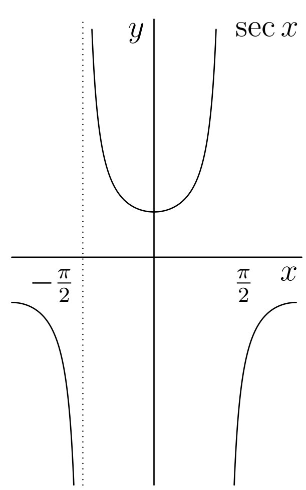


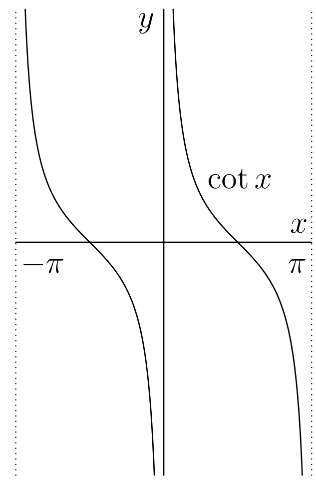




Problem 1.6: Verify that the graphs of the functions $y = \csc x = 1/\sin x$, $y = \sec x = 1/\cos x$, and $y = \cot x = 1/\tan x$ are periodic extensions of the illustrated graphs.

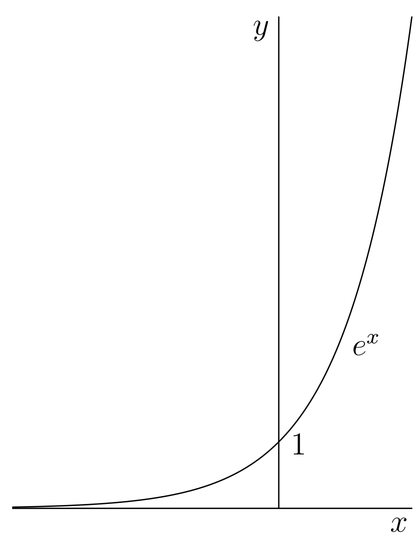






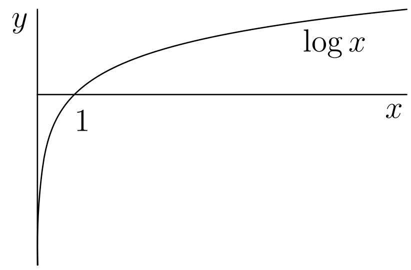
1.4 Exponential and Logarithmic Functions

The graph of the natural exponential function e^x, sometimes written $\exp(x)$, is shown below:



Remark: Notice that $e^x > 0$ for all real x.

The inverse of the exponential function is the natural logarithm $\log x$, sometimes written $\ln x$. It is defined for all positive x:



Remark: There are other exponential function (e.g. 10^x or 2^x) corresponding to other choices of the base (e.g. 10 or 2). The natural logarithm corresponds to the base $e \approx 2.718281828459\ldots$.

Definition: The general exponential function to the base b is defined as

$$b^x \doteq e^{x \log b}.$$

(We use the symbol $\doteq$ to emphasize a definition, although the notation $:=$ is more common.)

Remark: For a positive base b and real x and y:

1. $b^{x+y} = b^x b^y$.

2. $b^{x-y} = \dfrac{b^x}{b^y}$.

3. $(b^x)^y = b^{xy}$.

4. $(ab)^x = a^x b^x$.

Remark: We can also define a logarithm function to the base b:

$$\log_b x \doteq \frac{\log x}{\log b}.$$

Remark: If x, y, and b are positive numbers,

1. $\log_b(xy) = \log_b x + \log_b y$.

2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$.

3. $\log_b(x^r) = r\log_b x$.

1.5 Induction

Suppose that the weather office makes a long-term forecast consisting of two statements:

(A) If it rains on any given day, then it will also rain on the following day.

(B) It will rain today.

What would we conclude from these two statements? We would conclude that it will rain every single day from now on!

Or, consider a secret passed along an infinite line of people, $P_1 P_2 \ldots P_n P_{n+1} \ldots$, each of whom enjoys gossiping. If we know for every $n \in \mathbb{N}$ that P_n will always pass on a secret to P_{n+1}, then the mere act of telling a secret to the first person in line will result in everyone in the line eventually knowing the secret!

These amusing examples encapsulate the axiom of *Mathematical Induction*:

If a subset $\mathcal{S} \subset \mathbb{N}$ satisfies

(i) $1 \in \mathcal{S}$,

(ii) $k \in \mathcal{S} \Rightarrow k+1 \in \mathcal{S}$,

then $\mathcal{S} = \mathbb{N}$.

For example, suppose we wish to find the sum of the first n natural numbers. For small values of n, we could just compute the total of these n numbers directly. But for large values of n, this task could become quite time consuming! The great mathematician and physicist Carl Friedrich Gauss (1777–1855) at age 10 noticed that the rate of increase of the terms in the sum

$$1 + 2 + \ldots + n$$

could be exactly compensated by first writing the sum backwards, as

$$n + (n-1) + \ldots + 1,$$

and then averaging the two equal expressions term by term to obtain a sum of n identical terms:

$$\underbrace{\frac{n+1}{2}+\frac{n+1}{2}+\ldots+\frac{n+1}{2}}_{n \text{ terms}}=n\left(\frac{n+1}{2}\right).$$

We will use mathematical induction to verify Gauss' claim that

$$1+2+\ldots+n \equiv \sum_{i=1}^{n} i=\frac{n(n+1)}{2}. \tag{1.2}$$

Let $\mathcal{S}$ be the set of numbers n for which Eq. (1.2) holds.

Step 1: Check $1 \in \mathcal{S}$:

$$1=\frac{1(1+1)}{2}=1.$$

Step 2: Suppose $k \in \mathcal{S}$, i.e.

$$\sum_{i=1}^{k} i=\frac{k(k+1)}{2}.$$

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right)+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) \\
&= (k+1)\left(\frac{k}{2}+1\right) \\
&= \frac{(k+1)(k+2)}{2}.
\end{aligned}$$

Hence $k+1 \in \mathcal{S}$.
That is, $k \in \mathcal{S} \Rightarrow k+1 \in \mathcal{S}$.

By the Axiom of Mathematical Induction, we know that $\mathcal{S}=\mathbb{N}$.
In other words,

$$\sum_{i=1}^{n} i=\frac{n(n+1)}{2}, \quad \text{for all } n \in \mathbb{N}.$$

- Prove that for all natural numbers n,

$$\sum_{i=1}^{n} i^3=\frac{n^2(n+1)^2}{4}. \tag{1.3}$$

Step 1: We see for $n = 1$ that $1 = 1^2(1+1)^2/4$.

Step 2: Suppose

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \doteq S_n.$$

Then

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^{n} i^3\right) + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2}{4}(n^2+4n+4) \\
&= \frac{(n+1)^2(n+2)^2}{4} = S_{n+1}.
\end{aligned}$$

Hence by induction, Eq. (1.3) holds.

Problem 1.7: Use induction to prove that $22^n - 15$ is a multiple of 7 for every natural number n.

Step 1: We see for $n = 1$ that $22 - 15 = 7$ is a multiple of 7.

Step 2: Assume that $22^n - 15$ is a multiple of 7, say $7m$. We need only show that $22^{n+1} - 15$ is also a multiple of 7:

$$22^{n+1} - 15 = 22^n \cdot 22 - 15 = (7m+15) \cdot 22 - 15 = 7m \cdot 22 + 15 \cdot 21 = 7(m \cdot 22 + 15 \cdot 3),$$

which is indeed a multiple of 7. By mathematical induction, we see that $22^n - 15$ is multiple of 7 for every $n \in \mathbb{N}$.

1.6 Summation Notation

Recall

$$\sum_{k=1}^{k=n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

Q. What is $\sum\limits_{k=0}^{k=n} k$?

A.

$$\sum_{k=0}^{k=n} k = 0 + \sum_{k=1}^{k=n} k = 0 + \frac{n(n+1)}{2} = \frac{n(n+1)}{2}.$$

Q. How about $\sum_{k=1}^{k=n+1} k$?

A.

$$\sum_{k=1}^{k=n+1} k = \left(\sum_{k=1}^{k=n} k\right) + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}.$$

Q. How about $\sum_{k=1}^{k=n}(k+1)$?

A.

Method 1:

$$\sum_{k=1}^{k=n}(k+1) = \sum_{k=1}^{k=n} k + \sum_{k=1}^{k=n} 1 = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}.$$

Method 2: First, let $k' = k+1$:

$$\sum_{k=1}^{k=n}(k+1) = \sum_{k'=2}^{k'=n+1} k'.$$

Next, it is convenient to replace the symbol k' with k (since it is only a dummy index anyway):

$$\sum_{k'=2}^{k'=n+1} k' = \sum_{k=2}^{k=n+1} k = \left(\sum_{k=1}^{k=n+1} k\right) - 1 = \frac{(n+1)(n+2)}{2} - 1 = \frac{n(n+3)}{2}.$$

In general,

$$\boxed{\sum_{k=L}^{k=U} a_{k+m} = \sum_{k=L+m}^{k=U+m} a_k.}$$

Verify this by writing out both sides explicitly.

Problem 1.8: For any real numbers a_1, a_2, $\ldots$, a_n, b_1, b_2, $\ldots$, b_n, and c prove that

$$\sum_{k=1}^{n} c(a_k + b_k) = c\sum_{k=1}^{n} a_k + c\sum_{k=1}^{n} b_k.$$

- *Telescoping sum*:

$$\begin{aligned}
\sum_{k=1}^{n}(a_{k+1} - a_k) &= \sum_{k=1}^{n} a_{k+1} - \sum_{k=1}^{n} a_k \\
&= \sum_{k=2}^{n+1} a_k - \sum_{k=1}^{n} a_k \\
&= \cancel{\sum_{k=2}^{n}} a_k + a_{n+1} - \left(a_1 + \cancel{\sum_{k=2}^{n}} a_k \right) \\
&= a_{n+1} - a_1.
\end{aligned}$$

Chapter 2

Limits

2.1 Sequence Limits

Definition: A *sequence* is a function on the domain $\mathbb{N}$. The value of a function f at $n \in \mathbb{N}$ is often denoted by a_n,

$$a_n = f(n).$$

The consecutive function values are often written in a list:

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \ldots\} \quad \leftarrow \text{Repeated values are allowed.}$$

- $a_n = f(n) = n^2$,
  $\{a_n\}_{n=1}^{\infty} = \{1, 4, 9, 16 \ldots\}$.

- The *Fibonacci* sequence,
  $$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\},$$
  begins with the numbers 1 and 1, with subsequent numbers defined as the sum of the two immediately preceding numbers.

- $$\{\cos(n\pi)\}_{n=0}^{\infty} = \{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, \ldots\}.$$

- $$\{\sin(n\pi)\}_{n=0}^{\infty} = \{0, 0, 0, 0, \ldots\}.$$

- $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty} = \left\{-1, \dfrac{1}{2}, -\dfrac{1}{3}, \dfrac{1}{4}, \ldots\right\}$.
  Notice that as n gets large, the terms of this sequence get closer and closer to zero. We say that they *converge* to 0. However, a_n is not equal to 0 for any $n \in \mathbb{N}$.

We can formalize this observation with the following concept:

Definition: The sequence $\{a_n\}_{n=1}^{\infty}$ is *convergent* with limit L if, for each $\epsilon > 0$, there exist a number N such that

$$n > N \Rightarrow |a_n - L| < \epsilon.$$

We abbreviate this as: $\lim_{n\to\infty} a_n = L$.

If no such number L exists, we say $\{a_n\}_{n=1}^{\infty}$ *diverges.*

Remark: The statement $\lim_{n\to\infty} a_n = L$ means that $|a_n - L|$ can be made as small as we please, simply by choosing n large enough.

Remark: Equivalently, as illustrated in Fig. 2.1, $\lim_{n\to\infty} a_n = L$ means that any open interval about L contains all but a finite number of terms of $\{a_n\}_{n=1}^{\infty}$.

Remark: If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to L, the previous remark implies that every open interval $(L - \epsilon, L + \epsilon)$ will contain an infinite number of terms of the sequence (there cannot be only a finite number of terms inside the interval since a sequence has infinitely many terms and only finitely many of them are allowed to lie outside the interval).

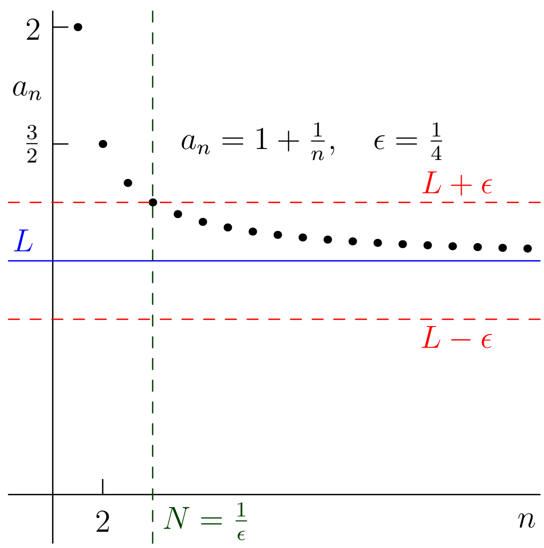


Figure 2.1: Limit of a sequence

- Let $a_n = 1, \quad$ for all $n \in \mathbb{N}$
  i.e. $\{1, 1, 1, \ldots\}$.
  Let $\epsilon > 0$. Choose $N = 1$.

$$n > 1 \Rightarrow |a_n - 1| = |1 - 1| = 0 < \epsilon.$$

That is, $L = 1$. Write $\lim_{n\to\infty} a_n = 1$.

Remark: Here N does not depend on ϵ, but normally it will.

- The sequence $a_n = \dfrac{1}{n}$ converges to 0 since given $\epsilon > 0$, we may force $|a_n - 0| < \epsilon$ for $n > N$ simply by picking $N \geq \dfrac{1}{\epsilon}$:

$$n > N \Rightarrow |a_n - 0| = \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

- $a_n = \dfrac{(-1)^n}{n}$.

  $\lim_{n\to\infty} a_n = 0$ since $|a_n - 0| = \left|\dfrac{(-1)^n}{n}\right| = \dfrac{1}{n} < \dfrac{1}{N}$ if $n > N$.

  So, given $\epsilon > 0$, we may force $|a_n - 0| < \epsilon$ for $n > N$ simply by picking $N \geq \dfrac{1}{\epsilon}$:

$$n > N \Rightarrow |a_n - 0| = \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

- The sequence $a_n = \dfrac{1}{n+1}$ converges to 0 since given $\epsilon > 0$, we may force $|a_n - 0| < \epsilon$ for $n > N$ simply by picking $N \geq \dfrac{1}{\epsilon}$:

$$n > N \Rightarrow |a_n - 0| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

Problem 2.1: Show that the sequence

$$a_n = \frac{n}{n+1}$$

converges to 1.

Given $\epsilon > 0$, we may force $|a_n - 1| < \epsilon$ for $n > N$ simply by picking $N \geq \dfrac{1}{\epsilon}$:

$$n > N \Rightarrow |a_n - 1| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

Thus $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Remark: Such limit calculations can quickly become quite technical. Fortunately, many limit questions can be greatly simplified using the following properties.

Properties of Limits: Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences. Denote $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$.

$$\lim_{n\to\infty} (a_n + b_n) = L + M;$$

$$\lim_{n\to\infty} a_n b_n = LM;$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M} \text{ if } M \neq 0.$$

- An easier way to show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ is to use the fact that the limit of a difference is the difference of limits, **provided that each individual limit exists:**

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \frac{n+1-1}{n+1} = \lim_{n\to\infty} \left[\frac{n+1}{n+1} - \frac{1}{n+1}\right] \\ &= \lim_{n\to\infty} \frac{n+1}{n+1} - \lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1 - 0 = 1.\end{aligned}$$

- Another way to show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ is divide numerator and denomator by the highest power of n appearing (in this case, n^1) and use the fact that the ratio of limits is the limit of the ratio, **provided that each individual limit exists:**

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1+1/n} \\ &= \frac{1}{1+\lim_{n\to\infty} 1/n} = \frac{1}{1+\lim_{n\to\infty} 1/n} = \frac{1}{1+0} = 1.\end{aligned}$$

Problem 2.2: Show by first dividing numerator and denominator by n^2 that

$$\lim_{n\to\infty} \frac{2n^2-5}{3n^2+n+1} = \frac{2}{3}.$$

Problem 2.3: Find $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$.

We find

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) &= \lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\\ &= \lim_{n\to\infty}(n+1-n)\cdot\frac{1}{\sqrt{n+1}+\sqrt{n}}\\ &= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}\\ &= 0\end{aligned}$$

since the final fraction is less than ϵ whenever $n > 1/\epsilon^2$.

Remark: The *Squeeze Theorem* states that if $x_n \leq z_n \leq y_n$ for all $n \in \mathbb{N}$ and the sequences $\{x_n\}$ and $\{y_n\}$ both converge **to the same number** c, then $\{z_n\}$ is also convergent to c.

- The Squeeze Theorem provides an alternative means to show $\lim_{n\to\infty} a_n = \frac{(-1)^n}{n} = 0$: Since
$$-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$$
and $\lim_{n\to\infty} -\frac{1}{n} = -\lim_{n\to\infty}\frac{1}{n} = -0 = 0 = \lim_{n\to\infty}\frac{1}{n}$, the Squeeze Theorem guarantees that $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$ too.

Definition: A sequence is *bounded* if there exists a number B such that

$$|a_n| \leq B \quad \text{for all } n \in \mathbb{N}.$$

Recall that this means that a_n lies within some interval $(-B, B)$.

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ is *increasing* if

$$a_1 \leq a_2 \leq a_3 \leq \ldots, \quad \text{i.e.} \quad a_n \leq a_{n+1} \quad \text{for all } n \in \mathbb{N}$$

and *decreasing* if

$$a_1 \geq a_2 \geq a_3 \geq \ldots, \quad \text{i.e.} \quad a_n \geq a_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Definition: A sequence is *monotone* if it is either (i) an increasing sequence or (ii) a decreasing sequence.

The following theorem can be helpful in establishing the convergence of monotone sequences:

Remark: Every bounded, monotone sequence is convergent.

- The sequence $a_n = \sum_{k=0}^{n} 2^{-k} = 1 + 1/2 + 1/4 + \ldots + 1/2^n$ is bounded below by 1 and above by 2. Since each new term in the sum is positive, a_n is also a monotone (increasing) sequence and is thus convergent.

Problem 2.4: Let $x \in [0,1]$. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined inductively by $a_1 = x$, and $a_{n+1} = a_n(1-a_n)$ for $n \geq 1$.

(a) Show that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence.

$$a_{n+1} - a_n = -a_n^2 \leq 0.$$

(b) Prove that $\{a_n\}_{n=1}^{\infty}$ is bounded.

We show that $0 \leq a_n \leq 1$ for all n. For $n = 1$, we are given that $a_1 = x \in [0,1]$. Suppose that $0 \leq a_n \leq 1$. Then $0 \leq 1 - a_n \leq 1$ and hence $a_{n+1} = a_n(1-a_n)$ is also in $[0,1]$. By induction, we see that $a_n \in [0,1]$ for all n.

(c) Does $\{a_n\}_{n=1}^{\infty}$ converge? Why or why not? If it does converge, find its limit.

The sequence converges because it is a decreasing bounded sequence. To find the limit, let

$$L = \lim_{n\to\infty} a_n.$$

The sequences $\{a_{n+1}\}$ and $\{a_n\}$ converge to the same limit since $n + 1 \to \infty$ as $n \to \infty$. Hence

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} [a_n(1-a_n)] = \lim_{n\to\infty} a_n \lim_{n\to\infty} (1-a_n) = L(1-L) = L - L^2.$$

This implies that $L^2 = 0$, from which we deduce $\lim_{n\to\infty} a_n = 0$.

2.2 Function Limits

Consider the function $f(x) = \frac{1}{x}$ $(x \neq 0)$.

Notice that as x gets large, $f(x)$ gets closer to, but never quite reaches, 0, very much like the terms of the sequence $\{\frac{1}{n}\}$ as $n \to \infty$. In fact, at integer values of x, f evaluates to a member of the sequence $\{\frac{1}{n}\}$:

$$f(n) = \frac{1}{n} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Unlike a sequence, f is defined also for nonintegral values of x. We therefore need to generalize our definition of a limit:

Definition: We say $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$ we can find a real number N such that

$$x > N \Rightarrow |f(x) - L| < \epsilon.$$

- Let $f(x) = 1/x$. Given any $\epsilon > 0$, we can make

$$|f(x) - 0| = \left|\frac{1}{x}\right| < \frac{1}{N} = \epsilon$$

for $x > N$ simply by picking $N = \dfrac{1}{\epsilon}$.

Hence $\lim_{x\to\infty} f(x) = 0$.

- $$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}.$$

Remark: As with sequence limits, we have the following properties:

Properties: Suppose $L = \lim_{x\to\infty} f(x)$ and $M = \lim_{x\to\infty} g(x)$. Then

$$\lim_{x\to\infty} (f(x) + g(x)) = L + M;$$

$$\lim_{x\to\infty} f(x)g(x) = LM;$$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ if } M \neq 0;$$

Remark: We can also introduce the notion of a limit of a function $f(x)$ as x approaches some real number a.

- Cnsider the function $f(x) = \sin x$. Notice for all real numbers near $x = 0$ that $\sin x$ is very close to 0. That is, if δ is a small positive number, the value of $\sin x$ is very close to zero for all $x \in (-\delta, \delta)$. Given $\epsilon > 0$, we can in fact always find a small region $(-\delta, \delta)$ about the origin such that

$$x \in (-\delta, \delta) \Rightarrow |\sin x| < \epsilon.$$

For example, we could choose $\delta = \epsilon$ since we have already shown that $|\sin x| \leq |x|$ for all real x:

$$|x| < \delta \Rightarrow |\sin x| \leq |x| < \delta = \epsilon.$$

We express this fact with the statement $\lim_{x\to 0} \sin x = 0$.

Definition: We say $\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$ we can find a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Remark: In the previous example we see that $a = 0$ and the limit L is 0. Notice in this case that $\lim_{x \to 0} f(x) = 0 = f(0)$. However, this is not true for all functions f. The value of a limit as $x \to a$ might be quite different from the value of the function at $x = a$. Sometimes the point a might not even be in the domain of the function, but the limit may still be defined. This is why we restrict $0 < |x - a|$ (that is, $x \neq a$) in the above definition.

Remark: The value of a function at a itself is irrelevant to its limit at a. We don't need to evaluate the function at $x = a$ any more than we need to evaluate the function $f(x) = 1/x$ at $x = \infty$ to find $\lim_{x \to \infty} f(x) = 0$.

- Let
$$f(x) = \begin{cases} x & \text{if } x > 0, \\ -x & \text{if } x < 0. \end{cases}$$
When we say $\lim_{x \to 0} f(x) = 0$ we mean the following. Given $\epsilon > 0$, we can make
$$|f(x)| < \epsilon$$
for all x satisfying $0 < |x| < \delta$ just by choosing $\delta = \epsilon$. That is,
$$0 < |x| < \delta \Rightarrow |f(x)| = |x| < \delta = \epsilon.$$

- How about
$$f(x) = |x| = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0, \end{cases}$$
Is $\lim_{x \to 0} f(x) = 0$? Yes, the value of f at $x = 0$ does not matter.

- Consider now
$$f(x) = \begin{cases} x & \text{if } x > 0, \\ 1 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases}$$
Is $\lim_{x \to 0} f(x) = 0$? Yes, the value of f at $x = 0$ does not matter.

- Let
$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$
This function is defined everywhere. Does $\lim_{x\to 0} f(x)$ exist?

No, given $\epsilon = \frac{1}{2}$, there are values of $x \neq 0$ in every interval $(-\delta, \delta)$ with very different values of f:

$$f\left(\frac{\delta}{2}\right) = 1,$$
$$f\left(-\frac{\delta}{2}\right) = 0.$$

Thus $\lim\limits_{x\to 0} f(x)$ does not exist.

- Let $f(x) = 7x - 3$. Show that $\lim\limits_{x\to 1} f(x) = 4$.
Let $\epsilon > 0$. Our task is to produce a $\delta > 0$ such that
$$0 < |x - 1| < \delta \Rightarrow |f(x) - 4| < \epsilon.$$
Well, $|f(x) - 4| = |7x - 7| = 7\,|x - 1| < 7\delta$.
How can we make $|f(x) - 4| < \epsilon$?
No matter what ϵ we are given, we can easily choose $\delta = \epsilon/7$, so that $7\delta = \epsilon$.

Q. Suppose
$$f(x) = \begin{cases} 7x - 3 & x \neq 1, \\ 5 & x = 1. \end{cases}$$
What is $\lim\limits_{x\to 1} f(x)$?

A. The limit is still 4; the value of $f(x)$ at $x = 1$ is completely irrelevant. The function need not even be defined at $x = 1$.

Remark: $\lim\limits_{x\to a}$ describes the behaviour of a function near a, not at a.

- Let $f(x) = x^2$, $x \in \mathbb{R}$.
Show $\lim\limits_{x\to 3} f(x) = 9$.

$$\begin{aligned} |x - 3| < \delta \Rightarrow |f(x) - 9| = \left|x^2 - 9\right| &= |x - 3|\,|x + 3| = |x - 3|\,|x - 3 + 6| \\ &< \delta(\delta + 6) \quad \text{from the Triangle Inequality.} \end{aligned}$$

We could solve the quadratic equation $\delta(\delta+6)=\epsilon$, but it is easier to restrict $\delta \leq 1$ so that
$$\delta(\delta+6) \leq \delta(1+6) = 7\delta \leq \epsilon \quad \text{if} \quad \delta \leq \frac{\epsilon}{7}.$$
Note here that we must allow for the possibility that $\delta < \epsilon/7$ instead of just setting $\delta = \epsilon/7$, in order to satisfy our simplifying restriction that $\delta \leq 1$.

Hence
$$|x-3| < \min\left(1, \frac{\epsilon}{7}\right) \Rightarrow |f(x)-9| < \epsilon.$$

- Let $f(x) = \frac{1}{x}$, $x \neq 0$.

  Show $\lim\limits_{x\to 2} f(x) = \dfrac{1}{2}$.

  Given $\epsilon > 0$, try to find a δ such that
$$0 < |x-2| < \delta \Rightarrow \left|f(x) - \frac{1}{2}\right| < \epsilon.$$
Note $\left|f(x) - \dfrac{1}{2}\right| = \left|\dfrac{1}{x} - \dfrac{1}{2}\right| = \left|\dfrac{2-x}{2x}\right|$ becomes very large near $x = 0$.

  Is this a problem? No, we are only interested in the behaviour of the function near $x = 2$.

  Let us restrict $\delta \leq 1$, to keep the factor $2x$ in the denominator from getting really small (and hence the whole expression from getting really large). Then
$$|x-2| < 1 \Rightarrow -1 < x-2 < 1 \Rightarrow 1 \leq x \leq 3 \Rightarrow \frac{1}{x} \leq 1.$$
So
$$\left|f(x) - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| \leq \frac{1}{2}|x-2| < \frac{1}{2}\delta \leq \epsilon,$$
if we take $\delta = \min(1, 2\epsilon)$.

Properties: Suppose $L = \lim\limits_{x\to a} f(x)$ and $M = \lim\limits_{x\to a} g(x)$. Then
$$\lim_{x\to a}(f(x)+g(x)) = L + M;$$
$$\lim_{x\to a} f(x)g(x) = LM;$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ if } M \neq 0;$$

Remark: If $M = 0$, we need to simplify a result before we can use the final property:

- $$\lim_{x\to 1}\frac{x-1}{x^2-1}=\lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}=\lim_{x\to 1}\frac{1}{(x+1)}=\frac{\lim\limits_{x\to 1}1}{\lim\limits_{x\to 1}(x+1)}=\frac{1}{2}.$$

Remark: The *Squeeze Theorem for Functions* states that if $f(x)\le h(x)\le g(x)$ when $x\in(a-\delta,a+\delta)$, for some $\delta>0$, then

$$\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=L\Rightarrow\lim_{x\to a}h(x)=L.$$

Remark: If $a>0$, then $\lim\limits_{x\to a}\sqrt{x}=\sqrt{a}$. To see this, consider

$$0\le\left|\sqrt{x}-\sqrt{a}\right|=\left|\sqrt{x}-\sqrt{a}\right|\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\frac{|x-a|}{\sqrt{x}+\sqrt{a}}\le\frac{|x-a|}{\sqrt{a}}.$$

The Squeeze Theorem then implies that

$$\lim_{x\to a}\left|\sqrt{x}-\sqrt{a}\right|=0,$$

or equivalently,

$$\lim_{x\to a}\left(\sqrt{x}-\sqrt{a}\right)=0.$$

Thus

$$\lim_{x\to a}\sqrt{x}=\lim_{x\to a}\sqrt{a}.$$

Remark: Similarly, it can be shown that

$$\lim_{x\to a}\sqrt{g(x)}=\sqrt{\lim_{x\to a}g(x)}$$

for any non-negative function $g(x)$.

Problem 2.5: Let $f(x)>0$ be a positive function, defined everywhere except perhaps at $x=0$. Suppose that

$$\lim_{x\to 0}\left(f(x)+\frac{1}{f(x)}\right)=2.$$

Prove that $\lim\limits_{x\to 0}f(x)$ exists and equals 1. Hint: First note that

$$\lim_{x\to 0}\left(\sqrt{f(x)}\pm\frac{1}{\sqrt{f(x)}}\right)=\sqrt{\lim_{x\to 0}\left(\sqrt{f(x)}\pm\frac{1}{\sqrt{f(x)}}\right)^2}.$$

Then sum these two expressions to find $\lim\limits_{x\to 0}\sqrt{f(x)}$.

Definition: If for every $M > 0$ there exists a $\delta > 0$ such that $x \in (a - \delta, a + \delta) \Rightarrow f(x) > M$, we say $\lim_{x \to a} f(x) = \infty$.

- $$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Definition: We say $\lim_{x \to \infty} f(x) = \infty$ if for every $M > 0$ we can find a real number N such that

$$x > N \Rightarrow f(x) > M.$$

- $$\lim_{x \to \infty} e^x = \infty.$$

2.3 Continuity

Definition: Let $D \subset \mathbb{R}$. A point c is an *interior point* of D if it belongs to some open interval (a, b) entirely contained in D: $c \in (a, b) \subset D$.

- $\frac{1}{10}, \frac{1}{2}, \frac{2}{3}, \frac{9}{10}$ are interior points of $[0, 1]$ but 0 and 1 are not.

- All points of $(0, 1)$ are interior points of $(0, 1)$.

Recall that the value of f at $x = a$ is completely irrelevant to the value of its limit as $x \to a$. Sometimes, however, these two values will happen to agree. In that case, we say that $f(x)$ is *continuous* at $x = a$.

Definition: A function f is *continuous* at an interior point a of its domain if

$$\lim_{x \to a} f(x) = f(a).$$

Remark: Otherwise, if

(a) the limit fails to exist, or

(b) the limit exists and equals some number $L \neq f(a)$,

the function is said to be *discontinuous*.

Remark: f is continuous at $a \iff$ for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

Note that when $x = a$ we have $|f(a) - f(a)| = 0 < \epsilon$.

- $f(x) = x$ is continuous at every point a of its domain ($\mathbb{R}$) since $\lim_{x\to a} x = a = f(a)$ for all $a \in \mathbb{R}$.

- $f(x) = x^2$ is continuous at all points a since

$$\lim_{x\to a} f(x) = \lim_{x\to a} x^2 = \lim_{x\to a} x \cdot \lim_{x\to a} x = a \cdot a = a^2 = f(a).$$

- Likewise, we see that every polynomial is continuous at all real numbers a.

Remark: Suppose f and g are continuous at a. Then $f+g$ and fg are continuous at a and f/g is continuous at a if $g(a) \neq 0$.

Remark: A rational function is continuous at all points of its domain.

- $f(x) = \dfrac{1}{x}$ is continuous at all $x \neq 0$.

- $f(x) = \dfrac{1}{x^2+1}$ is continuous everywhere.

- $f(x) = \dfrac{1}{x^2-1}$ is continuous on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

- We have seen that $\lim_{x\to a} \sqrt{x} = \sqrt{a}$. This means that $f(x) = \sqrt{x}$ is continuous at all $a > 0$.

Remark: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

2.4 One-Sided Limits

Definition: We write $\lim_{x \to a^+} f(x) = L$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{0 < x - a < \delta}_{\text{i.e. } a<x<a+\delta} \Rightarrow |f(x) - L| < \epsilon.$$

- For the function
$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$
we see that $\lim_{x \to 0^+} H(x) = 1$.

Definition: We write $\lim_{x \to a^-} f(x) = L$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon.$$

- In the above example, we see that $\lim_{x \to 0^-} H(x) = 0$.

Remark: $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

Definition: A function f is *continuous from the right* at a if

$$\lim_{x \to a^+} f(x) = f(a).$$

Definition: A function f is *continuous from the left* at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

- $f(x) = \sqrt{x}$ is continuous from the right at $x = 0$.

Remark: A function is continuous at an interior point a of its domain if and only if it is continuous both from the left and from the right at a.

Definition: A function f is said to be *continuous on* $[a, b]$ if f is continuous at each point in (a, b) and continuous from the right at a and from the left at b.

- $f(x) = \sqrt{x}$ is continuous on $[\,0, \infty)$.

Remark: Continuous functions are free of sudden jumps. This property may be exploited to help locate the roots of a continuous function. Suppose we want to know whether the continuous function $f(x) = x^3 + x^2 - 1$ has a root in $(0, 1)$. We might notice that $f(0)$ is negative and $f(1)$ is positive. Since f has no jumps, it would then seem plausible that there exists a number $c \in (0, 1)$ where $f(c) = 0$. The following theorem establishes that this is indeed the case.

Theorem 2.1 (Intermediate Value Theorem [IVT]): *Suppose*

(i) *f is continuous on [a,b],*

(ii) *$f(a) < y < f(b)$.*

Then there exists a number $c \in (a, b)$ such that $f(c) = y$.

Problem 2.6: Show that $f(x) = x^7 + x^5 + 2x - 1$ has at least one real root in $(0, 1)$.

Since f is a polynomial, it is continuous. Noting that $-1 = f(0) < f(1) = 3$, we then know by the Intermediate Value Theorem that there exists an $c \in (0, 1)$ for which $f(c) = 0$.

Problem 2.7: Let $f(x) = 2x^3 + x^2 - 1$. Show that there exists $x \in (0, 1)$ such that $f(x) = x$.

Consider the continuous function $g(x) = f(x) - x$. We see that $g(0) = -1$ and $g(1) = 1$. By the Intermediate Value Theorem, there exists a point $c \in (0, 1)$ for which $g(c) = 0$, so that $f(c) = c$.

Chapter 3

Differentiation

3.1 Tangent Lines

Definition: Given a function f and a fixed point a of its domain, we can construct the *secant line* joining the points $(a, f(a))$ and $(b, f(b))$ for every point $b \neq a$ in the domain of f. The precise equation for this line depends on the value of b:

$$y = f(a) + m(b) \cdot (x - a),$$

where the slope $m(b)$ is

$$m(b) = \frac{f(b) - f(a)}{b - a}.$$

- If $f(x) = x^2$, the equation of the secant line through $(3, 9)$ and (b, b^2) for $b \neq 3$ is

$$y = 9 + \frac{b^2 - 9}{b - 3}(x - 3).$$

Definition: The *tangent line* of f at an interior point a of its domain, is obtained as the limit of the *secant line* as b approaches a:

$$y = f(a) + m \cdot (x - a),$$

where the limiting slope m is a number (independent of b):

$$m = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$

- If $f(x) = x^2$, the slope of the tangent line through $(3, 9)$ is

$$m = \lim_{b \to 3} \frac{b^2 - 9}{b - 3} = \lim_{b \to 3} \frac{(b - 3)(b + 3)}{b - 3} = \lim_{b \to 3}(b + 3) = 6.$$

The equation of the tangent line to f through $(2, 4)$ is thus

$$y = 9 + 6(x - 3).$$

3.2 The Derivative

Definition: Let a be an interior point of the domain of a function f. If

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists, then f is said to be *differentiable* at a. The limit is denoted $f'(a)$ and is called the *derivative* of f at a. If f is differentiable at every point a of its domain, we say that f is differentiable.

Written in this way, we see that the derivative is the limit of the *slope*

$$m(x) = \frac{f(x)-f(a)}{x-a}$$

of a secant line joining the points $(a, f(a))$ and $(x, f(x))$, where $x \neq a$. The limit is taken as x gets closer to a; that is,

$$f'(a) = \lim_{x\to a} m(x).$$

Remark: Using the substition $h = x - a$, we see that

$$\lim_{x\to a} m(x) = L \iff \lim_{h\to 0} m(a+h) = L.$$

This subsitution allows us to rewrite the definition of a derivative as

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

- Let $f(t)$ be the position of a particle on a curve at time t. The *average velocity* of the particle between time t and $t+h$ is the ratio of the distance travelled over the time interval, h:

$$\frac{\text{change in position}}{\text{change in time}} = \frac{f(t+h)-f(t)}{h} \quad (h \neq 0).$$

The *instantaneous velocity* at t is calculated by taking the limit $h \to 0$:

$$\lim_{h\to 0} \frac{f(t+h)-f(t)}{h} = f'(t).$$

- If $f(x) = c$, where c is a constant, then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0 \quad \text{for all } a \in \mathbb{R}.$$

- The derivative of the *affine* function $f(x) = mx + b$, where m and b are constants, (the graph of which is a straight line) has the constant value m:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{m(a+h) - ma}{h} = m.$$

  In the case where $b = 0$, the function $f(x) = mx$ is said to be *linear*. A function that is neither linear nor affine is said to be *nonlinear*.

Remark: The derivative is the natural generalization of the slope of linear and affine functions to nonlinear functions. In general, the value of the local (or instantaneous) slope of a nonlinear function will depend on the point at which it is evaluated.

- Consider the function $f(x) = x^2$. Then

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h\to 0} \frac{\cancel{a^2} + 2ha + h^2 - \cancel{a^2}}{h} \\ &= \lim_{h\to 0} (2a + h) = 2a. \end{aligned}$$

  We see here that the value of the derivative of f at the point a depends on a. Note that

$$\begin{aligned} f'(a) &< 0 \text{ for } \ a < 0, \\ f'(a) &= 0 \text{ for } \ a = 0, \\ f'(a) &> 0 \text{ for } \ a > 0. \end{aligned}$$

  It is convenient to think of the derivative as a function on its own, which in general will depend on exactly where we evaluate it. We emphasize this fact by writing the derivative in terms of a dummy argument such as a or x. In this case, we can express this functional relationship as $f'(a) = 2a$ for all a, or with equal validity, $f'(x) = 2x$ for all x.

- If $f(x) = \sqrt{x}$, we find that

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h\to 0} \left(\frac{x+h-x}{h} \cdot \frac{1}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

- Let $f(x) = x^n$, where $n \in \mathbb{N}$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n - \cancel{x^n}}{h} \\ &= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1} \right] \\ &= nx^{n-1}. \end{aligned}$$

Remark: An alternative proof of the above result relies on the factorization

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1}),$$

which may be established either by long division, summing a geometric series, or by multiplying out the right-hand-side, exploiting the collapse of this Telescoping sum to just two end terms:

$$\begin{aligned} (x-a)\sum_{k=0}^{n-1} x^{n-1-k}a^k &= \sum_{k=0}^{n-1} x^{n-k}a^k - \sum_{k=0}^{n-1} x^{n-1-k}a^{k+1} = \sum_{k=0}^{n-1} x^{n-k}a^k - \sum_{k=1}^{n} x^{n-k}a^k \\ &= x^n - a^n. \end{aligned}$$

For example, when $n = 2$ we recover the result $x^2 - a^2 = (x-a)(x+a)$ and when $n = 3$ we obtain $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$.

- If $f(x) = x^n$, we then find that

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a} \frac{x^n - a^n}{x-a} \\ &= \lim_{x\to a} \left(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1} \right) \\ &= na^{n-1}. \end{aligned}$$

- We can compute the derivative of the function $f(x) = x^{1/n}$ where $x > 0$ and $n \in \mathbb{N}$

by applying the above factorization to $x-a=(x^{1/n})^n-(a^{1/n})^n$:

$$
\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\
&= \lim_{x\to a}\frac{x^{1/n}-a^{1/n}}{x-a}\\
&= \lim_{x\to a}\frac{x^{1/n}-a^{1/n}}{(x^{1/n}-a^{1/n})(x^{(n-1)/n}+x^{(n-2)/n}a^{1/n}+\ldots+x^{1/n}a^{(n-2)/n}+a^{(n-1)/n})}\\
&= \frac{1}{\underbrace{\lim_{x\to a}x^{(n-1)/n}+\lim_{x\to a}x^{(n-2)/n}a^{1/n}+\ldots+\lim_{x\to a}a^{(n-1)/n}}_{n\text{ terms}}}\\
&= \frac{1}{na^{(n-1)/n}}=\frac{1}{n}a^{\frac{1-n}{n}}\\
&= \frac{1}{n}a^{\frac{1}{n}-1}.
\end{aligned}
$$

Remark: The derivative of an exponential function $f(x)=b^x$ can be found by using the properties of exponentials:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{b^{x+h}-b^x}{h}=b^x\lim_{h\to 0}\frac{b^h-1}{h}=b^xf'(0),$$

on noting that $f'(0)=\frac{b^h-1}{h}$. This emphasizes the important property that the slope of an exponential function is proportional to the value of the function itself.

Remark: For the special choice of base $b=e$, this proportionality constant equals one; that is, $f'(0)=1$. That is, if $f(x)=e^x$, then $f'(x)=e^x$. The natural exponential function is thus its own derivative.

Problem 3.1: At what point on the graph of $f(x)=e^x$ is the tangent line parallel to the line $y=2x$?

On setting $f'(x)=e^x=2$ we find that $x=2$. The required point is thus $(\log 2,2)$.

Q. Are all functions differentiable?

A. No, consider

$$f(x)=\begin{cases}0 & \text{if } x<0,\\ 1 & \text{if } x\ge 0.\end{cases}$$

We see that

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{1-1}{x}=\lim_{x\to 0^+}\frac{0}{x}=0,$$

but

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{0-1}{x} \text{ does not exist.}$$

So $\lim_{x\to 0} \frac{f(x)-1}{x-0}$ does not exist. It appears, at the very least, that we must avoid jumps, as the following theorem points out.

Theorem 3.1 (Differentiable $\Rightarrow$ Continuous): *If f is differentiable at a then f is continuous at a.*

Proof: For $x \neq a$, we may write

$$f(x) = f(a) + \frac{f(x)-f(a)}{x-a}(x-a).$$

If $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, then

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a} f(a) + \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a) \\ &= f(a) + f'(a)\cdot 0 \\ &= f(a),\end{aligned}$$

so f is continuous at a.

Q. Are all **continuous** functions differentiable?

A. No, consider $f(x) = |x|$:

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|-0}{x-0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Hence $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist; f is not differentiable at 0, even though f is continuous at 0.

Derivative Notation

Three equivalent notations for the derivative have evolved historically. Letting $y = f(x)$, $\Delta y = f(x+h) - f(x)$, and $\Delta x = (x+h) - x = h$, we may write

$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

To help us remember this, we sometimes denote the derivative by dy/dx (Leibniz notation).

The operator notation Df (or $D_x f$, which reminds us that the derivative is with respect to x) is also occasionally used to emphasize that the derivative Df is a function derived from the original function, f.

Remark: When the derivative f' of a function f is itself differentiable we will use either the notation f'' or $f^{(2)}$ to denote the second derivative of f. In general, we will let $f^{(n)}$ denote the n-th derivative of f, obtained by differentiating f with respect to its argument n times (the parentheses help us avoid confusion with powers). It is also convenient to define $f^{(0)} = f$ itself.

Remark: Observe that $f^{(n+1)} = (f^{(n)})'$ and that if $f^{(n+1)}$ exists at a point x, then $f^{(n)}$ and all lower-order derivatives must also exist at x.

3.3 Properties

Theorem 3.2 (Properties of Differentiation): *If f and g are both differentiable at a, then*

(a) $(f+g)'(a) = f'(a) + g'(a)$,

(b) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$,

(c) $\left(\dfrac{f}{g}\right)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$ *if* $g(a) \neq 0$.

Proof: We are given that $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists and $g'(a) = \lim\limits_{x\to a} \dfrac{g(x)-g(a)}{x-a}$ exists.

(a)

$$\begin{aligned}
\lim_{x\to a} \frac{(f+g)(x)-(f+g)(a)}{x-a} &= \lim_{x\to a} \frac{f(x)+g(x)-f(a)-g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= f'(a) + g'(a).
\end{aligned}$$

(b)

$$\begin{aligned}
&\lim_{x\to a} \frac{(fg)(x)-(fg)(a)}{x-a} = \lim_{x\to a} \frac{f(x)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \underbrace{\lim_{x\to a} g(x)}_{\text{exists } =g(a) \text{ by Theorem 3.1}} + f(a) \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= f'(a)g(a) + f(a)g'(a).
\end{aligned}$$

(c) Let $h(x) = \dfrac{1}{g(x)}$. Then

$$\begin{aligned} h'(a) &= \lim_{x\to a} \frac{h(x)-h(a)}{x-a} = \lim_{x\to a} \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} \\ &= \lim_{x\to a} \frac{\frac{g(a)-g(x)}{g(x)g(a)}}{x-a} \\ &= -\frac{1}{g^2(a)} \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = -\frac{g'(a)}{g^2(a)}. \end{aligned}$$

Then from **(b)**,

$$\begin{aligned} \left(\frac{f}{g}\right)'(a) = (fh)'(a) &= f'(a)h(a) + f(a)h'(a) \\ &= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g^2(a)} \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}. \end{aligned}$$

Remark: Any polynomial is differentiable on $\mathbb{R}$.

Remark: A rational function is differentiable at every point of its domain.

Problem 3.2: If $f(x) = e^x - x$, find f' and $f'' \doteq (f')'$.

Since $f'(x) = e^x - 1$, we see that $f''(x) = e^x$.

Problem 3.3: Use the quotient rule to show that the rule $dx^n/dx = nx^{n-1}$ is valid for all $n \in \mathbb{Z}$, including $n = 0$ and $n < 0$.

For $n = 0$, the derivative evaluates to $\lim\limits_{h\to 0} \dfrac{1-1}{h} = 0$. For $n < 0$, we have

$$\frac{d}{dx}x^n = \frac{d}{dx}\frac{1}{x^{-n}} = \frac{0 \cdot x^{-n} - 1 \cdot \frac{d}{dx}x^{-n}}{(x^{-n})^2} = \frac{-(-n)x^{-n-1}}{(x^{-n})^2} = nx^{n-1}.$$

Problem 3.4: Use the following procedure to show that the derivative of $\sin x$ is $\cos x$.

(a) Use the inequality $\sin x \le x \le \tan x$ for $0 \le x < \pi/2$ to prove that

$$\cos x \le \frac{\sin x}{x} \le 1 \text{ for } 0 < |x| < \frac{\pi}{2}.$$

For $0 < x < \pi/2$, we know that both x and $\cos x$ are positive, which allows us rewrite the inequalities $x \le \tan x$ and $\sin x \le x$ as

$$\cos x \le \frac{\sin x}{x} \le 1.$$

Since each of these expressions are even functions of x, the inequality also holds for $-\pi/2 < x < 0$.

(b) Prove that

$$\lim_{x \to 0} \frac{\sin x}{x}$$

exists and evaluate the limit.

Since $\cos x$ is a continuous function, we know that $\lim_{x \to 0} \cos x = \cos 0 = 1$. We then deduce from the Squeeze Theorem that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

(c) Prove that

$$1 - \cos x \leq \frac{x^2}{2} \quad \text{for all } x \in \mathbb{R}.$$

Hint: Try replacing x by $2x$.

$$1 - \cos x = 2 \sin^2 \frac{x}{2} = 2 \left| \sin \frac{x}{2} \right|^2 \leq 2 \left| \frac{x}{2} \right|^2 = \frac{x^2}{2}.$$

(d) Prove that

$$\lim_{x \to 0} \frac{1 - \cos x}{x}$$

exists and evaluate the limit.

Since $\cos x \leq 1$ for all x, we know for $x \neq 0$ that

$$\left| \frac{1 - \cos x}{x} \right| = \frac{1 - \cos x}{|x|} \leq \frac{|x|}{2}$$

and hence

$$-\frac{|x|}{2} \leq \frac{1 - \cos x}{x} \leq \frac{|x|}{2}.$$

As $x \to 0$, we deduce from the Squeeze Principle that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0.$$

Alternatively,

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x(1 + \cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)} \\
&= \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \frac{\sin x}{1 + \cos x} = 1 \cdot \frac{0}{2} = 0.
\end{aligned}$$

(e) Use the above results to prove that $\sin x$ is differentiable at any real number a and find its derivative. That is, show that

$$\lim_{h\to 0} \frac{\sin(a+h) - \sin a}{h}$$

exists and evaluate the limit.

$$\lim_{h\to 0} \frac{\sin a \cos h + \cos a \sin h - \sin a}{h} = \sin a \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos a \lim_{h\to 0} \frac{\sin h}{h} = 0 + \cos a = \cos a.$$

Problem 3.5: Compute

$$\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right)$$

Hint: let $y = 1/x$. As $x \to \infty$, what happens to y?

$$= \lim_{y\to 0^+} \frac{\tan y}{y} = \lim_{y\to 0^+} \left(\frac{\sin y}{y}\right) \lim_{y\to 0^+} \left(\frac{1}{\cos y}\right) = 1 \cdot 1 = 1.$$

Theorem 3.3 (Chain Rule): *Suppose $h = f \circ g$, i.e. $h(x) = f(g(x))$. Let a be an interior point of the domain of h and define $b = g(a)$. If $f'(b)$ and $g'(a)$ both exist, then h is differentiable at a and*

$$h'(a) = f'(b)g'(a).$$

That is, if $y = f(u)$ and $u = g(x)$, then

$$\left.\frac{dy}{dx}\right|_a = \left.\frac{dy}{du}\right|_b \left.\frac{du}{dx}\right|_a.$$

- Consider that $\frac{d}{dx}(x^2+1)^2 = \frac{d}{dx} f(g(x))$, where $u = g(x) = x^2 + 1$ and $f(u) = u^2$. We let $h(x) = f(g(x))$:

$$\begin{aligned} h'(x) &= f'(u)g'(x) \\ &= 2u \cdot 2x \\ &= 2(x^2+1) \cdot 2x = 4x^3 + 4x. \end{aligned}$$

As a check, we could also work out this derivative directly:

$$\frac{d}{dx}(x^2+1)^2 = \frac{d}{dx}(x^4 + 2x^2 + 1) = 4x^3 + 4x.$$

- The Chain Rule makes it easy to find

$$\begin{aligned}\frac{d}{dx}(x^3+1)^{100} &= 100(x^3+1)^{99}3x^2\\ &= 300x^2(x^3+1)^{99}.\end{aligned}$$

- Let$f(u) = u^{\frac{1}{n}} \Rightarrow f'(u) = \frac{1}{n}u^{\frac{1}{n}-1}$ and $g(x) = x^m \Rightarrow g'(x) = mx^{m-1}$.

  Then $h(x) = f(g(x)) = x^{\frac{m}{n}} \Rightarrow h'(x) = f'(u)g'(x)$ where $u = g(x)$. Thus

$$\begin{aligned}h'(x) &= f'(g(x))g'(x)\\ &= \frac{1}{n}(x^m)^{\frac{1}{n}-1}mx^{m-1}\\ &= \frac{m}{n}x^{\frac{m}{n}-\cancel{m}+\cancel{m}-1}.\end{aligned}$$

Hence $\frac{d}{dx}x^q = qx^{q-1}$ for all $q \in \mathbb{Q}$.

- Find $\frac{d}{dx}\frac{1}{g(x)}$ (cf. Theorem 3.2(c)).

  Let $f(x) = x^{-1}$, $f'(x) = -x^{-2}$, and $h(x) = \frac{1}{g(x)} = f(g(x))$. Then

$$\begin{aligned}h'(x) &= f'(g(x))g'(x)\\ &= -\frac{1}{[g(x)]^2}g'(x),\end{aligned}$$

We may express this using an alternative notation. Letting $y = \frac{1}{u}$ and $u = g(x)$, we find

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{u^2}g'(x) = -\frac{g'(x)}{g^2(x)}.$$

-

$$\begin{aligned}\frac{d}{dx}\sqrt{\frac{1}{1+x^3}} &= \frac{1}{2\sqrt{\frac{1}{1+x^3}}}\left[-\frac{1}{(1+x^3)^2}\cdot 3x^2\right]\\ &= -\frac{3x^2(1+x^3)^{\frac{1}{2}}}{2(1+x^3)^2} = -\frac{3}{2}x^2(1+x^3)^{-\frac{3}{2}}.\end{aligned}$$

  Here is an even easier way to find this derivative:

$$\begin{aligned}\frac{d}{dx}\sqrt{\frac{1}{1+x^3}} &= \frac{d}{dx}(1+x^3)^{-\frac{1}{2}}\\ &= -\frac{1}{2}(1+x^3)^{-\frac{3}{2}}\cdot 3x^2.\end{aligned}$$

- $$\frac{d}{dx}\sin(\sin(x)) = \cos(\sin(x))\cos(x).$$

- $$\frac{d}{dx}\sin(\sin(\sin(x))) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x).$$

Remark: To prove the Chain Rule it is not enough to argue

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$$

because $\Delta u = g(x) - g(a)$ might be zero for values of x close to (but not equal to) a. However, we can easily fix up this argument as follows.

Proof (of Theorem 3.3):
Let $b = g(a)$ and define

$$m(u) = \begin{cases} \dfrac{f(u) - f(b)}{u - b} & \text{if } u \neq b, \\ f'(b) & \text{if } u = b. \end{cases}$$

Then

$$\begin{aligned} &f'(b) \text{ exists} \Rightarrow \lim_{u\to b} m(u) = f'(b) = m(b) \Rightarrow m \text{ is continuous at } b \\ &\qquad\text{and} \\ &g'(a) \text{ exists} \Rightarrow g \text{ is continuous at } a \Rightarrow m\circ g \text{ is continuous at } a, \\ &\qquad\qquad \Rightarrow \lim_{x\to a} m(g(x)) = m(g(a)) = m(b) = f'(b). \end{aligned}$$

Note that

$$f(u) - f(b) = m(u)(u - b) \text{ for all } u.$$

Letting $u = g(x)$, we then find that

$$\begin{aligned} \lim_{x\to a}\frac{f(g(x)) - f(g(a))}{x - a} &= \lim_{x\to a} m(g(x))\lim_{x\to a}\frac{g(x) - g(a)}{x - a} \\ &= f'(b)g'(a). \end{aligned}$$

- With the help of the Chain Rule, the derivative of $\cos x$ can be calculated as:

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - x\right)(-1) = -\sin x,$$

- We can also find the derivative of $\tan x$:

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Problem 3.6: Compute

$$\frac{d}{dx}(x\cos x)$$

$$= \cos x - x\sin x.$$

Problem 3.7: Find

$$\frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \left(\frac{1}{\cos^2 x}\right)\sin x.$$

Problem 3.8: Let f be a differentiable function. Find the following derivatives

(a)

$$\frac{d}{dx}f(f(f(x)))$$

$$= f'(f(f(x)))f'(f(x))f'(x).$$

(b)

$$\frac{d}{dx}\left[\frac{f^3(x)+1}{f^2(x)}\right]$$

$$= \frac{d}{dx}\left[f(x) + \frac{1}{f^2(x)}\right] = \left[1 - \frac{2}{f^3(x)}\right]f'(x).$$

Problem 3.9: If $f(x) = b^x = e^{x\log b}$, show that $f'(x) = b^x \log b$. Note for $b = e$ that this reduces to $f'(x) = e^x$ since $\log e = 1$.

Problem 3.10: Calculate the following derivatives:

(a)

$$\frac{d}{dx}\frac{x^2}{x^3+1}$$

$$=\frac{2x(x^3+1)-x^2(3x^2)}{(x^3+1)^2}=\frac{-x^4+2x}{(x^3+1)^2}.$$

(b)

$$\frac{d}{dx}\left(\frac{\sqrt{2x+1}}{\sin x}\right)$$

$$\frac{\sin x\frac{1}{\sqrt{2x+1}}-\sqrt{2x+1}\cos x}{\sin^2 x}=\frac{\sin x-(2x+1)\cos x}{\sqrt{2x+1}\sin^2 x}.$$

(c)

$$\frac{d}{dx}\frac{1}{\sin^3(\sin x)}$$

$$\frac{d}{dx}\sin^{-3}(\sin x)=-3\sin^{-4}(\sin x)\cos(\sin x)\cos x=-3\frac{\cos(\sin x)\cos x}{\sin^4(\sin x)}.$$

Problem 3.11: Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function. Consider $g(x)=f(-x)$.

(a) Compute g' in terms of f'.
Using the Chain Rule, we find that $g'(x)=-f'(-x)$.
(b) If f is an even function, show that f' is odd.
If f is even then $g(x)=f(-x)=f(x)$; that is, g and f are the same function. From part (a) we then see that f is odd: $f'(x)=-f'(-x)$.
(c) If f is an odd function, show that f' is even.
If f is odd then $g(x)=f(-x)=-f(x)$. From part (a) we then see that f is even: $f'(x)=f'(-x)$.

Problem 3.12:

(a) A spherical balloon is being inflated at the rate of 10 cm^3/s. Given that the volume V of the balloon is related to the radius by $V=\frac{4}{3}\pi r^3$, use the Chain Rule to compute how fast the radius of the balloon is growing when the volume has reached 100 cm^3.

The rate of inflation $r(t)$ must equal the rate of volume increase: $dV/dt=10\,\text{cm}^3/\text{s}$. We will need to know the formula for r in terms of V,

$$r=\left(\frac{3}{4\pi}\right)^{1/3}V^{1/3}.$$

and its derivative,

$$\frac{dr}{dV} = \left(\frac{3}{4\pi}\right)^{1/3} \frac{1}{3} V^{-2/3} = \left(\frac{1}{36\pi}\right)^{1/3} V^{-2/3}.$$

When the volume of the balloon is 100 cm^3, we can use the Chain Rule to determine that the radius is growing at the rate

$$\frac{dr}{dt} = \frac{dr}{dV}\frac{dV}{dt} = \left(\frac{1}{36\pi}\right)^{1/3} (100\,\mathrm{cm}^3)^{-2/3} \times 10\,\frac{\mathrm{cm}^3}{\mathrm{s}} = 0.096\frac{\mathrm{cm}}{\mathrm{s}}.$$

Incidentally, the derivative of r with respect to V can also be calculated by first calculating $dV/dr = 4\pi r^2$, taking the reciprocal to get dr/dV, and finally expressing the result in terms of V. (What justifies that one can calculate dr/dV in this way?)

(b) Suppose now that the (constant) inflation rate of the balloon is unknown, but it is known that when the volume is 100 cm^3, the radius is growing at a rate of 1 cm/s. How fast is the radius of the balloon growing when the volume has reached 1000 cm^3?

We are given that at a time t_1, the volume $V(t_1) = 100$ cm^3 and $\frac{dr}{dt} t_1| = 1$ cm/s. By the Chain Rule,

$$\frac{dr}{dt} t_1| = \frac{dr}{dV} t_1| \frac{dV}{dt}$$

and

$$\frac{dr}{dt} t_2| = \frac{dr}{dV} t_2| \frac{dV}{dt},$$

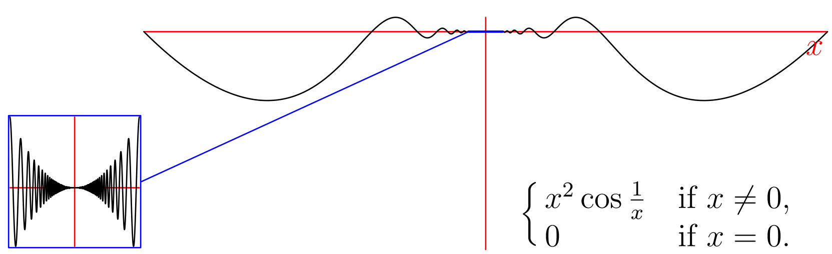
since the inflation rate $\frac{dV}{dt}$ is constant. Hence

$$\frac{dr}{dt} t_2| = \left(\frac{\frac{dr}{dV} t_2|}{\frac{dr}{dV} t_1|}\right) \frac{dr}{dt} t_1| = \left(\frac{V(t_2)}{V(t_1)}\right)^{-2/3} \frac{dr}{dt} t_1| = \left(\frac{1}{10}\right)^{2/3} \times 1\,\frac{\mathrm{cm}}{\mathrm{s}} = 0.215\,\frac{\mathrm{cm}}{\mathrm{s}}.$$

Problem 3.13: Let

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is differentiable for all $x \in \mathbb{R}$ and find $f'(x)$. A graph of f is shown below, including an inset where the y axis is stretched to show more detail around the origin.



Since $\cos(x)$ is differentiable at all x and $1/x$ is differentiable on $(-\infty, 0) \cup (0, \infty)$, the composite function $\cos(1/x)$, and hence f, is differentiable on $(-\infty, 0) \cup (0, \infty)$. Moreover, f is also differentiable at $x = 0$, with derivative 0:

$$\lim_{x\to 0} \frac{x^2 \cos\left(\frac{1}{x}\right) - 0}{x - 0} = \lim_{x\to 0} x \cos\left(\frac{1}{x}\right) = 0$$

by the Squeeze Theorem since

$$0 \le \left| x \cos\left(\frac{1}{x}\right) \right| \le |x|$$

and $\lim_{x\to 0} 0 = 0 = \lim_{x\to 0} |x|$.

Hence f is differentiable on $\mathbb{R}$ and

$$f'(x) = \begin{cases} 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Problem 3.14: Suppose that two functions f and g are differentiable n times at the point a. Use induction to prove *Leibniz's formula*:

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

3.4 Implicit Differentiation

Suppose that a variable y is defined implicitly in terms of x and we wish to know dy/dx. For example, given the *implicit equation*

$$y^3 + 3y^2 + 3y + 1 = x^5 + x, \tag{3.1}$$

we could solve for y to find

$$\begin{aligned} (y+1)^3 &= x^5 + x \\ \Rightarrow y + 1 &= (x^5 + x)^{\frac{1}{3}} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{3}(x^5 + x)^{-\frac{2}{3}}(5x^4 + 1). \end{aligned} \tag{3.2}$$

But what happens if you can't (or don't want to) solve for y? You might try first to solve for x in terms of y and then find the derivative dx/dy of the inverse function. But what if this is also difficult?

It is often easier in these cases to differentiate both sides of Eq. (3.1) with respect to x, noting that $y = y(x)$:

$$\frac{d}{dx}\left[y^3(x) + 3y^2(x) + 3y(x) + 1\right] = \frac{d}{dx}(x^5 + x).$$

By the Chain Rule, we find

$$\big(3y^2 + 6y + 3\big)y'(x) = 5x^4 + 1,$$

which we can easily solve to obtain dy/dx as a function of x and y,

$$\frac{dy}{dx} = \frac{5x^4 + 1}{3y^2 + 6y + 3} = \frac{5x^4 + 1}{3(y+1)^2}. \tag{3.3}$$

Once we know an (x, y) pair that satisfies Eq. (3.1), we can immediately compute the derivative from Eq. (3.3).

It is instructive to verify that Eqs. (3.2) and (3.3) agree:

$$\frac{dy}{dx} = \frac{5x^4 + 1}{3(y+1)^2} = \frac{5x^4 + 1}{3(x^5 + x)^{\frac{2}{3}}}.$$

Problem 3.15: If $x^2 + y^2 = 25$, find dy/dx. Then find an equation for the tangent line to this circle through the point $(3, 4)$.

On implicitly differentiating both sides with respect to x, we find

$$2x + 2y\frac{dy}{dx} = 0.$$

When $y \neq 0$ we can then solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Since the slope of the tangent line at $(3, 4)$ is $-3/4$, an equation of the tangent line is

$$y - 4 = -\frac{3}{4}(x - 3).$$

Problem 3.16: Find dy/dx if $x^3 + y^3 = 6xy$.

3.5 Inverse Functions and Their Derivatives

This section addresses the question: given a function f, when is it possible to find a function g that undoes the effect of f, so that

$$y = f(x) \iff x = g(y)?$$

Recall that a function is a collection of pairs of numbers (x, y) such that if (x, y_1) and (x, y_2) are in the collection, then $y_1 = y_2$.

Definition: A function $f : A \to B$ is *one-to-one* on its domain A if, whenever (x_1, y) and (x_2, y) are in the collection, then $x_1 = x_2$. That is,

$$x_1 = x_2 \iff f(x_1) = f(x_2).$$

We say that such a function is 1–1 or *invertible*.

This can be restated using the *horizontal line test*: a set of ordered pairs (x, y) is a one-to-one function if every horizontal and every vertical line intersects their graph at most once.

Remark: Equivalently, a 1–1 function f satisfies

$$x_1 \neq x_2 \iff f(x_1) \neq f(x_2).$$

- $f(x) = x$ and $f(x) = x^3$ are 1–1 functions.

- $f(x) = x^2$ and $f(x) = \sin x$ are not 1–1 functions.

Remark: Sometimes a *noninvertible* function can be made invertible by restricting its domain.

- $f = \sin x$ restricted to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is 1–1.

Remark: If $f : A \to B$ is 1–1 then the collection of pairs of numbers (y, x) such that (x, y) belong to f is also a function.

Definition: The function defined by the pairs $\{(y, x) : (x, y) \in f\}$ is the *inverse* function $f^{-1} : B \to A$ of f.

Problem 3.17: Show that the inverse of a 1–1 function is itself an invertible function; that is, it satisfies both the horizontal and vertical line tests.

- The inverse of the function $\sin x$ restricted to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is denoted $\arcsin x$ or $\sin^{-1} x$; it is itself a 1–1 function on $[-1, 1]$, yielding values in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Remark: Do not confuse the notation $\sin^{-1} x$ with $\frac{1}{\sin x}$; they are not the same function! Because of this rather unfortunate notational ambiguity, we will use the short-hand notation $f^n(x)$ to denote $(f(x))^n$ only when $n \geq 0$; in particular, we reserve the notation $f^{-1}(x)$ for the inverse of f.

Problem 3.18: Suppose that f and g are inverse functions of each other. Show that $g(f(x)) = x$ for all x in the domain of f and $f(g(y)) = y$ for all y in the range of f.

Theorem 3.4 (Continuous Invertible Functions): *Suppose f is continuous on I. Then f is one-to-one on $I \iff f$ is strictly monotonic on I.*

Theorem 3.5 (Continuity of Inverse Functions): *Suppose f is continuous and one-to-one on an interval I. Then its inverse function f^{-1} is continuous on $f(I) = \{f(x) : x \in I\}$.*

Theorem 3.6 (Differentiability of Inverse Functions): *Suppose f is continuous and one-to-one on an interval I and differentiable at $a \in I$. Let $b = f(a)$ and denote the inverse function of f on I by g. If*

(i) *$f'(a) = 0$, then g is* **not** *differentiable at b;*

(ii) *$f'(a) \neq 0$, then g is differentiable at b and $g'(b) = \dfrac{1}{f'(a)}$.*

- The inverse of the function $f(x) = x^3$ is $f^{-1}(y) = y^{1/3}$ since $y = x^3 \Rightarrow x = y^{\frac{1}{3}}$. Notice that $f'(x) = 3x^2 \neq 0$ for $x \neq 0$ (i.e. $y \neq 0$). We can then verify that

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3y^{\frac{2}{3}}} = \frac{1}{3[f^{-1}(y)]^2} = \frac{1}{f'(f^{-1}(y))}.$$

- What is the derivative of $y = \arctan x$ (or $y = \tan^{-1} x$, the inverse function of $x = \tan y$?

Theorem 3.6 $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ where $x = \tan y$ and $\dfrac{dx}{dy} = \dfrac{1}{\cos^2 y}$. That is,

$$\frac{dy}{dx} = \frac{1}{\frac{1}{\cos^2 y}} = \cos^2 y.$$

Normally, we will want to re-express the derivative in terms of x. Recalling that $\tan^2 y + 1 = \frac{1}{\cos^2 y}$ and $x = \tan y$, we see that

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

$$\therefore \boxed{\frac{d}{dx} \arctan x = \frac{1}{1 + x^2} \text{ on } (-\infty, \infty).}$$

Remark: Although $f(x) = \tan x$ does not satisfy the horizontal line test on $\mathbb{R}$, it does if we restrict $\tan x$ to the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$. We call $\tan x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ the *principal branch* of $\tan x$, which is sometimes denoted $\operatorname{Tan} x$. Its inverse, which is sometimes written $\operatorname{Arctan} x$ or $\operatorname{Tan}^{-1} x$, maps $\mathbb{R}$ to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

- Consider $f(x) = \sqrt{1-x^2}$, which is 1–1 on $[0, 1]$.

  Note that $f'(x) = -\frac{x}{\sqrt{1-x^2}}$ exists on $[0, 1)$.
  Now $y = \sqrt{1-x^2} \Rightarrow x = \sqrt{1-y^2} \Rightarrow x = f^{-1}(y) = f(y)$.
  In this case f and f^{-1} are identical functions of their respective arguments!

$$\begin{aligned}
\frac{d}{dy} f^{-1}(y) &= \frac{1}{f'(x)} \\
&= -\frac{\sqrt{1-x^2}}{x} \\
&= -\frac{\sqrt{1-[f^{-1}(y)]^2}}{f^{-1}(y)} \\
&= -\frac{\sqrt{1-[f(y)]^2}}{f(y)} \\
&= -\frac{\sqrt{1-(1-y^2)}}{\sqrt{1-y^2}} \\
&= -\frac{y}{\sqrt{1-y^2}} \qquad \text{on } [0, 1).
\end{aligned}$$

- $y = \sin x$ is 1–1 on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

  $\frac{dy}{dx} = \cos x \neq 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.
  The inverse function (as a function of y) is

$$x = \arcsin y \text{ (or } x = \sin^{-1} y),$$

with derivative

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x}.$$

We can express $\cos x$ as a function of y:

$$\begin{aligned}
\cos x &= \sqrt{1-\sin^2 x} \\
&= \sqrt{1-y^2},
\end{aligned}$$

noting that $\cos x > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, to find

$$\frac{d}{dy} \arcsin y = \frac{1}{\sqrt{1-y^2}} \text{ on } (-1,1).$$

That is,

$$\boxed{\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \text{ on } (-1,1).}$$

- $y = \cos x$ is 1–1 on $[0, \pi]$.
  $\dfrac{dy}{dx} = -\sin x \neq 0$ on $(0, \pi)$.
  The inverse function $x = \arccos y$ (or $x = \cos^{-1} y$) has derivative

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{-\sin x},$$

which we can express as a function of y, noting that $\sin x > 0$ on $(0, \pi)$,

$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1-y^2}.$$

$$\therefore \ \frac{d}{dy} \arccos y = -\frac{1}{\sqrt{1-y^2}},$$

i.e.

$$\boxed{\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}} \text{ on } (-1,1).}$$

It is not surprising that $\frac{d}{dx} \arccos x = -\frac{d}{dx} \arcsin x$ since $\arccos x = \frac{\pi}{2} - \arcsin x$, as can readily be seen by taking the cosine of both sides and using $\cos y = \sin(\frac{\pi}{2} - y)$.

- Prove that $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$ for all $x \in [-1, 1]$. Let

$$\begin{aligned} f(x) &= \cos^{-1} x + \sin^{-1} x \\ f'(x) &= \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = 0 \\ \Rightarrow f(x) &= c, \quad \text{a constant.} \end{aligned}$$

Set $x = 0$ to find c:

$$c = f(0) = \cos^{-1} 0 = \frac{\pi}{2}.$$

$$\therefore \ f(x) = \frac{\pi}{2} \text{ for all } x \in [-1, 1].$$

Problem 3.19: Let $f(x) = \sin^{-1}(x^2 - 1)$. Find

(a) the domain of f;

The inverse function $y = \sin^{-1} x$ has domain $[-1, 1]$, and $x^2 - 1 \in [-1, 1]$ implies $x^2 \in [0, 2]$. Hence, the domain of f is $[-\sqrt{2}, \sqrt{2}]$.

(b) $f'(x)$;

Letting $y = \sin^{-1}(x^2 - 1)$, we first find the derivative for $x > 0$:

$$\begin{aligned} x^2 - 1 &= \sin y \\ \Rightarrow x &= \sqrt{\sin y + 1} \\ \Rightarrow \frac{dx}{dy} &= \frac{\cos y}{2\sqrt{\sin y + 1}} \\ &= \frac{\sqrt{1 - (x^2 - 1)^2}}{2\sqrt{x^2}} = \frac{\sqrt{2x^2 - x^4}}{2x} \\ \Rightarrow \frac{dy}{dx} &= \frac{2x}{\sqrt{2x^2 - x^4}}. \end{aligned}$$

Since the derivative of an even function is odd (and *vice-versa*) we see that the same result holds for $x < 0$ as well.

Alternatively, one could use the formula for the derivative of $\sin^{-1} x$ together with the Chain Rule.

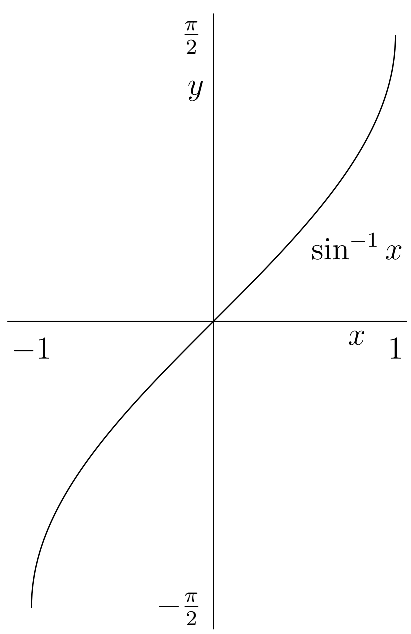
(c) the domain of f'.

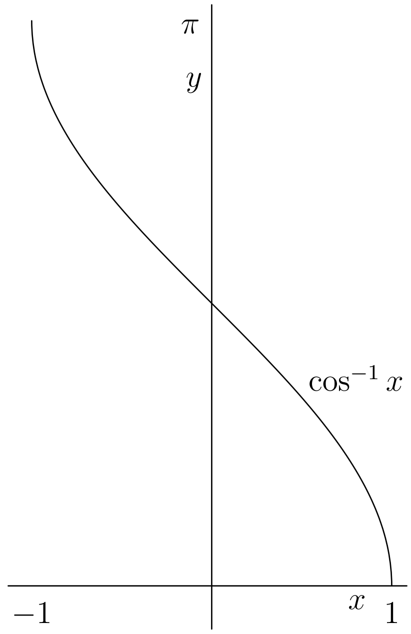
The domain of $f' = dy/dx$ is the set of x such that $2x^2 - x^4 > 0$:

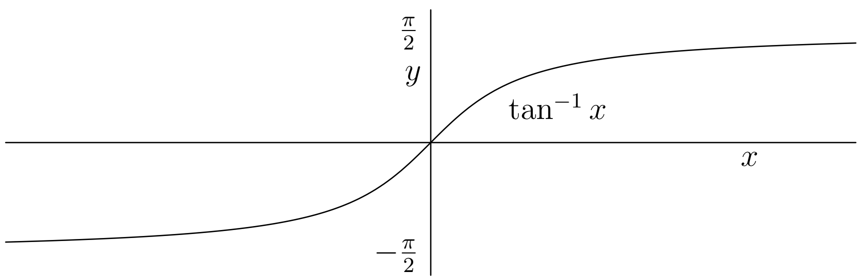
$$2x^2 > x^4 \Rightarrow 2 > x^2 \quad \text{if } x \neq 0.$$

$\therefore$ domain of f' is $\left\{x : 0 < |x| < \sqrt{2}\right\} = \left(-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right)$.

Problem 3.20: Verify that the graphs of the functions $y = \sin^{-1} x$, $y = \cos^{-1} x$, and $y = \tan^{-1} x$ are as shown below.







Remark: We may use the fact that the derivative of the exponential function $y = f(x) = e^x$ is $f(x)$ itself to find the derivative of the inverse function $x = g(y) = \log y$. Since

$$f'(x) = \frac{dy}{dx} = y,$$

we see that

$$g'(y) = \frac{dx}{dy} = \frac{1}{y},$$

Thus for $x > 0$ we find that

$$\frac{d}{dx} \log x = \frac{1}{x}.$$

That is, the derivative of the natural logarithm is the reciprocal function.

Remark: The derivative of the logarithm to the base b follows immediately:

$$\frac{d}{dx} \log_b x = \frac{d}{dx} \frac{\log x}{\log b} = \frac{1}{x \log b}.$$

- Consider

$$F(x) = \log|x| = \begin{cases} \log x & x > 0, \\ \log(-x) & x < 0. \end{cases}$$

Then

$$F'(x) = \begin{cases} \dfrac{1}{x} & x > 0, \\ \\ \dfrac{1}{-x}(-1) & x < 0 \end{cases}$$
$$= \frac{1}{x} \quad \text{for all } x \neq 0.$$

That is, for $x \neq 0$ we find that

$$\frac{d}{dx} \log|x| = \frac{1}{x}.$$

Problem 3.21: Suppose that f and its inverse g are twice differentiable functions on $\mathbb{R}$. Let $a \in \mathbb{R}$ and denote $b = f(a)$.

(a) Implicitly differentiate both sides of the identity $g(f(x)) = x$ with respect to x.
By the Chain Rule,

$$g'(f(x))f'(x) = 1.$$

(b) Using part(a), prove that $f'(a) \neq 0$.
If $f'(a) = 0$, we would obtain a contradiction:

$$0 = g'(f(a))f'(a) = 1.$$

(c) Using parts (a) and (b), find a formula expressing $g'(b)$ in terms of $f'(a)$.

$$g'(b) = \frac{1}{f'(a)}.$$

(d) Show that

$$g''(b) = -\frac{f''(a)}{[f'(a)]^3}.$$

On differentiating the expression in part (a), we find that

$$g''(f(x))[f'(x)]^2 + g'(f(x))f''(x) = 0.$$

On setting $x = a$ and using part(c), we find that

$$g''(f(a))[f'(a)]^2 + \frac{f''(a)}{f'(a)} = 0,$$

from which the desired result immediately follows.

3.6 Logarithmic Differentiation

Because they can be used to transform multiplication problems into addition problems, logarithms are frequently exploited in calculus to facilitate the calculation of derivatives of complicated products or quotients. For example, if we need to calculate the derivative of a *positive* function $f(x)$, the following procedure may simplify the task:

1. Take the logarithm of both sides of $y = f(x)$.
2. Differentiate each side implicitly with respect to x.
3. Solve for dy/dx.

- Differentiate $y = x^{\sqrt{x}}$ for $x > 0$.

  We have
$$\log y = \log x^{\sqrt{x}} = \sqrt{x}\log x.$$
Thus
$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{1}{2\sqrt{x}}\log x + \sqrt{x}\left(\frac{1}{x}\right).\\ \Rightarrow \frac{dy}{dx} &= y\left(\frac{\log x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)\\ &= x^{\sqrt{x}}\left(\frac{\log x + 2}{2\sqrt{x}}\right).\end{aligned}$$

Problem 3.22: Show that the same result follows on differentiating $y = e^{\sqrt{x}\log x}$ directly.

- For $x > 0$ differentiate
$$y = -\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}.$$

  Since
$$\log(-y) = \frac{3}{4}\log x + \frac{1}{2}\log(x^2+1) - 5\log(3x+2),$$
  we find
$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{3}{4}\left(\frac{1}{x}\right) + \frac{1}{2}\left(\frac{1}{x^2+1}\right)(2x) - \frac{5}{3x+2}(3)\\ \Rightarrow \frac{dy}{dx} &= -\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).\end{aligned}$$

Remark: We can use logarithmic differentiation to show that if $y = f(x) = x^n$ for some real number n, then $f'(x) = nx^{n-1}$. First, we take the absolute value of y to ensure that the argument of the logarithm is non-negative:

$$\log|y| = \log|x|^n = n\log|x|.$$

We then implicitly differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x},$$

from which we find that

$$\frac{dy}{dx} = \frac{ny}{x} = \frac{nx^n}{x} = nx^{n-1}.$$

Problem 3.23: Alternatively, show directly from the definition $x^n = e^{n\log x}$ that the rule $dx^n/dx = nx^{n-1}$ is valid for any real n.

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\log x} = e^{n\log x}n\frac{1}{x} = nx^{n-1}.$$

Remark: Recall that

$$\frac{1}{y} = \frac{d}{dy}\log(y) = \lim_{h\to 0}\frac{\log(y+h) - \log(y)}{h}.$$

In particular, at $y = 1/x$ we find

$$x = \lim_{h\to 0}\frac{\log\left(\frac{1}{x}+h\right) + \log x}{h} = \lim_{h\to 0}\log(1+xh)^{\frac{1}{h}} = \log\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

We thus obtain another expression for e^x:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

Remark: In particular at $x = 1$ we obtain a limit expression for the number e:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

3.7 Rates of Change: Physics

Problem 3.24: The position at time t of a particle in meters is given by the equation

$$x = f(t) = t^3 - 6t^2 + 9t.$$

Find (a) the velocity at time t.

$$v(t) = \frac{dx}{dt} = f'(t) = 3t^2 - 12t + 9.$$

(b) When is the particle at rest? Setting

$$0 = v(t) = 3(t^2 - 4t + 3) = 3(t - 3)(t - 1),$$

we see that the particle is at rest when $t = 1$ or $t = 3$.

(c) When is the particle moving forward?

The particle is moving forward when $v(t) > 0$; that is when $t - 3$ and $t - 1$ have the same sign. This happens when $t < 1$ and $t > 3$. For $t \in (1, 3)$ we see that $v(t) < 0$, so the particle moves backwards.

(d) Find the total distance travelled during the first five seconds.

Because the particle retraces it path for $t \in (1, 3)$, we must calculate these distance travelled during $[0, 1]$, $[1, 3]$, and $[3, 5]$ separately. From $t = 0$ to $t = 1$, the distance travelled is $|f(1) - f(0)| = |4 - 0| = 4m$. From $t = 1$ to $t = 3$, the distance travelled is $|f(3) - f(1)| = |0 - 4| = 4m$. From $t = 3$ to $t = 5$, the distance travelled is $|f(5) - f(3)| = |20 - 0| = 20m$. The total distance travelled is therefore $28m$.

(e) Determine the acceleration $a = dv/dt$ of the particle as a function of t.

$$a(t) = 6t - 12.$$

3.8 Related Rates

The Chain Rule is useful for solving problems with two variables that are related to one another. In this, the rate of change of one variable may be related to the rate of change of the other.

Problem 3.25: A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 3m higher than the bow of the boat. If the rope is pulled in at a rate of 2m/s, how fast is the boat approaching the dock when it is 4m from the dock?

Let r denote the length of the rope, from bow to pulley, and x the (horizontal) distance between the bow and the dock. Then $r(x) = \sqrt{x^2 + 3^2}$ so that $dr/dx = x/\sqrt{x^2 + 3^2}$. Thus

$$\frac{dx}{dt} = \frac{dx}{dr} \cdot \frac{dr}{dt} = \frac{\sqrt{4^2 + 3^2}}{4} \times 2 = \frac{5}{2} \text{ m/s}.$$

Problem 3.26: A stone thrown into a pond produces a circular ripple which expands from the point of impact. When the radius is 8m it is observed that the radius is increasing at a rate of 1.5m/s. How fast is the area increasing at that instant?

Problem 3.27: Water is leaking out of a tank shaped like an inverted cone (pointed end at the bottom) at a rate of 10 m^3/min. The tank has a height of 6m and a diameter at the top of 4m. How fast is the water level dropping when the height of the water in the tank is 2m?

3.9 Hyperbolic Functions

Hyperbolic functions are combinations of e^x and e^{-x}:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}, \tanh x = \frac{\sinh x}{\cosh x},$$

$$\mathrm{csch} x = \frac{1}{\sinh x}, \quad \mathrm{sech}\, x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}.$$

Recall that the points $(x, y) = (\cos t, \sin t)$ generate a circle, as t is varied from 0 to 2π, since $x^2 + y^2 \cos^2 t + \sin^2 t = 1$. In contrast, the points $(x, y) = (\cosh t, \sinh t)$ generate a hyperbola, as t is varied over all real values, since $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$ (hence the name hyperbolic functions). That is,

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1.$$

Note that

$$\frac{d}{dx} \sinh x = \frac{e^x + e^{-x}}{2} = \cosh x,$$

but

$$\frac{d}{dx} \cosh x = \frac{e^x - e^{-x}}{2} = \sinh x$$

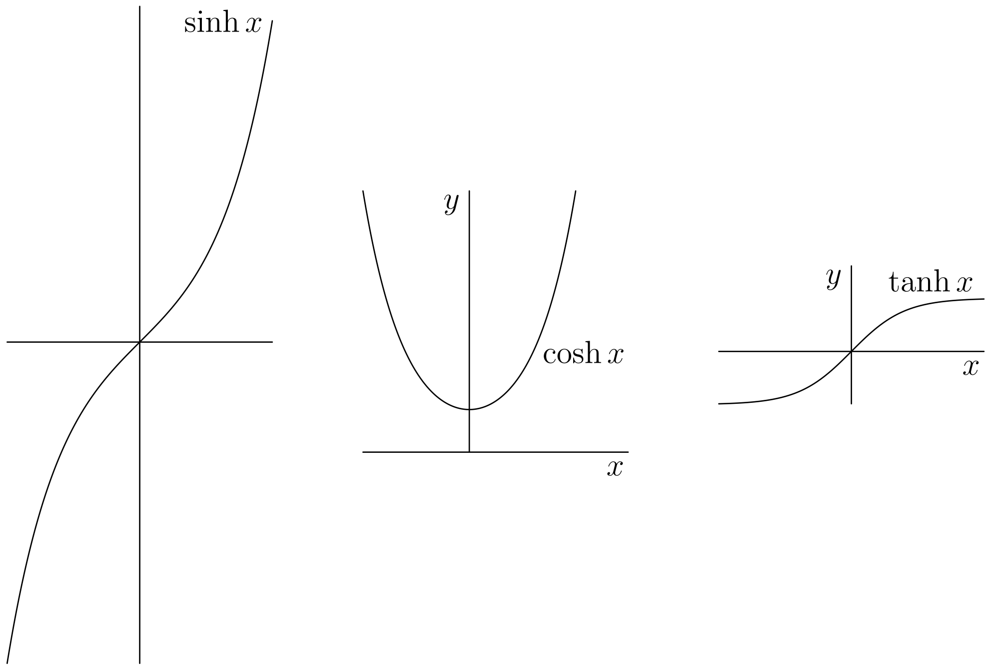
(without any minus sign). Also,

$$\frac{d}{dx} \tanh x = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}.$$

Note that $\sinh x$ and $\tanh x$ are strictly monotonic, whereas $\cosh x$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$.

Just as the inverse of e^x is $\log x$, the inverse of $\sinh x$ also involves $\log x$. Letting $y = \sinh^{-1} x$, we see that

$$x = \sinh y = \frac{e^y - e^{-y}}{2},$$



so that $e^y - e^{-y} - 2x = 0$. To solve for y, it is convenient to make the substitution $z = e^y$:

$$z - \frac{1}{z} - 2x = 0$$
$$\Rightarrow z^2 - 2xz - 1 = 0.$$

Thus

$$z = \frac{2x \pm \sqrt{(2x)^2 + 4}}{2},$$

so that $e^y = x \pm \sqrt{x^2 + 1}$. But since $e^y > 0$ for all $y \in \mathbb{R}$, only the positive square root is relevant. That is, for all real x,

$$\sinh^{-1} x = \log(x + \sqrt{x^2 + 1}).$$

Problem 3.28: Prove that the two solutions for $\cosh^{-1} x$ are given by $\log(x \pm \sqrt{x^2 - 1})$. Show directly that $\log(x + \sqrt{x^2 - 1}) = -\log(x - \sqrt{x^2 - 1})$.

Problem 3.29: Show that

$$\tanh^{-1} x = \frac{1}{2} \log\left(\frac{1 + x}{1 - x}\right).$$

Problem 3.30: Show that

$$\frac{d}{dx}\sinh^{-1}x = \frac{d}{dx}\log\left(x+\sqrt{x^2+1}\right) = \frac{1}{\sqrt{x^2+1}}.$$

Also verify this result directly from the fact that

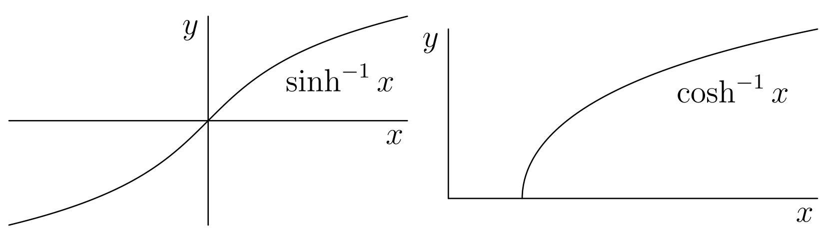
$$\frac{d}{dy}\sinh y = \cosh y.$$

- Thus

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \left[\sinh^{-1}x\right]_0^1 = \left[\log(x+\sqrt{x^2+1})\right]_0^1 = \log\left(1+\sqrt{2}\right).$$

- To find $\frac{d}{dx}\cosh^{-1}x$, we can use the relation $\cosh^2 y - \sinh^2 y = 1$:

$$\begin{aligned} y &= \cosh^{-1}x \\ \Rightarrow x &= \cosh y \\ \Rightarrow \frac{dx}{dy} &= \sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2-1} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{x^2-1}}. \end{aligned}$$



Problem 3.31: Prove that

(a)

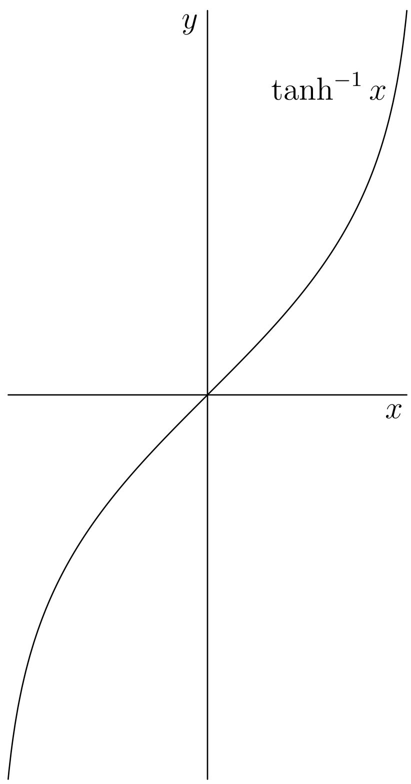
$$\cosh^2 t = \frac{\cosh 2t + 1}{2}$$

(b)

$$\sinh^2 t = \frac{\cosh 2t - 1}{2}$$

(c)

$$2\sinh t\cosh t = \sinh 2t$$


y
tanh⁻¹ x
x

Chapter 4

Applications of Differentiation

4.1 Maxima and Minima

Definition: f has a *global maximum* (*global minimum*) at c if

$$f(x) \leq f(c) \quad (f(x) \geq f(c))$$

for all x in the domain of f. A global maximum or global minimum is sometimes called an *absolute maximum* or *absolute minimum*.

Definition: A function f has an *interior local maximum* (*interior local minimum*) at an interior point c of its domain if for some $\delta > 0$,

$$\begin{aligned} x \in (c-\delta, c+\delta) \Rightarrow\ & f(x) \leq f(c) \\ & (f(x) \geq f(c)). \end{aligned}$$

Definition: An *extremum* is either a maximum or a minimum.

Remark: A global extremum is always a *local extremum* (but not necessarily an *interior local extremum*).

Remark: The following theorem guarantees that a continuous function always has a global maximum over a closed interval.

Theorem 4.1 (Extreme Value Theorem): *If f is continuous on $[a, b]$ then it achieves both a global maximum and minimum value on $[a, b]$. That is, there exists numbers c and d in $[a, b]$ such that*

$$f(c) \leq f(x) \leq f(d) \quad \textit{for all } x \in [a, b].$$

Theorem 4.2 (Fermat's Theorem): *Suppose*

(i) *f has an interior local extremum at c,*

(ii) *$f'(c)$ exists.*

Then $f'(c) = 0$.

Proof: Without loss of generality we can consider the case where f has an interior local maximum, i.e. there exists $\delta > 0$ such that

$$\begin{aligned} x \in (c-\delta, c+\delta) &\Rightarrow f(x) \le f(c) \\ &\Rightarrow \frac{f(x)-f(c)}{x-c} \begin{cases} \ge 0 & \text{if } x \in (c-\delta, c), \\ \le 0 & \text{if } x \in (c, c+\delta) \end{cases} \\ &\Rightarrow f'_L(c) \doteq \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} \ge 0, \\ & f'_R(c) \doteq \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \le 0 \\ & \Rightarrow f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0. \end{aligned}$$

Remark: Theorem 4.2 establishes that the condition $f'(c) = 0$ is *necessary* for a differentiable function to have an interior local extremum. However, this condition alone is not *sufficient* to ensure that a differentiable function has an extremum at c; consider the behaviour of the function $f(x) = x^3$ near the point $c = 0$.

Remark: If a function is continuous on a closed interval, we know from Theorem 4.1 that it must achieve global maximum and minimum values somewhere in the interval. We know from Theorem 4.2 that if these extrema occur in the interior of the interval, the derivative of the function must either vanish there or else not exist. However, it is possible that the global maximum or minimum occurs at one of the endpoints of the interval; at these points, it is not at all necessary that the derivative vanish, even if it exists. It is also possible that an extremum occurs at a point where the derivative doesn't exist. For example, consider the fact that $f(x) = |x|$ has a minimum at $x = 0$.

Extrema can occur either at

(i) an end point,

(ii) a point where f' does not exist,

(iii) a point where $f' = 0$.

- Find the maxima and minima of

$$f(x) = 2x^3 - x^2 + 1 \text{ on } [0, 1].$$

Since f is continuous on $[0, 1]$ we know that it has a global maximum and minimum value on $[0, 1]$. Note that $f'(x) = 6x^2 - 2x = 2x(3x - 1) = 0$ in $(0, 1)$ only at the point $x = 1/3$. Theorem 4.2 implies that the only possible global interior extremum (which is of course also a local interior extremum) is at the point $x = 1/3$. By comparing the function values $f(1/3) = 26/27$ with the endpoint function values $f(0) = 1$ and $f(1) = 2$ we see that f has an (interior) global minimum value of $26/27$ at $x = 1/3$ and an (endpoint) global maximum value of 2 at $x = 1$. Hence $26/27 \le f(x) \le 2$ for all $x \in [0, 1]$.

4.2 The Mean Value Theorem

Theorem 4.3 (Rolle's Theorem): *Suppose*

(i) *f is continuous on $[a, b]$,*

(ii) *f' exists on (a, b),*

(iii) *$f(a) = f(b)$.*

Then there exists a number $c \in (a, b)$ for which $f'(c) = 0$.

Proof:

Case I: $f(x) = f(a) = f(b)$ for all $x \in [a, b]$ (i.e. f is constant on $[a, b]$)
$\Rightarrow f'(c) = 0$ for all $c \in (a, b)$.

Case II: $f(x_0) > f(a) = f(b)$ for some $x_0 \in (a, b)$. Theorem 4.1 $\Rightarrow f$ achieves its maximum value $f(c)$ for some $c \in [a, b]$. But

$$f(c) \ge f(x_0) > f(a) = f(b) \Rightarrow c \in (a, b).$$

$\therefore f$ has an interior local maximum at c.
Theorem 4.2 $\Rightarrow f'(c) = 0$.

Case III (Exercise): $f(x_0) < f(a) = f(b)$ for some $x_0 \in (a, b)$.

- $f(x) = x^3 - x + 1$.

$f(0) = 1$, $f(1) = 1 \Rightarrow$ there exists $c \in (0, 1)$ such that $f'(c) = 0$.
In this case we can actually find the point c. Since $f'(x) = 3x^2 - 1$, we can solve the equation $0 = f'(c) = 3c^2 - 1$ to deduce $c = \dfrac{1}{\sqrt{3}} \in (0, 1)$.

- Recall that $\sin n\pi = 0$, for all $n \in \mathbb{N}$. Rolle's Theorem tells us that $\frac{d}{dx} \sin x = \cos x$ must *vanish* (become zero) at some point $x \in (n\pi, (n+1)\pi)$. Indeed, we know that

$$\cos\left[\left(n + \frac{1}{2}\right)\pi\right] = \cos\left(\frac{2n+1}{2}\pi\right) = 0 \quad \text{for all } n \in \mathbb{N}.$$

- We can use Rolle's Theorem to show that the equation

$$f(x) = x^3 - 3x^2 + k = 0$$

never has 2 distinct roots in $[0, 1]$, no matter what value we choose for the real number k. Suppose that there existed two numbers a and b in $[0, 1]$, with $a \neq b$ and $f(a) = f(b) = 0$. Then Rolle's Theorem $\Rightarrow$ there exists $c \in (a, b) \subset (0, 1)$ such that $f'(c) = 0$. But $f'(x) = 3x^2 - 6x = 3x(x - 2)$ has no roots in $(0, 1)$; this is a contradiction.

Q. What happens when the condition $f(a) = f(b)$ is dropped from Rolle's Theorem? Can we still deduce something similar?

A. Yes, the next theorem addresses precisely this situation.

Theorem 4.4 (Mean Value Theorem [MVT]): *Suppose*

(i) *f is continuous on $[a, b]$,*

(ii) *f' exists on (a, b).*

Then there exists a number $c \in (a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark: Notice that when $f(a) = f(b)$, the Mean Value Theorem reduces to Rolle's Theorem.

Proof: Consider the function

$$\varphi(x) = f(x) - M(x - a),$$

where M is a constant. Notice that $\varphi(a) = f(a)$. We choose M so that $\varphi(b) = f(a)$ as well:

$$M = \frac{f(b) - f(a)}{b - a}.$$

Then φ satisfies all three conditions of Rolle's Theorem:

(i) φ is continuous on $[a, b]$,

(ii) φ' exists on (a, b),

(iii) $\varphi(a) = \varphi(b)$.

Hence there exists $c \in (a, b)$ such that

$$0 = \varphi'(c) = f'(c) - M = f'(c) - \frac{f(a) - f(b)}{b - a}.$$

Q. We know that when $f(x)$ is constant that $f'(x) = 0$. Does the converse hold?

A. No, a function may have zero slope somewhere without being constant (e.g. $f(x) = x^2$ at $x = 0$). However, if $f'(x) = 0$ for all $x \in [a, b]$, where $a \neq b$, we may then make use of the following result.

Theorem 4.5 (Zero Derivative on an Interval): *Suppose $f'(x) = 0$ for every x in an interval I (of nonzero length). Then f is constant on I.*

Proof: Let x, y be any two elements of I, with $x < y$. Since f is differentiable at each point of I, we know by Theorem 3.1 that f is continuous on I. From the MVT, we see that

$$\frac{f(x) - f(y)}{x - y} = f'(c) = 0$$

for some $c \in (x, y) \subset I$. Hence $f(x) = f(y)$. Thus, f is constant on I.

Theorem 4.6 (Equal Derivatives): *Suppose $f'(x) = g'(x)$ for every x in an interval I (of nonzero length). Then $f(x) = g(x) + k$ for all $x \in I$, where k is a constant.*

Theorem 4.7 (Monotonic Test): *Suppose f is differentiable on an interval I. Then*

(i) *f is increasing on $I \iff f'(x) \geq 0$ on I;*

(ii) *f is decreasing on $I \iff f'(x) \leq 0$ on I.*

Proof:
"$\Rightarrow$" Without loss of generality let f be increasing on I. Then for each $x \in I$,

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \geq 0.$$

"$\Leftarrow$" Suppose $f' \geq 0$ on I. Let $x, y \in I$ with $x < y$. The MVT $\Rightarrow$ there exists $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \geq 0$$
$$\Rightarrow f(y) - f(x) \geq 0.$$

Hence f is increasing on I.

Remark: Theorem 4.7 only provides sufficient, not necessary, conditions for a function to be increasing (since it might not be differentiable).

- Consider the function $f(x) = \lfloor x \rfloor$, which returns the greatest integer less than or equal to x. Note that f is increasing (on $\mathbb{R}$) but $f'(x)$ does not exist at integer values of x.

Q. If we replace "increasing" with "strictly increasing" in Theorem 4.7 (i), can we then change "$\geq$" to "$>$"?

A. No, consider the strictly increasing function $f(x) = x^3$. We can only say $f'(x) = 3x^2 \geq 0$ since $f'(0) = 0$.

Problem 4.1: Prove that if f is continuous on $[a, b]$ and $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.

4.3 First Derivative Test

We have seen that points where the derivative of a function vanishes may or may not be extrema. How do we decide which ones are extrema and, of those, which are maxima and which are minima? One answer is provided by the First Derivative Test.

Definition: A point where the derivative of f is zero or does not exist is called a *critical* point.

Theorem 4.2 $\Rightarrow$ Local interior maxima and minima occur at critical points.

Remark: Not all critical points are extrema: consider $f(x) = x^3$ at $x = 0$.

Q. How do we decide which critical points c correspond to maxima, to minima, or neither?

A. If f is differentiable near c, look at the first derivative.

Theorem 4.8 (First Derivative Test): *Let c be a critical point of a continuous function f. If*

(i) *$f'(x)$ changes from negative to positive at c, then f has a local minimum at c;*

(ii) *$f'(x)$ changes from positive to negative at c, then f has a local maximum at c;*

(iii) *$f'(x)$ is positive on both sides of c or negative on both sides of c then f does not have a local extremum at c.*

Proof: (i) This follows directly from the fact that f is then decreasing to the left of c and increasing to the right of c.

(ii)-(iii) Exercises.

Problem 4.2: Give examples of differentiable functions which have the behaviours described in each of the cases above.

4.4 Second Derivative Test

In cases where the second derivative of f can be easily computed, the following test provides simple conditions for classifying critical points.

Theorem 4.9 (Second Derivative Test): *Suppose f is twice differentiable at a critical point c (this implies $f'(c) = 0$). If*

(i) $f''(c) > 0$, *then f has a local minimum at c;*

(ii) $f''(c) < 0$, *then f has a local maximum at c.*

Proof:

(i) $f''(c) > 0 \Rightarrow \lim\limits_{x \to c} \dfrac{f'(x) - f'(c)}{x - c} > 0 \Rightarrow \lim\limits_{x \to c} \dfrac{f'(x)}{x - c} > 0$

$\Rightarrow$ there exists $\delta > 0$ such that $f'(x) \begin{cases} < 0 & \text{for all } x \in (c - \delta, c), \\ > 0 & \text{for all } x \in (c, c + \delta) \end{cases}$

$\Rightarrow f$ has a local minimum at c by the First Derivative Test.

(ii) Exercise.

Remark: If $f''(c) = 0$, then anything is possible.

- $f(x) = x^3$,
  $f'(x) = 3x^2 = 0$ at $x = 0$,
  $f''(x) = 6x = 0$ at $x = 0$,
  f has neither a maximum nor minimum at $x = 0$.

- $f(x) = x^4$,
  $f'(x) = 4x^3 = 0$ at $x = 0$,
  $f''(x) = 12x^2 = 0$ at $x = 0$,
  f has a minimum at $x = 0$.

- $f(x) = -x^4$ has a maximum at $x = 0$.

Remark: The First Derivative Test can sometimes be helpful in cases where the Second Derivative Test fails, e.g. in showing that $f(x) = x^4$ has a minimum at $x = 0$.

Remark: The Second Derivative Test establishes only the local behaviour of a function, whereas the First Derivative Test can sometimes be used to establish that an extremum is global:

$$f(x) = x^2, \qquad f'(x) = 2x \begin{cases} < 0 & \text{for all } x < 0, \\ > 0 & \text{for all } x > 0. \end{cases}$$

Since f is decreasing for $x < 0$ and increasing for $x > 0$, we see that f has a global minimum at $x = 0$.

4.5 Convex and Concave Functions

Definition: A function is *convex* (sometimes called *concave up*) on an interval I if the secant line segment joining $(a, f(a))$ and $(b, f(b))$ lies on or above the graph of f for all $a, b \in I$.

Definition: A function f is *concave* (sometimes called *concave down*) on an interval I if $-f$ is convex on I.

Definition: An *inflection point* is a point on the graph of a function f at which the behaviour of f changes from convex to concave. For example, since $f(x) = x^3$ is concave on $(-\infty, 0]$ and convex on $[0, \infty)$, the point $(0, 0)$ is an inflection point.

Remark: Since the equation of the line through $(a, f(a))$ and $(b, f(b))$ is

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a),$$

the definition of convex says

$$f(x) \le f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \qquad \text{for all } x \in [a, b], \qquad \text{for all } a, b \in I. \quad (4.1)$$

The *linear interpolation* of f between $[a, b]$ on the right-hand side of Eq. (4.1) may be rewritten as:

$$f(x) \le \left(\frac{b - x}{b - a}\right) f(a) + \left(\frac{x - a}{b - a}\right) f(b) \qquad \text{for all } x \in [a, b], \qquad \text{for all } a, b \in I \quad (4.2)$$

or as

$$f(x) \le f(b) + \frac{f(b) - f(a)}{b - a}(x - b) \qquad \text{for all } x \in [a, b], \qquad \text{for all } a, b \in I. \quad (4.3)$$

The convexity condition may also be expressed directly in terms of the slope of a secant:

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x} \qquad \text{for all } x \in (a,b), \quad \text{for all } a,b \in I. \tag{4.4}$$

The left-hand inequality follows directly from Eq. (4.1) and the right-hand inequality follows from Eq. (4.3).

Theorem 4.10 (First Convexity Test): *Suppose f is differentiable on an interval I. Then*

(i) *f is convex $\iff$ f' is increasing on I;*

(ii) *f is concave $\iff$ f' is decreasing on I.*

Proof: Without loss of generality we only need to consider the case where f is convex.

"$\Rightarrow$" Suppose f is convex. Let $a, b \in I$, with $a < b$, and define

$$m(x) = \frac{f(x)-f(a)}{x-a} \quad (x \neq a), \qquad M(x) = \frac{f(b)-f(x)}{b-x} \quad (x \neq b).$$

From Eq. (4.4) we know that

$$m(x) \le m(b) = M(a) \le M(x)$$

whenever $a < x < b$. Hence

$$f'(a) = \lim_{x\to a} m(x) = \lim_{x\to a^+} m(x) \le m(b) = M(a) \le \lim_{x\to b^-} M(x) = \lim_{x\to b} M(x) = f'(b).$$

Thus f' is increasing on I.

"$\Leftarrow$" Suppose f' is increasing on I. Let $a, b \in I$, with $a < b$ and $x \in (a,b)$. By the MVT,

$$\frac{f(x)-f(a)}{x-a} = f'(c_1), \qquad \frac{f(b)-f(x)}{b-x} = f'(c_2)$$

for some $c_1 \in (a,x)$ and $c_2 \in (x,b)$. Since f' is increasing and $c_1 < c_2$, we know that $f'(c_1) \le f'(c_2)$. Hence

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(x)}{b-x}$$

$$\Rightarrow f(x)\left[\frac{1}{x-a} + \frac{1}{b-x}\right] \le \frac{f(b)}{b-x} + \frac{f(a)}{x-a} = \frac{f(b)(x-a)+f(a)(b-x)}{(b-x)(x-a)},$$

which reduces to Eq. (4.2), so f is convex.

Theorem 4.11 (Second Convexity Test): *Suppose f'' exists on an interval I. Then*

(i) *f is convex on $I \iff f''(x) \geq 0 \quad$ for all $x \in I$;*

(ii) *f is concave on $I \iff f''(x) \leq 0 \quad$ for all $x \in I$.*

Proof: Apply Theorem 4.7 to f'.

Theorem 4.12 (Tangent to a Convex Function): *If f is convex and differentiable on an interval I, the graph of f lies above the tangent line to the graph of f at every point of I.*

Proof: Let $a \in I$. The equation of the tangent line to the graph of f at the point $(a, f(a))$ is $y = f(a) + f'(a)(x-a)$. Given $x \in I$, the MVT implies that $f(x) - f(a) = f'(c)(x-a)$, for some c between a and x. Since f is convex on I, we also know, from Theorem 4.10, that f' is increasing on I:

$$x < a \Rightarrow c < a \Rightarrow f'(c) \leq f'(a),$$

$$x > a \Rightarrow c > a \Rightarrow f'(c) \geq f'(a).$$

In either case $f(x) - f(a) = f'(c)(x-a) \geq f'(a)(x-a)$. Hence

$$f(x) \geq f(a) + f'(a)(x-a) \qquad \text{for all } x \in I.$$

- Consider $f(x) = \dfrac{1}{1+x^2}$ on $\mathbb{R}$.

  Observe that $f(0) = 1$ and $f(x) > 0$ for all $x \in \mathbb{R}$ and $\lim\limits_{x\to\pm\infty} f(x) = 0$. Note that f is even: $f(-x) = f(x)$. Also, $1 + x^2 \geq 1 \Rightarrow f(x) \leq 1 = f(0)$, so f has a maximum at $x = 0$. Alternatively, we can use either the First Derivative Test or the Second Derivative Test to establish this. We find

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

  and,

$$f''(x) = \frac{-2}{(1+x^2)^2} + \frac{2(2x)2x}{(1+x^2)^3} = \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3} = \frac{2(3x^2-1)}{(1+x^2)^3}.$$

  First Derivative Test: $\begin{cases} f'(x) > 0 \text{ on } (-\infty, 0) \Rightarrow f \text{ is increasing on } (-\infty, 0), \\ f'(x) < 0 \text{ on } (0, \infty) \Rightarrow f \text{ is decreasing on } (0, \infty) \end{cases}$
  $\Rightarrow f$ has a maximum at 0.

  Second Derivative Test: $f'(0) = 0, f''(0) = -2 < 0 \Rightarrow f$ has a maximum at 0.

$$\text{Convexity: } \begin{cases} f''(x) \geq 0 \text{ for } |x| \geq \dfrac{1}{\sqrt{3}}, \text{ i.e. } f \text{ is convex on } \left(-\infty, -\dfrac{1}{\sqrt{3}}\right] \cup \left[\dfrac{1}{\sqrt{3}}, \infty\right), \\ f''(x) \leq 0 \text{ for } |x| \leq \dfrac{1}{\sqrt{3}}, \text{ i.e. } f \text{ is concave on } \left[-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right], \\ f''(x) = 0 \text{ at } \pm\dfrac{1}{\sqrt{3}}; \text{ these correspond to inflection points.} \end{cases}$$

Problem 4.3: Consider the function $f(x) = (x+1)x^{2/3}$ on $[-1, 1]$.

(a) Find $f'(x)$.
On rewriting $f(x) = x^{5/3} + x^{2/3}$, we find

$$f'(x) = \frac{5}{3}x^{2/3} + \frac{2}{3}x^{-1/3} = \frac{x^{-1/3}}{3}(5x+2) \quad (x \neq 0).$$

(b) Determine on which intervals f is increasing and on which intervals f is decreasing.
Since

$$f'(x) \begin{cases} > 0, & -1 \leq x < -2/5, \\ = 0, & x = -2/5, \\ < 0, & -2/5 < x < 0, \\ \text{does not exist} & x = 0, \\ > 0, & 0 < x \leq 1, \end{cases}$$

we know that f is increasing on $[-1, -2/5]$ and $[0, 1]$. It is decreasing on $[-2/5, 0]$.

(c) Does f have any interior local extrema on $[-1, 1]$? If so, where do these occur? Which are maxima and which are minima?

Note that f has two critical points: $x = -2/5$ and $x = 0$. By the First Derivative Test, f has a local maximum at $x = -2/5$ and a local minimum at $x = 0$.

(d) What are the global minimum and maximum values of f and at what points do these occur?

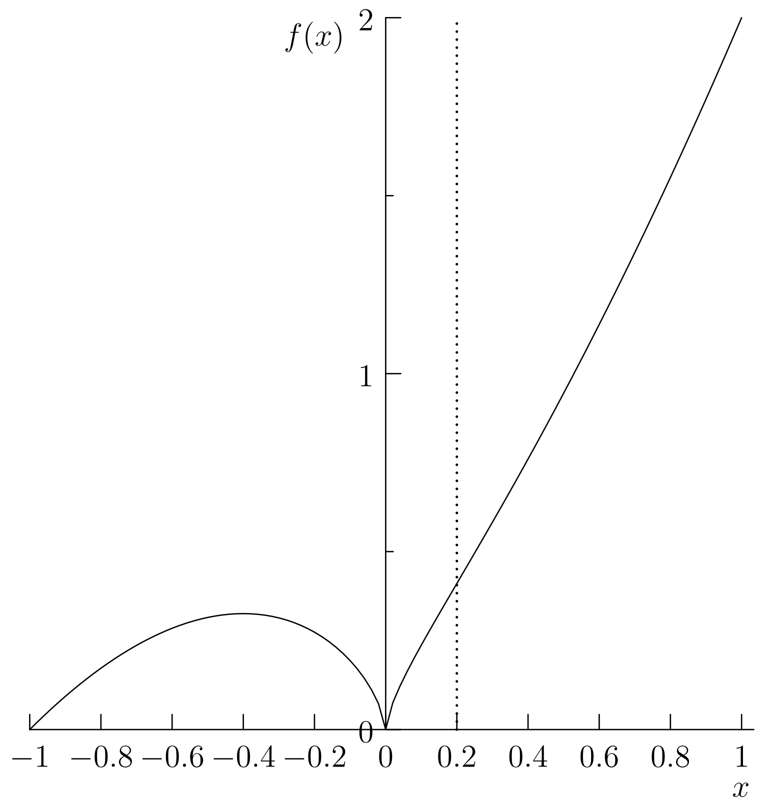
On comparing the endpoint function values to the function values at the critical points, we conclude that f achieves its global minimum value of 0 at $x = -1$ and at $x = 0$. It has a global maximum value of 2 at $x = 1$.

(e) Determine on which intervals f is convex and on which intervals f is concave.
Since $f''(x) = \frac{10}{9}x^{-1/3} - \frac{2}{9}x^{-4/3} = \frac{2}{9}x^{-4/3}(5x-1)$, we see that

$$f''(x) \begin{cases} < 0, & -1 \leq x < 0, \\ \text{does not exist,} & x = 0, \\ < 0, & 0 < x < 1/5, \\ = 0, & x = 1/5, \\ > 0, & 1/5 < x \leq 1. \end{cases}$$

Thus, f is concave on $[-1, 0]$ and $[0, 1/5]$ and convex on $[1/5, 1]$. Note that f is not concave on the interval $[-1, 1/5]$.

(f) Does f have any inflection points? If so, where?
Yes: f has an inflection point at $x = 1/5$.
(g) Sketch a graph of f using the above information.



4.6 L'Hôpital's Rule

Theorem 4.13 (Cauchy Mean Value Theorem): *Suppose*

(i) *f and g are continuous on $[a, b]$,*

(ii) *f' and g' exist on (a, b).*

Then there exists a number $c \in (a, b)$ for which

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Proof: Consider

$$\phi(x) = [f(x) - f(a)][g(b) - g(a)] - [f(b) - f(a)][g(x) - g(a)].$$

Note that ϕ is continuous on $[a, b]$ and differentiable on (a, b). Since $\phi(a) = \phi(b) = 0$, we know from Rolle's Theorem that $\phi'(c) = 0$ for some $c \in (a, b)$; from this we immediately deduce the desired result.

Theorem 4.14 (L'Hôpital's Rule for $\frac{0}{0}$): *Suppose f and g are differentiable on (a,b), $g'(x) \neq 0$ for all $x \in (a,b)$, $\lim\limits_{x\to b^-} f(x) = 0$, and $\lim\limits_{x\to b^-} g(x) = 0$. Then*

$$\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to b^-} \frac{f(x)}{g(x)} = L.$$

This result also holds if

(i) $\lim\limits_{x\to b^-}$ *is replaced by* $\lim\limits_{x\to a^+}$*;*

(ii) $\lim\limits_{x\to b^-}$ *is replaced by* $\lim\limits_{x\to\infty}$ *and b is replaced by ∞;*

(iii) $\lim\limits_{x\to b^-}$ *is replaced by* $\lim\limits_{x\to-\infty}$ *and a is replaced by $-\infty$.*

Proof: Theorem 3.1 $\Rightarrow$ f and g are continuous on (a,b). Consider

$$F(x) = \begin{cases} f(x) & a < x < b, \\ 0 & x = b. \end{cases}$$

$$G(x) = \begin{cases} g(x) & a < x < b, \\ 0 & x = b. \end{cases}$$

Since $\lim_{x\to b^-} f(x) = 0$, and $\lim_{x\to b^-} g(x) = 0$, we know for any $x \in (a,b)$ that F and G are continuous on $[x,b]$ and differentiable on (x,b). We can also be sure that G is nonzero on (a,b): if $G(x) = 0 = G(b)$ for some $x \in (a,b)$, Rolle's Theorem would imply that G', and hence g', vanishes somewhere in (x,b).

Given $\epsilon > 0$, we know there exists a number δ with $0 < \delta < b-a$ such that

$$x \in (b-\delta, b) \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon.$$

If $x \in (b-\delta, b)$, Theorem 4.13 then implies that there exists a point $c \in (x,b)$ such that

$$\frac{f(x)}{g(x)} = \frac{F(x)}{G(x)} = \frac{F(x)-F(b)}{G(x)-G(b)} = \frac{F'(c)}{G'(c)} = \frac{f'(c)}{g'(c)},$$

so that

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon.$$

That is, $\lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)} = L$.

- Using L'Hôpital's Rule, we find

$$\lim_{x\to 0} \frac{\tan x}{x} = 1 \Leftarrow \lim_{x\to 0} \frac{\sec^2 x}{1} = 1,$$

- $$\lim_{x\to 1} \frac{x^n - 1}{x - 1} = n \Leftarrow \lim_{x\to 1} \frac{nx^{n-1}}{1} = n.$$

Remark: L'Hôpital's Rule should only be used where it applies. For example, it should not be used for when the limit does not have the $\frac{0}{0}$ form. For example, $0 = \lim_{x\to 1} \dfrac{x-1}{x} \neq \lim_{x\to 1} \dfrac{1}{1} = 1.$

Theorem 4.15 (L'Hôpital's Rule for $\frac{\infty}{\infty}$): *Suppose f and g are differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, and $\lim_{x\to b^-} f(x) = \infty$, and $\lim_{x\to b^-} g(x) = \infty$. Then*

$$\lim_{x\to b^-} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to b^-} \frac{f(x)}{g(x)} = L.$$

This result also holds if

(i) $\lim_{x\to b^-}$ *is replaced by* $\lim_{x\to a^+}$*;*

(ii) $\lim_{x\to b^-}$ *is replaced by* $\lim_{x\to\infty}$ *and b is replaced by ∞;*

(iii) $\lim_{x\to b^-}$ *is replaced by* $\lim_{x\to -\infty}$ *and a is replaced by $-\infty$.*

Proof: We only need to make minor modifications to the proof used to establish Theorem 4.14. Choose δ such that $f(x) > 0$ and $g(x) > 0$ on $(b - \delta, b)$ and redefine

$$F(x) = \begin{cases} \frac{1}{f(x)} & b - \delta < x < b, \\ 0 & x = b, \end{cases} \qquad G(x) = \begin{cases} \frac{1}{g(x)} & b - \delta < x < b, \\ 0 & x = b. \end{cases}$$

Problem 4.4: Determine which of the following limits exist as a finite number, which are ∞, which are $-\infty$, and which do not exist at all. Where possible, compute the limit.

(a)

$$\lim_{x\to 1} \frac{\log x}{x - 1}$$

$$= \lim_{x\to 1} \frac{1/x}{1} = 1.$$

(b)

$$\lim_{x\to 1} \frac{e^x - 1 - x}{x^2}$$

$$= \lim_{x\to 1} \frac{e^x - 1}{2x} = \lim_{x\to 1} \frac{e^x}{2} = \frac{1}{2}.$$

(c)

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

$$= \lim_{x\to 1} \frac{e^x}{2x} = \lim_{x\to 1} \frac{e^x}{2} = \infty.$$

(d)

$$\lim_{x\to 0} x^x$$

$$= \lim_{x\to 0} e^{x\log x} = e^{\lim_{x\to 0} x\log x} = e^{\lim_{x\to 0}} \frac{\log x}{1/x} = e^{\lim_{x\to 0}} \frac{1/x}{-1/x^2} = e^{\lim_{x\to 0}} - x = e^0 = 1.$$

(e)

$$\lim_{x\to 1} \frac{\sin(x^{99}) - \sin(1)}{x-1}$$

One could use L'Hôpital's Rule here, but it is even simpler to note that this is just the definition of the derivative of the function $f(x) = \sin(x^{99})$ at $x = 1$. Since

$$f'(x) = \cos(x^{99})99x^{98},$$

the limit reduces to $f'(1) = 99\cos(1)$.

(f)

$$\lim_{x\to\pi/4} \frac{\tan x - 1}{x - \pi/4}$$

Letting $f(x) = \tan x$, we see that this is just the definition of $f'(\pi/4) = \sec^2(\pi/4) = 2$.

(g)

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}$$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^3 x \sin x}{6x} = \lim_{x\to 0} \frac{-6\sec^2 x \sin^2 x + 2\sec^2 x \cos x}{6} = \frac{1}{3},$$

on applying the 0/0 form of L'Hôpital's Rule three times. Alternatively, after the second application of L'Hôpital's Rule, one can use the fact that $\lim\limits_{x\to 0} \sin x/x = 1$.

4.7 Slant Asymptotes

Definition: A function f is asymptotic to a *slant asymptote* $y = mx + b$ if

$$\lim_{x\to\infty}[f(x) - (mx + b)] = 0.$$

or

$$\lim_{x\to-\infty}[f(x) - (mx + b)] = 0$$

This means that the vertical distance between the function f and the line $y = mx+b$ approaches zero in the limit as $x \to \infty$ or $x \to -\infty$, respectively.

Remark: If

$$\lim_{x\to\infty}[f(x) - (mx + b)] = 0,$$

then

$$\lim_{x\to\infty}[\frac{f(x) - (mx + b)}{x}] = 0.$$

We also know that $\lim_{x\to\infty} \frac{b}{x} = 0$. On adding these equations we find that

$$\lim_{x\to\infty}[\frac{f(x)}{x} - m] = 0.$$

Hence

$$m = \lim_{x\to\infty} \frac{f(x)}{x}.$$

Once we know m then we can find

$$b = \lim_{x\to\infty}[f(x) - mx].$$

If **both** of the limits for m and b exist, then f has a slant asymptote.

- The function $f(x) = x^3/(x^2 + 1)$, has no vertical asymptotes (since the denominator is never zero) and no horizontal asymptotes since $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. However, it does have a slant asymptote $y = mx + b$ where

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x^2}{(x^2 + 1)} = 1$$

and

$$b = \lim_{x\to\infty} \frac{x^3}{(x^2 + 1) - x} = \lim_{x\to\infty} \frac{x^3 - x(x^2 + 1)}{(x^2 + 1)} = \lim_{x\to\infty} \frac{x}{(x^2 + 1)} = 0.$$

So f has a slant asymptote of $y = x$ as $x \to \infty$. Note that f also has a slant asymptote of $y = x$ as $x \to -\infty$.

Problem 4.5: Show that the function $f(x) = x + \frac{\sin x}{x}$ has a slant asymptote of $y = x$ as $x \to \pm\infty$.

4.8 Optimization Problems

- Determine the rectangle having the largest area that can be inscribed inside a right-angle triangle of side lengths a, b, and $\sqrt{a^2+b^2}$, if the sides of the rectangle are constrained to be parallel to the sides of length a and b.

  Let the vertices of the triangle be $(0,0)$, $(a,0)$, $(0,b)$ and those of the rectangle be $(0,0)$, $(0,x)$, (x,y), $(0,y)$, where $0 \le x \le a$. By similar triangles we see that

$$\frac{y}{a-x} = \frac{b}{a}.$$

  The area A of the rectangle is given by

$$A(x) = xy = \frac{b}{a}x(a-x) = bx - \frac{b}{a}x^2,$$

  so that

$$A'(x) = b - \frac{2b}{a}x = \frac{b}{a}(a-2x).$$

  Since A is continuous on the closed interval $[0,a]$ we know that A must achieve maximum and minimum values in $[0,a]$. Since $A'(x)$ exists everywhere in $(0,a)$, the only points we need to check are $x = a/2$, where $A'(x) = 0$, and the endpoints $x = 0$ and $x = a$; at least one of these must represent a maximum area and one must represent a minimum area. Since $A(a/2) = ab/4$ and $A(0) = A(a) = 0$ we see that the maximum area is $ab/4$ and the minimum area is 0. Thus, the largest rectangle that can be inscribed has side lengths $a/2$ and $b/2$.

Problem 4.6: A canoeist is at the southwest corner of a square lake of side 1 km. She would like to travel to the northeast corner of the lake by rowing to a point on the north shore at a speed of 3 km/h in a straight line at an angle θ measured relative to north. She then plans to walk east along the north shore at a speed of 6 km/h until she arrives at her destination.

At what angle θ should the canoeist row in order to arrive at her destination in the shortest possible time? What is this minimum time? Prove that your answer corresponds to a minimum.

The time to reach her destination is

$$T(\theta) = \frac{1}{3\cos\theta} + \frac{1 - \tan\theta}{6}.$$

The continuous function T must achieve global minimum and maximum values on $[0, \frac{\pi}{4}]$. First, we look for critical points of this function on $(0, \frac{\pi}{4})$:

$$0 = T'(\theta) = \frac{\sin\theta}{3\cos^2\theta} - \frac{1}{6\cos^2\theta} \Rightarrow \sin\theta = \frac{1}{2}.$$

The only critical point in $(0, \frac{\pi}{4})$ is at $\theta = \frac{\pi}{6}$. By simply comparing values, we see that the endpoint value $T(0) = 1/2$ is an exterior global maximum, the endpoint value $T(\frac{\pi}{4}) = \sqrt{2}/3$ is an endpoint local maxima, and $T(\frac{\pi}{6}) = \frac{1+\sqrt{3}}{6}$ is the global minimum value. Thus the canoeist should row at an angle $\frac{\pi}{6}$ relative to north.

Problem 4.7: Maximize the total surface area (including the top and bottom) of a can with volume 1000cm^3 and the shape of a circular cylinder.

The total surface area of cylinderical can of radius r and height h is given by $2\pi r^2 + 2\pi rh$, where the volume is constrained to be $\pi r^2 h = 1000$. We can use the volume constraint to eliminate one variable, say h, from the problem:

$$h = \frac{1000}{\pi r^2},$$

allowing us to express the area A solely as a function of r:

$$A(r) = 2\pi r^2 + \frac{2000}{r}.$$

We need to maximize $A(r)$ on the interval $(0, \infty)$. On noting that $A'(r) = 4\pi r - 2000/r^2 = 4r(\pi - 500/r^3)$, we see that the only critical point of A occurs at $c = \sqrt[3]{\frac{500}{\pi}}$. Since $(0, \infty)$ is not a closed interval, we cannot use the Extreme Value Theorem. Instead, we note that the first derivative of A is negative for $r < c$ and positive for c. This means that A is decreasing on $(0, c]$ and increasing on $[c, \infty)$. Thus A has a global minimum at $r = c$.

4.9 Newton's Method

In 1823, Abel proved that no general algebraic solution (involving only arithmetic operations and radicals) exists for finding roots of fifth-degree polynomials. This result was generalized by Galois to all degrees above four.

If we need to find the roots to a polynomial of degree five or higher, or to a transcendental (non-algebraic) equation like

$$\cos x - x = 0,$$

then we must resort to a numerical method.

Newton's method (also called the *Newton-Raphson* method for finding a root to a function $f(x)$ with a continuous derivative begins with an initial guess x_1. The equation for the tangent line to the graph of f at $(x_1, f(x_1)$ is

$$y - f(x_1) = f'(x_1)(x - x_1).$$

This line intersects the x axis when $y = 0$, at a point $(x_2, 0)$:

$$0 - f(x_1) = f'(x_1)(x_2 - x_1).$$

On solving for x_2 we find

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Newton's method doesn't always work, but when it does, the point x_2 will be closer to the root of f than x_1. On repeating this procedure using x_2 as initial guess, we obtain our next guess x_3 for the root:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

Inductively, we define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If $\lim_{n\to\infty} x_n$ exists and equals c then we see that

$$0 = \lim_{n\to\infty} x_{n+1} - \lim_{n\to\infty} x_n + \lim_{n\to\infty} \frac{f(x_n)}{f'(x_n)} = c - c + \lim_{n\to\infty} \frac{f(x_n)}{\lim_{n\to\infty} f'(x_n)} = \frac{f(c)}{f'(c)}$$

since f and f' are continuous. Thus $f(c) = 0$ so we have determined that c is a root of f $f(c) = 0$.

- To find a root of $f(x) = \cos x - x$, first compute $f'(x) = -\sin x - 1$. The Newton iteration appears as

  $$x_{n+1} = x_n + \frac{\cos x - x}{\sin x + 1}.$$

  Starting with an initial guess $x_1 = 1$, we find

  $$x_2 = 0.750363867840,$$

  $$x_3 = 0.739112890911,$$

  $$x_4 = 0.739085133385,$$

  $$x_5 = 0.739085133215,$$

  $$x_6 = 0.739085133215,$$

  from which it appears that there is a root of f very close to $x = 0.739085133215$. Indeed we verify that $\cos x - x$ is about 2.7×10^{-13}.

Chapter 5

Integration

5.1 Areas

Suppose, given a function $f(x) \geq 0$ on $[a, b]$ that we wish to determine the area of the region bounded by the graph of $f(x)$, the x axis, and the lines $x = a$ and $x = b$. That is, we want to find the area of the region

$$\mathcal{S} = \{(x, y) : a \leq x \leq b,\ 0 \leq y \leq f(x)\}.$$

We could approximate the area as a sum of areas of rectangles as in Figure 5.1 or as in Figure 5.2, determining the height of each rectangle by the function value at the left or right endpoint of each subinterval, respectively.

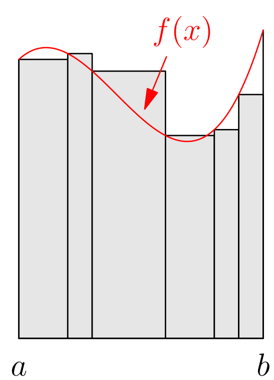


Figure 5.1: Left-endpoint approximation.

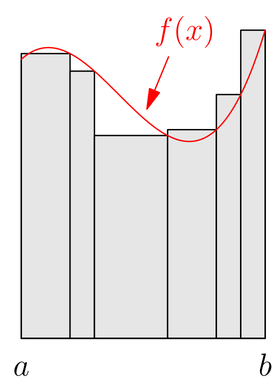


Figure 5.2: Right-endpoint approximation.

Definition: Let f be a function on $[a, b]$. Divide the interval $[a, b]$ into subintervals $[x_{i-1}, x_i]$ for $i = 1, 2, \dots n$. A *Riemann sum* of f is any sum of the form

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}),$$

where the *sample points* x_i^* are arbitrarily chosen from $[x_{i-1}, x_i]$. Here $f(x_i^*)$ refers to the height of a rectangle of width $x_i - x_{i-1}$ that approximates the contribution to the area coming from the ith subinterval $[x_{i-1}, x_i]$. On summing up these contributions from all n subintervals, we obtain an approximation to the total area under the function f from $x = a$ to $x = b$.

Definition: The *left Riemann sum*

$$\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

is obtained by choosing $x_i^* = x_{i-1}$.

Definition: The *right Riemann sum*

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

is obtained by choosing $x_i^* = x_i$.

Definition: A common choice for the subinterval endpoints is $x_i = a + i(b - a)/n$, which generates subintervals with a uniform width:

$$x_i - x_{i-1} = \frac{b - a}{n}.$$

- The right Riemann sum corresponding to $f(x) = x$ on $[0, 1]$ partitioned into n uniform subintervals $[x_{i-1}, x_i]$, where $x_i = i/n$, is

$$\sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \frac{n(n+1)}{2} = \frac{n+1}{2n}$$

since the subinterval widths $x_i - x_{i-1} = \dfrac{1-0}{n} = \dfrac{1}{n}$.

We now use the concept of a Riemann sum to obtain a precise definition for the notion of the *area* under a function.

Definition: The area of the region bounded by the graph of a positive function $f(x)$, the x axis, and the lines $x = a$ and $x = b$ is given by the *Riemann Integral*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}),$$

whenever this limit exists and is independent of the choice of sample points x_i^* and subintervals $[x_{i-1}, x_i]$, provided the subinterval widths $x_i - x_{i-1}$ approach zero as $n \to \infty$. In this case we say that f is *integrable* on $[a, b]$ and define the *definite integral* of f on $[a, b]$ as

$$\int_a^b f = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

The next theorem tells us that the Riemann integral of any continuous function on a bounded interval $[a, b]$ always exists.

Theorem 5.1 (Integrability of Continuous Functions): *If f is continuous on $[a, b]$ then $\int_a^b f$ exists.*

Remark: By the definition of integrability, all uniform Riemann sums of a continuous function f on $[a, b]$ will converge to $\int_a^b f$ as the width of the subintervals approaches zero.

- The definite integral of the continous function $f(x) = x$ on $[0, 1]$ can be computed as the limit of its right Riemann sum:

$$\int_a^b f = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

  Notice that this result agrees with the area of a right triangle with two sides of length one.

- The definite integral of the continous function $f(x) = x$ on $[0, 1]$ can also be computed as the limit of its left Riemann sum:

$$\begin{aligned}\int_a^b f &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} (i-1) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=0}^{n-1} i \\ &= \lim_{n\to\infty} \frac{1}{n^2} \left(0 + \sum_{i=1}^{n-1} i \right) = \lim_{n\to\infty} \frac{1}{n^2} \left(\frac{(n-1)n}{2} \right) = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}.\end{aligned}$$

  This result agrees with the definite integral computed using the right Riemann sum.

Definition: If $a \le b$, define $\int_b^a f = -\int_a^b f$.

Remark: This implies that $\int_a^a f = 0$.

Theorem 5.2 (Piecewise Integration): *Let c be a real number. If $\int_a^c f$ and $\int_c^b f$ exists then $\int_a^b f$ exists and equals $\int_a^c f + \int_c^b f$.*

Remark: Using piecewise integration, we can integrate any function with a finite number of jump discontinuities over a closed interval.

Theorem 5.3 (Linearity of Integral Operator): *Suppose $\int_a^b f$ and $\int_a^b g$ exist. Then*

(i) $\int_a^b (f+g)$ exists and equals $\int_a^b f + \int_a^b g$,

(ii) $\int_a^b (cf)$ exists and equals $c\int_a^b f$ for any constant $c \in \mathbb{R}$.

Theorem 5.4 (Integral Bounds): *Suppose for $a < b$ that*

(i) $\int_a^b f$ exists,

(ii) $m \le f(x) \le M$ for $x \in [a, b]$.

Then

$$m(b-a) \le \int_a^b f \le M(b-a).$$

Theorem 5.5 (Preservation of Non-Negativity): *If $f(x) \ge 0$ for all $x \in [a, b]$, where $a < b$, and $\int_a^b f$ exists then $\int_a^b f \ge 0$.*

Proof: Set $m = 0$ in Theorem 5.4.

Theorem 5.6 (Absolute Integral Bounds): *If $|f(x)| \le M$ for all x between a and b and $\int_a^b f$ exists then $\left|\int_a^b f\right| \le M\,|b-a|$.*

Proof: Set $m = -M$ in Theorem 5.4.

Theorem 5.7 (Triangle Inequality for Integrals): *Let f be an integrable function on $[a, b]$. Then*

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Proof: Consider the integrable functions $f(x)$, $-|f(x)|$ and $|f(x)|$. For every $x \in [a, b]$,

$$-|f(x)| \le f(x) \le |f(x)|.$$

Thus

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|.$$

This means that $\left|\int_a^b f\right| \le \int_a^b |f|$.

Remark: When we write $\int f$ we understand that f is a function of some variable. Let's call it x. We want to partition a portion of the x axis into $\{x_0, x_1, \ldots, x_n\}$ and compute Riemann sums based on function values $f(x_i^*)$ and interval widths $x_i - x_{i-1}$. Similarly, when we write f', it is clear that we mean the derivative of f with respect to its argument, whatever that may be. However, if we want to differentiate the function $y = f(u)$, where $u = x^2$, it is important to know whether we are differentiating with respect to u or with respect to x. Likewise, suppose we wish to calculate the integral of f. It is equally important to know whether we are calculating the integral with respect to u or with respect to x, because the area under the graph of $y = f(u)$ with respect to u will in general differ from the area under the graph of $y = f(x^2)$ with respect to x. Since we can differentiate with respect to different variables, it is only reasonable that we should be able to integrate with respect to different variables as well. It will often be helpful to indicate explicitly with respect to which variable we are integrating, that is, which variable do we use to construct the differences $x_i - x_{i-1}$ in the Riemann sums.

Definition: We can specify the integration variable by writing $\int_0^1 f(x)\,dx$ instead of just $\int_0^1 f$. The notation $f(x)\,dx$ reminds us that the Riemann sums consists of function values multiplied by interval widths, $x_i - x_{i-1}$.

5.2 Fundamental Theorem of Calculus

Definition: A differentiable function F is called an *antiderivative* of f at an interior point x of its domain if $F'(x) = f(x)$.

Remark: If $F(x)$ is an antiderivative of f, then so is $F(x) + C$ for any constant C.

Theorem 5.8 (Families of Antiderivatives): *Let $F_0(x)$ be an antiderivative of f on an interval I. Then F is an antiderivative of f on I $\iff$ $F(x) = F_0(x) + C$ for some constant C.*

Proof:

"$\Leftarrow$" Let $F(x) = F_0(x) + C$. Then $F'(x) = F_0'(x) = f(x)$; that is, F is an antiderivative of f on I.

"$\Rightarrow$" Since

$$\frac{d}{dx}[F(x) - F_0(x)] = F'(x) - F_0'(x) = f(x) - f(x) = 0,$$

we see by Theorem 4.5 that $F(x) - F_0(x)$ is constant on I.

Theorem 5.9 (Antiderivatives at Points of Continuity): *Suppose*

(i) $\int_a^b f$ *exists;*

(ii) f *is continuous at* $c \in (a, b)$.

Then f *has the antiderivative* $F(x) = \int_a^x f$ *at* $x = c$.

Proof: Given $\epsilon > 0$, we know from the continuity of f at c that there exists a $\delta > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

We use this bound in Theorem 5.6 to conclude for $|h| < \delta$ that

$$\left| \int_c^{c+h} [f(x) - f(c)]\, dx \right| \le \epsilon\, |c + h - c| = \epsilon\, |h|\, .$$

Consider $F(x) = \int_a^x f$ for $x \in [a, b]$. Then for $0 < |h| < \delta$ we see that

$$\begin{aligned} \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| &= \frac{1}{|h|} \left| \int_a^{c+h} f(x)\, dx - \int_a^c f(x)\, dx - f(c)h \right| \\ &= \frac{1}{|h|} \left| \int_c^{c+h} f(x)\, dx - f(c) \int_c^{c+h} 1\, dx \right| \\ &= \frac{1}{|h|} \left| \int_c^{c+h} [f(x) - f(c)]\, dx \right| \le \epsilon. \end{aligned}$$

But this is just the statement that the limit

$$F'(c) = \lim_{h \to 0} \frac{F(c+h) - F(c)}{h}$$

exists and equals $f(c)$.

Remark: In particular, Theorem 5.9 says that, at any point $x \in (a, b)$ where an integrable function f is continuous,

$$\frac{d}{dx} \int_a^x f = f(x).$$

Thus we see that differentiation and integration are in a sense opposite processes. The actual situation is slightly complicated by the fact that antiderivatives are not unique, as we saw in Theorem 5.8. However, note that the arbitrary constant C in Theorem 5.8 disappears upon differentiation of the antiderivative.

Theorem 5.10 (Antiderivative of Continuous Functions): *If* f *is continuous on* $[a, b]$ *then* f *has an antiderivative on* $[a, b]$.

Proof: The antiderivative of f on $[a, b]$ is just the antiderivative $\int_a^x \bar{f}$ of the *continuous extension* $\bar{f}$ of f onto all of $\mathbb{R}$:

$$\bar{f}(x) = \begin{cases} f(a) & \text{if } x < a, \\ f(x) & \text{if } a \le x \le b, \\ f(b) & \text{if } x > b. \end{cases}$$

Theorem 5.11 (Fundamental Theorem of Calculus [FTC]): *Let f be integrable and have an antiderivative F on $[a, b]$. Then*

$$\int_a^b f = F(b) - F(a).$$

Proof: Parition $[a, b]$ into $\{x_0, x_1, \ldots, x_n\}$. Since F is differentiable on $[a, b]$, the MVT tells us that for each $i = 1, \ldots n$ there exists a $c_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1}).$$

Consider the Riemann sum

$$\sum_{i=1}^n f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^n [F(x_i) - F(x_{i-1})] = F(x_n) - F(x_0) = F(b) - F(a),$$

independent of n. Since f is integrable, the value of $\int_a^b f$ must equal the limit of this Riemann sum as $n \to \infty$. That is, $\int_a^b f = F(b) - F(a)$.

Remark: It is possible for a function to be integrable, but have no antiderivative. But by Theorem 5.9, we know that such a function cannot be continuous. An example is the function

$$f(x) = \begin{cases} -1 & \text{if } -1 \le x < 0, \\ \\ 1 & \text{if } 0 \le x \le 1. \end{cases}$$

Being piecewise continuous, f is integrable. However, the FTC implies that f cannot be the derivative of another function F. For if $f = F'$, then $\int_0^x f = F(x) - F(0)$, so that

$$\begin{aligned} F(x) = F(0) + \int_0^x f = F(0) + &\begin{cases} \int_0^x (-1) & \text{if } -1 \le x < 0, \\ \\ \int_0^x 1 & \text{if } 0 \le x \le 1, \end{cases} \\ = F(0) + &\begin{cases} -1(x - 0) & \text{if } -1 \le x < 0, \\ \\ 1(x - 0) & \text{if } 0 \le x \le 1, \end{cases} \\ = F(0) + &\left|x\right|, \end{aligned}$$

which we know is not differentiable at $x = 0$, regardless of what $F(0)$ is.

Remark: It is also possible for an integrable function f to be discontinuous at a point but still have an antiderivative F. Consider $f = F'$, where

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Although f is discontinuous at 0, is still integrable on any finite interval.

Theorem 5.12 (FTC for Continuous Functions): *Let f be continuous on $[a, b]$ and let F be any antiderivative of f on $[a, b]$. Then*

$$\int_a^b f = F(b) - F(a).$$

Proof: This follows directly from Theorem 5.1 and the FTC.

Remark: The FTC says that a *definite integral* $\int_a^b f$ is equal to the value of any antiderivative F of f at b minus the value of the **same** function F at a. That is, $\int_a^b f = [F(x)]_a^b$, where the notation $[F(x)]_a^b$ or $F(x)|_a^b$ is shorthand for the difference $F(b) - F(a)$.

- Let $f(x) = x$. Then

$$\int_0^1 f = \left[\frac{x^2}{2} + c\right]_0^1 = \frac{1}{2} + c - (0 + c) = \frac{1}{2}.$$

Remark: We need a convenient notation for an antiderivative.

Definition: If an integrable function f has antiderivative F, we write $F = \int f$ and say F is the *indefinite integral* of f.

$$\int f = F \text{ means } f = F'.$$

For example,

$$\int x \, dx = \frac{x^2}{2} + C \qquad \text{means} \qquad x = \frac{d}{dx}\left(\frac{x^2}{2} + C\right).$$

Remark: Remember that the definite integral $\int_a^b f(x) \, dx$ is a number, whereas the indefinite integral $\int f(x) \, dx$ represents a *family* of functions that differ from each other by a constant.

- Since
$$\frac{d}{dx}\frac{x^{p+1}}{p+1} = x^p,$$
we know that
$$\int_a^b x^p\,dx = \left.\frac{x^{p+1}}{p+1}\right|_a^b = \frac{b^{p+1}-a^{p+1}}{p+1} \qquad \text{if } p \neq -1.$$

- Also,
$$\int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = -[\cos x]_0^\pi = -[-1-1] = 2.$$

- But
$$\int_0^{2\pi} \sin x\,dx = [-\cos x]_0^{2\pi} = [-1-(-1)] = 0.$$

- $$\int_0^1 e^x\,dx = [e^x]_0^1 = e-1.$$

- $$\int_1^2 \frac{1}{x}\,dx = [\log x]_1^2 = \log 2 - \log 1 = \log 2.$$

- Consider
$$F(x) = \log|x| = \begin{cases} \log x & x > 0, \\ \log(-x) & x < 0. \end{cases}$$
Then
$$F'(x) = \begin{cases} \dfrac{1}{x} & x > 0, \\ \dfrac{1}{-x}(-1) & x < 0 \end{cases}$$
$$= \frac{1}{x} \quad \text{for all } x \neq 0.$$
Therefore, we see that
$$\int \frac{1}{x}\,dx = \log|x| + C \qquad \left(\textbf{not } \frac{x^0}{0}\right),$$

where C is an arbitrary constant. Thus
$$\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & \text{if } n \neq -1, \\ \log|x| + C & \text{if } n = -1. \end{cases}$$

- $$\int_{-2}^{-1} \frac{1}{x}\, dx = [\log|x|]_{-2}^{-1} = \log 1 - \log 2 = -\log 2.$$

- Consider the inverse trigonometric function $y = \sin^{-1} x$ for $x \in [-1, 1]$. Recall that

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \text{ for } (-1, 1)$$

and

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \text{ for } x \in (-\infty, \infty).$$

These results yield two important antiderivatives:

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C$$

and

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C.$$

- The function

$$F(x) = \int_0^x \frac{1}{\cos t}\, dt$$

is differentiable for $x \in [0, \frac{\pi}{2})$.

We don't yet know F, but we do know its derivative. Thm 5.9 $\Rightarrow$

$$F'(x) = \frac{1}{\cos x}.$$

Furthermore, suppose

$$G(x) = \int_0^{x^2} \frac{1}{\cos t}\, dt = F(x^2)$$

for $x \in \left[0, \sqrt{\frac{\pi}{2}}\right)$. Then

$$G'(x) = F'(x^2)\, 2x = \frac{2x}{\cos(x^2)} \quad \text{by the Chain Rule.}$$

Problem 5.1: Suppose f is a continuous function and g and b are differentiable functions on $[a, b]$. Prove that

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))b'(x) - f(a(x))a'(x).$$

Let F be an antiderivative for f. Theorem FTC states that

$$\int_{a(x)}^{b(x)} f(t)\,dt = F(b(x)) - F(a(x)).$$

Hence, using the Chain Rule,

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = F'(b(x))b'(x) - F'(a(x))a'(x) = f(b(x))b'(x) - f(a(x))a'(x).$$

5.3 Substitution Rule

Q. What is $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$?

A. On differentiating $F(x) = -\log|\cos x| + C$, we see that $\int \tan x\,dx = F(x)$.

Q. Are there systematic ways of finding such antiderivatives?

A. Yes, the following theorem *Substitution Rule*) is often helpful.

Theorem 5.13 (Substitution Rule): *Suppose g' is continuous on $[a, b]$ and f is continuous on $g([a, b])$. Then*

$$\int_{x=a}^{x=b} f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = \int_{u=g(a)}^{u=g(b)} f(u)\,du.$$

Proof: Theorem 5.9 $\Rightarrow$ f has an antiderivative F:

$$F'(u) = f(u) \quad \text{for all } u \in g([a, b]).$$

Consider $H(x) = F(g(x))$. Then

$$\begin{aligned} H'(x) &= F'(g(x))g'(x) \\ &= f(g(x))g'(x), \end{aligned}$$

that is, H is an antiderivative of $(f \circ g)g'$. Letting $u = g(x)$, we may then write

$$\int f(g(x))g'(x)\,dx = H(x) = F(g(x)) = F(u) = \int f(u)\,du$$

and, using the FTC,

$$\int_a^b f(g(x))g'(x) = [H(x)]_a^b = [F(g(x))]_a^b = F(g(b)) - F(g(a)) = \int_{u=g(a)}^{u=g(b)} f(u)\,du.$$

- Suppose we wish to calculate $\int_0^1 (x^2+2)^{99}\, 2x\, dx$. One could expand out this polynomial and integrate term by term, but a much easier way to evaluate this integral is to make the substitution $u = g(x) = x^2+2$. To help us remember the factor $\frac{du}{dx} = g'(x) = 2x$ we formally write $du = g'(x)\, dx = 2x\, dx$,

$$\int_{x=0}^{x=1} (x^2+2)^{99} 2x\, dx = \int_{u=2}^{u=3} u^{99}\, du = \left.\frac{u^{100}}{100}\right|_2^3 = \frac{3^{100}-2^{100}}{100}.$$

- To compute $\int x\sqrt{x^2+1}\, dx$, it is helpful to substitute $u = x^2+1 \Rightarrow du = 2x\, dx$.

$$\int x\sqrt{x^2+1}\, dx = \int u^{1/2}\, \frac{du}{2} = \frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2} + C \qquad \longleftarrow \text{(don't leave in this form)}$$
$$= \frac{1}{3}(x^2+1)^{3/2} + C.$$

Check:

$$\frac{d}{dx}\left[\frac{1}{3}(x^2+1)^{3/2} + C\right] = \frac{1}{\cancel{3}}\frac{\cancel{3}}{\cancel{2}}(x^2+1)^{1/2}\, \cancel{2}\, x = x\sqrt{x^2+1}.$$

- The subsitution $u = 2x+1$ reduces the integral $\int_0^4 \sqrt{2x+1}\, dx$ to

$$\int_1^9 \sqrt{u}\, \frac{du}{2} = \left[\frac{1}{3}u^{3/2}\right]_1^9 = \frac{1}{3}\big(3^3 - 1\big) = \frac{26}{3}.$$

- The subsitution $u = \log x$ reduces the integral

$$\int_1^e \frac{\log x}{x}\, dx = \int_0^1 u\, du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}.$$

- The change of variables $u = e^t \Rightarrow du = e^t\, dt$ allows us to evaluate

$$\int \frac{e^t}{e^t+1}\, dt = \int \frac{du}{u+1} = \log|u+1| + C = \log(e^t+1) + C.$$

- The substitution $u = \frac{x}{a} \Rightarrow x = au \Rightarrow dx = adu$, where a is a constant, allows us to evaluate any integral of the form

$$\begin{aligned}
\int \frac{1}{x^2+a^2}\, dx &= \int \frac{1}{a^2\big(\frac{x^2}{a^2}+1\big)}\, dx \\
&= \frac{1}{a^2}\int \frac{1}{u^2+1}\, a\, du \\
&= \frac{1}{a}\int \frac{1}{u^2+1}\, du \\
&= \frac{1}{a}\tan^{-1} u + C \\
&= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.
\end{aligned}$$

Problem 5.2: Find

$$\int \cos t \sqrt{\sin t}\, dt.$$

$$= \frac{2}{3} \sin^{3/2} t + C.$$

Problem 5.3: Let α be a real number. Find

$$\int x^{-\alpha} e^{-\alpha x} \left(\frac{1}{x} + 1 \right) dx.$$

We use the substitution $u = \log x + x$ to rewrite

$$\int e^{-\alpha(\log x + x)} \left(\frac{1}{x} + 1 \right) dx = \int e^{-\alpha u}\, du = \begin{cases} \frac{-e^{-\alpha u}}{\alpha} + C & \text{if } \alpha \neq 0, \\ u + C & \text{if } \alpha = 0. \end{cases}$$
$$= \begin{cases} -\frac{x^{-\alpha} e^{-\alpha x}}{\alpha} + C & \text{if } \alpha \neq 0, \\ \log x + x + C & \text{if } \alpha = 0. \end{cases}$$

Problem 5.4: Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^{\pi/2} f(\sin x)\, dx = \int_0^{\pi/2} f(\cos x)\, dx.$$

This follows on using the substitution $u = \pi/2 - x$:

$$\int_0^{\pi/2} f(\sin x)\, dx = \int_0^{\pi/2} f\left(\cos\left(\frac{\pi}{2} - x \right) \right) dx = -\int_{\pi/2}^{0} f(\cos u)\, du = \int_0^{\pi/2} f(\cos u)\, du.$$

Problem 5.5:

(a) Show that any function $f : \mathbb{R} \to \mathbb{R}$ can be decomposed as a sum of an even function f_e and an odd function f_o. Hint: Construct explicit expressions for f_e and f_o in terms of $f(x)$ and $f(-x)$ and show that they are even and odd functions, respectively.

Let

$$f_e(x) = \frac{f(x) + f(-x)}{2},$$
$$f_o(x) = \frac{f(x) - f(-x)}{2},$$

Then $f(x) = f_e(x) + f_o(x)$.

(b) Show using Theorem 5.2 and an appropriate substitution that if f_e is an even integrable function on $[-a, a]$, then

$$\int_{-a}^{a} f_e = 2\int_0^a f_e.$$

$$\begin{aligned}\int_{-a}^{a} f_e(x)\,dx &= \int_{-a}^{0} f_e(x)\,dx + \int_0^a f_e(x)\,dx = -\int_a^0 f_e(-x)\,dx + \int_0^a f_e(x)\,dx \\ &= \int_0^a f_e(x)\,dx + \int_0^a f_e(x)\,dx = 2\int_0^a f_e.\end{aligned}$$

(c) Show that if f_o is an odd integrable function on $[-a, a]$ that

$$\int_{-a}^{a} f_o = 0.$$

$$\begin{aligned}\int_{-a}^{a} f_o(x)\,dx &= \int_{-a}^{0} f_o(x)\,dx + \int_0^a f_o(x)\,dx = -\int_a^0 f_o(-x)\,dx + \int_0^a f_o(x)\,dx \\ &= -\int_0^a f_o(x)\,dx + \int_0^a f_o(x)\,dx = 0.\end{aligned}$$

(d) Deduce that

$$\int_0^a f + \int_{-a}^{0} f = 2\int_0^a f_e$$

and

$$\int_0^a f - \int_{-a}^{0} f = 2\int_0^a f_o.$$

We find

$$\int_0^a f + \int_{-a}^{0} f = \int_0^a (f_e + f_o) + \int_{-a}^{0} (f_e + f_o) = \int_{-a}^{a} f_e + \int_{-a}^{a} f_o = 2\int_0^a f_e$$

and

$$\int_0^a f - \int_{-a}^{0} f = \int_0^a (f_e + f_o) - \int_{-a}^{0} (f_e + f_o) = \int_0^a f_e - \int_0^a f_e + \int_0^a f_o + \int_0^a f_o = 2\int_0^a f_o.$$

- Since the integrand is even, we can simplify

$$\int_{-2}^{2} (x^4 + 1)\,dx = 2\int_0^2 (x^4 + 1)\,dx = 2\left[\frac{x^5}{5} + x\right]_0^2 = 2\left(\frac{2^5}{5} + 2\right) = \frac{2^6}{5} + 4 = \frac{84}{5}.$$

- Since the integrand is odd, we can simplify

$$\int_{-\pi/2}^{\pi/2} \csc x\, dx = 0.$$

Problem 5.6: (a) Let f be an odd function with antiderivative F. Prove that F is an even function. Note: we do not assume that f is continuous or even integrable.

We are given that $f(-x) = -f(x)$ and $F'(x) = f(x)$. Hence

$$\frac{d}{dx}[F(x) - F(-x)] = f(x) + f(-x) = 0,$$

so that

$$F(x) - F(-x) = C$$

for some constant C. Evaluating this result at $x = 0$, we see that $C = 0$. Hence $F(x) = F(-x)$, that is, F is even.

(b) If f is an even function with antiderivative F, can one always find an antiderivative G of f that is odd? Are all antiderivatives of f odd? Prove or provide a counterexample for each of these statements.

We are given that $f(-x) = f(x)$ and $F'(x) = f(x)$. Hence

$$\frac{d}{dx}[F(x) + F(-x)] = f(x) - f(-x) = 0,$$

so that

$$F(x) + F(-x) = C,$$

where C is a constant. Let $G(x) = F(x) - C/2$. Then G is an antiderivative of f and $G(-x) = F(-x) - C/2 = -F(x) + C/2 = -G(x)$, so G is odd. However, not all antiderivatives of f are odd. Consider the even function $f(x) = 1$. The antiderivative $x + 1$ is not an odd function, although the antiderivative $G(x) = x$ is.

5.4 Numerical Approximation of Integrals

There are many continuous functions such as

$$\frac{e^x}{x}, \frac{\sin x}{x}, \text{ and } e^{x^2},$$

for which the antiderivative cannot be expressed in terms of the elementary functions introduced so far. For applications where one needs only the value of a definite integral, one possibility is to approximate the integral numerically.

To illustrate the numerical evaluation of definite integrals, it is helpful to consider an integral for which we know the exact answer, such as $\int_0^1 f\,dx$, where $f(x) = x^2$. For the partition $\{0, \frac{1}{2}, 1\}$ of $[0, 1]$ we can find a lower bound

$$L = 0\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right) = \frac{1}{8} = 0.125$$

and an upper bound

$$U = \frac{1}{4}\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) = \frac{5}{8} = 0.625$$

on the integral.

That is,

$$L \le \int_0^1 x^2\,dx \le U$$

but neither L nor U provides us with a very good approximation to the integral. Notice that the average of L and U, namely $(L+U)/2 = 3/8 = 0.375$, is much closer to the exact value $(1/3)$ of the definite integral and that since f is increasing, L is identical to the left Riemann sum $S_L = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$ and U is the right Riemann sum $S_R = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$. This suggests that it may be better to approximate the integral by using the *Trapezoidal Rule*

$$T_n \doteq \frac{S_L + S_R}{2} = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}(x_i - x_{i-1}),$$

Remark: For a uniform partition with fixed a, b, and f, T depends only on the number of subintervals.

Q. How accurately does T_n, for a uniform partition of $[a, b]$ into n subintervals, approximate $\int_a^b f$? How does the error depend on n?

A. First, we look at a special case of this question where there is only one subinterval.

Theorem 5.14 (Linear Interpolation Error): *Let f be a twice-differentiable function on $[0, h]$ satisfying $|f''(x)| \le M$ for all $x \in [0, h]$. Let*

$$L(x) = f(0) + \frac{f(h) - f(0)}{h}x.$$

Then

$$\int_0^h |L(x) - f(x)|\,dx \le \frac{Mh^3}{12}.$$

Proof: Let $x \in (0,h)$ and

$$\varphi(t) = L(t) - f(t) - Ct(t-h),$$

where C is chosen so that $\varphi(x) = 0$. Then

$$\varphi(0) = L(0) - f(0) = 0,$$

$$\varphi(h) = L(h) - f(h) = 0.$$

From Rolle's Theorem, we then know that there exists $x_1 \in (0,x)$ and $x_2 \in (x,h)$ such that

$$\varphi'(x_1) = \varphi'(x_2) = 0.$$

Again by Rolle's Theorem, we know that there exists $c \in (x_1, x_2)$ such that

$$0 = \varphi''(c) = -f''(c) - 2C,$$

noting that L is linear. Therefore $C = -f''(c)/2$ and since $\varphi(x) = 0$,

$$L(x) - f(x) = \frac{-1}{2} f''(c) x(x-h),$$

where $c \in (0,h)$ depends on x. That is, for every $x \in [0,h]$ we have

$$|L(x) - f(x)| \le \frac{1}{2} Mx(h-x),$$

so

$$\int_0^h |L(x) - f(x)|\, dx \le \frac{M}{2} \int_0^h x(h-x)\, dx = \frac{M}{2} \left[\frac{x^2 h}{2} - \frac{x^3}{3} \right]_0^h = \frac{Mh^3}{12}.$$

Theorem 5.15 (Trapezoidal Rule Error): *Consider a uniform partition of $[a,b]$ into n subintervals of width $h = (b-a)/n$, and f be a twice-differentiable function on $[a,b]$ satisfying $|f''(x)| \le M$ for all $x \in [a,b]$. Then the error $E_n^T \doteq T_n - \int_a^b f$ of the uniform Trapezoidal Rule*

$$T_n = h \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}$$

satisfies

$$\left| E_n^T \right| \le \frac{nMh^3}{12} = \frac{M(b-a)^3}{12n^2}.$$

Proof: We need to add up the contribution to the error from each subinterval. If we temporarily relabel the endpoints of each subinterval 0 and h, we may apply Theorem 5.14 to obtain a contribution, $\left| \int_0^h L - \int_0^h f \right| \le \int_0^h |L - f| \le Mh^3/12$, from each of the n subintervals.

Remark: We can rewrite the Trapezoidal Rule as

$$T_n = \frac{h}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)].$$

- We can use the Trapezoidal Rule to approximate $\int_1^2 \frac{1}{x}\,dx$ with $n = 5$ subintervals of width $h = 1/5$:

$$\int_1^2 \frac{1}{x}\,dx \approx T_n = \frac{1}{10}\left[\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right] \approx 0.6956.$$

  The exact value of the integral is $\log 2 = 0.6931\ldots$.

Remark: Typically, a more accurate method than the Trapezoidal Rule is the *Midpoint Rule*

$$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)(x_i - x_{i-1}),$$

which has the additional advantage of requiring one less function evaluation.

Problem 5.7: Show that the Midpoint Rule has an error $E_n^M \doteq M_n - \int_a^b f$ satisfying

$$\left|E_n^M\right| \le \frac{M(b-a)^3}{24n^2}.$$

Notice that this bound is a factor of 2 smaller than the error bound for the Trapezoidal Rule.

- Let us use the Midpoint Rule to approximate $\int_1^2 \frac{1}{x}\,dx$ with $n = 5$ subintervals of width $h = 1/5$:

$$\int_1^2 \frac{1}{x}\,dx \approx M_n = \frac{1}{5}\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right] \approx 0.6919,$$

  which is indeed closer than T_n to the exact value of $\log 2$ (by roughly a factor of 2).

Remark: Even better are the higher-order methods, such as *Simpson's Rule*, which fits parabolas rather than line segments to the data values $f(x_0), f(x_1), \ldots, f(x_n)$, where n is even. This approximation is given by

$$S_n = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)],$$

with an error $E_n^S \doteq S_n - \int_a^b f$ satisfying

$$\left|E_n^S\right| \le \frac{K(b-a)^5}{180n^4} \quad \text{if } \left|f^{(4)}(x)\right| \le K \text{ for all } x \in [a,b].$$

Problem 5.8: Consider the function $f(x) = 1/(1+x^2)$ on $[0,1]$. Let P be a uniform partition on $[0,1]$ with 2 subintervals of equal width.

(a) Compute the left Riemann sum S_L.
Since the partition is uniform,

$$S_L = \frac{1}{2}\left(\frac{4}{5} + \frac{1}{2}\right) = \frac{13}{20}.$$

(b) Compute the right Riemann sum S_R.

$$S_R = \frac{1}{2}\left(1 + \frac{4}{5}\right) = \frac{9}{10}.$$

(c) Use your results in part (a) and (b) to find lower and upper bounds for π.
We see that

$$\frac{13}{20} = S_L \le \frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2}\,dx \le S_R = \frac{9}{10}.$$

Thus $\dfrac{13}{5} \le \pi \le \dfrac{18}{5}$.

(d) Use the Trapezoidal Rule to find a numerical estimate for π.
We find that π is approximately

$$4\left(\frac{1}{2}\right)\left(\frac{1+\frac{4}{5}}{2} + \frac{\frac{4}{5}+\frac{1}{2}}{2}\right) = 2\left(\frac{9}{10} + \frac{13}{20}\right) = \frac{31}{10}.$$

(e) Obtain a better rational estimate for π by using the Midpoint Rule.
We find that π is approximately

$$4\left(\frac{1}{2}\right)\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right) = 2\left(\frac{16}{17} + \frac{16}{25}\right) = 32\left(\frac{1}{17} + \frac{1}{25}\right) = 32\left(\frac{42}{425}\right) = \frac{1344}{425}.$$

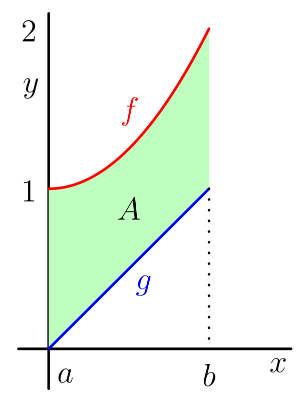
Chapter 6

Areas and Volumes

6.1 Areas between curves

The area A between two continuous functions $y = f(x)$ and $y = g(x)$ on $[a, b]$, where $f(x) \geq g(x)$, is given by the difference of the respective areas between these functions and the x axis:

$$A = \int_a^b f(x) - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx.$$



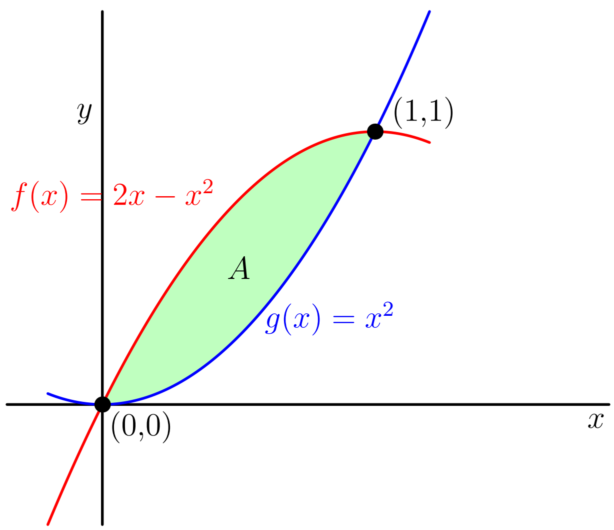
- Find the area bounded by $f(x) = x^2 + 1$ and $g(x) = x$ between $x = 0$ and $x = 1$.

$$A = \int_0^1 [f(x) - g(x)]\, dx = \int_0^1 [x^2 + 1 - x]\, dx$$
$$= \left[\frac{x^3}{3} + x - \frac{x^2}{2}\right]_0^1 = \frac{1}{3} + 1 - \frac{1}{2} = \frac{5}{6}.$$

Sometimes we are not given a and b, but we can determine them from the points of intersections of the two curves.

- Find the area enclosed by the curves $f(x) = 2x - x^2$ and $g(x) = x^2$. Here a and b are determined by the points of intersection of $f(x)$ and $g(x)$,

$$\begin{aligned} f(x) &= g(x) \\ 2x - x^2 &= x^2 \\ &\Rightarrow 2x = 2x^2 \Rightarrow 0 = 2x^2 - 2x = 2x(x-1) \\ &\Rightarrow x = 0 \text{ or } x = 1. \end{aligned}$$



Thus

$$\begin{aligned} A &= \int_0^1 [f(x) - g(x)]\, dx = \int_0^1 (2x - x^2 - x^2)\, dx = \int_0^1 (2x - 2x^2)\, dx \\ &= 2\int_0^1 (x - x^2)\, dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}. \end{aligned}$$

Q. What happens when $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values?

A. We simply take the absolute value of the integrand before integrating. That is, the general formula for the area A of the region bounded by two continuous functions f and g and the vertical lines $x = a$ and $x = b$ is

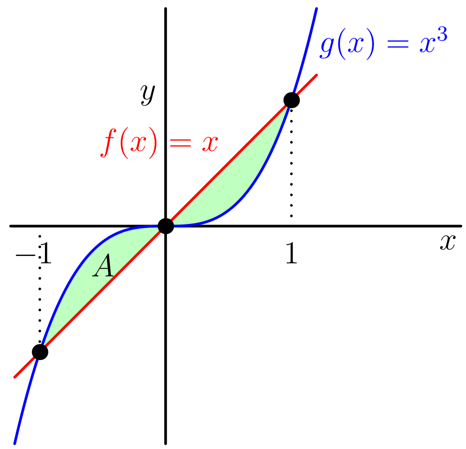
$$A = \int_a^b |f(x) - g(x)|\, dx,$$

where

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geq g(x), \\ g(x) - f(x) & \text{when } f(x) < g(x). \end{cases}$$

For continuous functions f and g, the regions where $f(x) > g(x)$ and $f(x) < g(x)$ are separated by the points where $f(x) = g(x)$.

- To find the area of the region bounded by $f(x) = x$ and $g(x) = x^3$, we first solve for the intersection points:



$$\begin{aligned}
&f(x) = g(x)\\
&\Rightarrow x = x^3\\
&\Rightarrow 0 = x^3 - x = x(x^2 - 1) = x(x-1)(x+1)\\
&\Rightarrow x = -1, 0, 1.
\end{aligned}$$

On $[-1, 0]$ we see that $f(x) \leq g(x)$ and on $[0, 1]$ we see that $f(x) \geq g(x)$. Thus

$$\begin{aligned}
A &= \int_{-1}^{1} |f(x) - g(x)|\, dx = \int_{-1}^{0} [g(x) - f(x)]\, dx + \int_{0}^{1} [f(x) - g(x)]\, dx\\
&= \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} (x - x^3)\, dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}\\
&= 0 - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) - 0 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\end{aligned}$$

Q. In the above example, what would happen if we tried to compute

$$\int_{-1}^{1} [f(x) - g(x)]\, dx$$

without first taking the absolute value of the integrand?

A. We would find

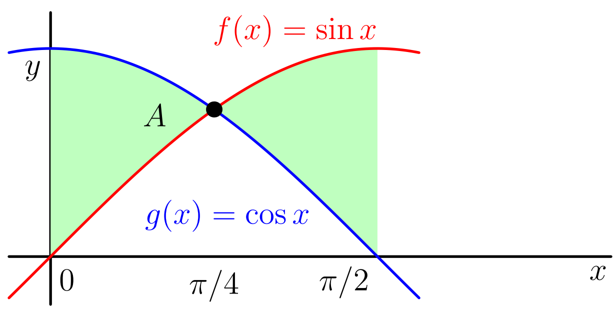
$$\int_{-1}^{1} [x - x^3]\, dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{1} = \left(\frac{1}{2} - \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{4}\right) = 0.$$

In general, whenever $f(x) - g(x)$ is an odd function we will find

$$\int_{-1}^{1} [f(x) - g(x)]\, dx = \int_{-1}^{0} [f(x) - g(x)]\, dx + \int_{0}^{1} [f(x) - g(x)]\, dx = 0,$$

because the two contributions are of opposite sign, even though the geometric area of the region bounded by the two functions will (normally) be positive.

- Find the area bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.



The intersection points occur when

$$\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}.$$

We need to split the integration interval $[0, \pi/2]$ into two parts:

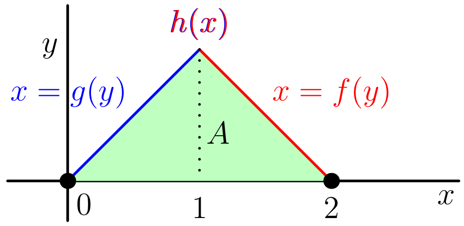
$$\begin{aligned} A = \int_0^{\frac{\pi}{2}} |\cos x - \sin x|\ dx &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\, dx \\ &= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 - 0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\sqrt{2} - 2. \end{aligned}$$

- If the function is defined piecewise, we can integrate it in two pieces. For example, to find the area bounded by $y = h(x)$ and $y = 0$ between $x = 0$ and $x = 2$, where

$$h(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 2 - x & \text{if } 1 < x \le 2, \end{cases}$$

we would perform the integral over $[0, 1]$ and $[1, 2]$ separately:

$$\int_0^2 |h(x)|\ dx = \int_0^1 x\, dx + \int_1^2 (2 - x)\, dx = \left[\frac{x^2}{2}\right]_0^1 + \left[-\frac{(2-x)^2}{2}\right]_1^2 = \frac{1}{2} + \frac{1}{2} = 1.$$



However, it is even easier to determine the area of this region by finding the area between the inverse functions $x = f(y) = 2 - y$ and $x = g(y) = y$, where y varies from 0 to 1:

$$\int_0^1 |f(y) - g(y)|\, dy = \int_0^1 |(2-y) - y|\, dy = 2\int_0^1 (1-y)\, dy = 2\left[-\frac{(1-y)^2}{2}\right]_0^1 = 1.$$

6.2 Volumes by Cross Sections

Single-variable calculus can sometimes be used to calculate more than just lengths and areas. If an expression for the cross-sectional area of an object is known, it is possible to compute its volume by the *method of cross sections* (also known as the method of slabs or slices).

For example, we can of course easily compute the volume of a loaf of bread, where each slice has same shape and size, using the definition

$$\text{volume} = \text{area} \times \text{length}.$$

But what if the slices of bread don't all have the same size (or even the same shape)? Maybe we have a conical loaf!

Q. What is the volume of such a strange loaf of bread?

A. Slice up the loaf and sum up the area (height $\times$ width) $\times$ thickness $(x_i - x_{i-1})$ of each slice to form the Riemann sum

$$\sum_{i=1}^{n} A(x_i^*)(x_i - x_{i-1}),$$

where $A(x)$ is the area of a cross section at x obtaining by slicing perpendicular to the x axis and x_i^* is a point in $[x_{i-1}, x_i]$ Assuming that $A(x)$ is integrable on $[a, b]$, we can take the limit as $n \to \infty$ to find the volume:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)(x_i - x_{i-1}) = \int_a^b A(x)\, dx.$$

- For a conical loaf of bread of length L, the middle slice, at $x = L/2$ has 1/2 the height and 1/2 the width of the largest slice, so its area is 1/4 the area of the largest slice. If we put the apex of the cone at $x = 0$ and the largest slice, with area A, at $x = L$, we see by similar triangles that the slice located at x has area $A(x) = (x/L)^2 A$. Thus

$$\begin{aligned} V &= \int_0^L A(x)\,dx = \int_0^L \frac{x^2}{L^2} A\,dx \\ &= \frac{A}{L^2}\int_0^L x^2\,dx = \frac{A}{L^2}\left[\frac{x^3}{3}\right]_0^L = \frac{A}{L^2}\frac{L^3}{3} = \frac{1}{3}AL. \end{aligned}$$

  We have thus established the formula:

$$V_{\text{cone}} = \frac{1}{3}\,\text{base area} \times \text{length}.$$

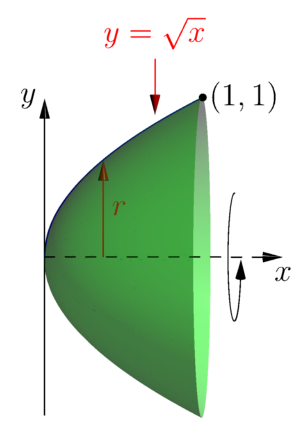
- We can compute the volume enclosed by a sphere of radius R, described by the equation $x^2+y^2+z^2 = R^2$, by partitioning the x axis. This produces circular cross sections of radius $r = r(x) > 0$. The value of r is the maximum possible value of y, which occurs when $z = 0$:

$$x^2 + y^2 = R^2 \Rightarrow y = \pm\sqrt{R^2 - x^2}.$$

  That is, $r(x) = \sqrt{R^2 - x^2}$, so that $A(x) = \pi r^2 = \pi(R^2 - x^2)$. Thus

$$V = \int_{-R}^{R} \pi(R^2 - x^2)\,dx = 2\pi\int_0^R (R^2 - x^2)\,dx = 2\pi\left[R^2 x - \frac{x^3}{3}\right]_0^R = \frac{4}{3}\pi R^3.$$

- Find the volume of the solid obtained by rotating the area bounded by the curves $y = \sqrt{x}$ and $y = 0$ from 0 to 1 about the x axis.



Since the radius of revolution is given by $r(x) = y = \sqrt{x}$, the cross-sectional area is given by

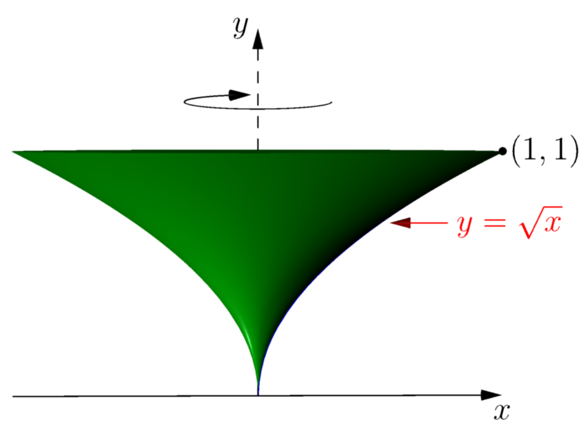
$$A(x) = \pi r^2 = \pi\left(\sqrt{x}\right)^2 = \pi x.$$

Thus

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi\left[\frac{x^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

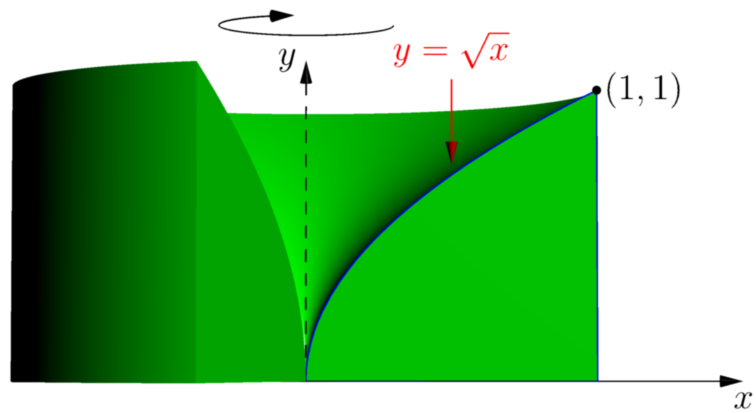
- We could instead compute the volume of the funnel-shaped object generated by rotating the region bounded by $y = \sqrt{x}$, $x = 0$, and $y = 1$ about the y axis. For the method of cross sections, we always slice the rotation axis (in this case the y axis) and express everything else in terms of the corresponding variable (y). We see that the radius of each circular cross section is $r(y) = x = y^2$, so that the cross-sectional area is $A(y) = \pi r^2 = \pi y^4$. The resulting volume of revolution is thus

$$V = \int_0^1 A(y)\,dy = \pi\int_0^1 y^4\,dy = \pi\left[\frac{y^5}{5}\right]_0^1 = \frac{\pi}{5}.$$



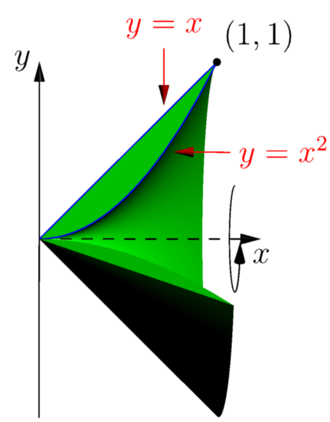
- We could also rotate the region bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$ about the y axis. If we slice the y axis, each cross section is just an annulus of outer radius $r_{\text{out}}(y) = 1$ and inner radius $r_{\text{in}}(y) = x = y^2$, with area $A(y) = \pi {r_{\text{out}}}^2 - \pi {r_{\text{in}}}^2 = \pi(1 - y^4)$. The volume of the resulting object is then

$$V = \int_0^1 A(y)\,dy = \pi\int_0^1 (1 - y^4)\,dy = \pi\left[y - \frac{y^5}{5}\right]_0^1 = \frac{4\pi}{5}.$$



Problem 6.1: Explain why the volumes calculated in the previous two examples add up to π, the volume of a cylinder of unit radius and unit height.

- If we rotate the region bounded by $f(x) = x$ and $g(x) = x^2$ between $x = 0$ and $x = 1$ about the x axis,



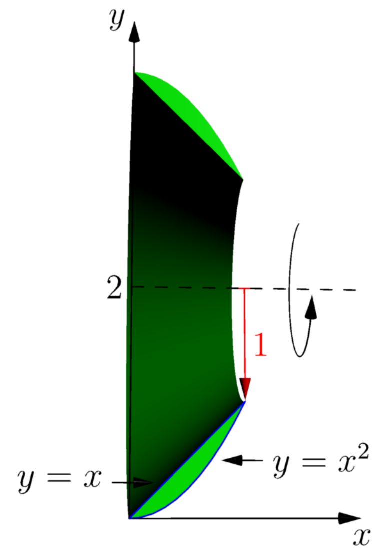
we need to find the area of the annular region with outer radius $r_{\text{out}}(x) = f(x) = x$ and inner radius $r_{\text{in}}(x) = g(x) = x^2$:

$$A(x) = \pi {r_{\text{out}}}^2 - \pi {r_{\text{in}}}^2 = \pi\left[x^2 - (x^2)^2\right] = \pi(x^2 - x^4).$$

Thus

$$V = \int_0^1 [\pi(x^2 - x^4)]\, dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}.$$

- We could also rotate the same area about the line $y = 2$ instead of $y = 0$.



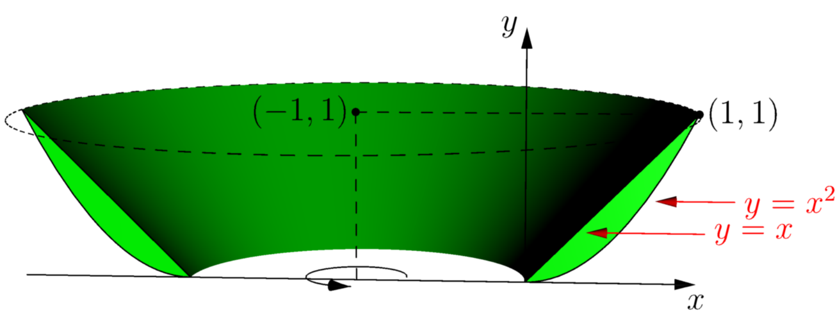
Now

$$A(x) = \pi {r_{\text{out}}}^2 - \pi {r_{\text{in}}}^2 = \pi \left(2 - x^2\right)^2 - \pi(2 - x)^2 = \pi(x^4 - 5x^2 + 4x).$$

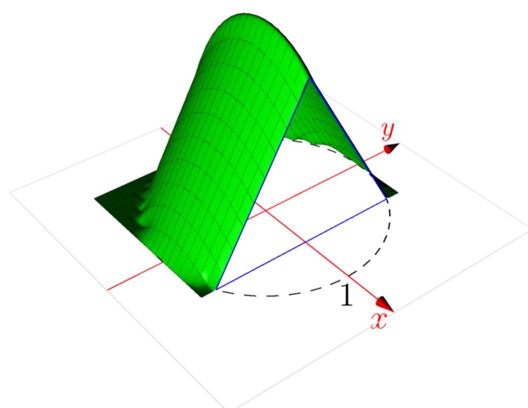
The generated volume is then

$$\begin{aligned} V &= \int_0^1 \mathrm{A}(x)\, dx = \pi \int_0^1 (x^4 - 5x^2 + 4x)\, dx = \pi \left[\frac{x^5}{5} - \frac{5x^3}{3} + \frac{4x^2}{2} \right]_0^1 \\ &= \pi \left(\frac{1}{5} - \frac{5}{3} + 2 \right) = \frac{8\pi}{15}. \end{aligned}$$

Problem 6.2: Find the volume generated by rotating the region bounded by $f(x) = x$ and $g(x) = x^2$ between $x = 0$ and $x = 1$ about the line $x = -1$.



- Consider the three-dimensional object formed by erecting an equilateral triangle, with altitude perpendicular to the xy plane, on every chord $x = \text{const}$ of the circle $x^2 + y^2 = 1$.



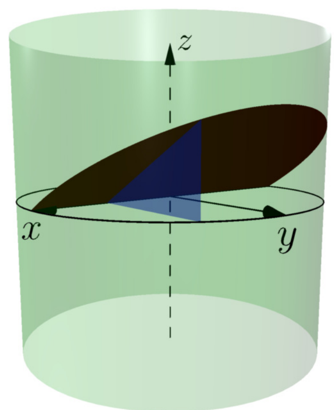
To find the volume of this object, we only need to find the cross-sectional area $A(x)$ of each equilateral triangle obtained by slicing the object along the planes $x = \text{const}$. The length of the base of this triangle, which has endpoints $(x, -y)$ and (x, y), where $x^2 + y^2 = 1$, is $2y$. Pythagoras' Theorem tell us that the altitude h of this equilateral triangle is $\sqrt{(2y)^2 - y^2} = \sqrt{3}y$. Hence

$$A(x) = \frac{1}{2}(2y)h = \sqrt{3}y^2 = \sqrt{3}(1 - x^2).$$

The volume of the object is then easily computed:

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}(1-x^2)\,dx = 2\int_{0}^{1} \sqrt{3}(1-x^2)\,dx = 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4\sqrt{3}}{3}.$$

- Consider the volume of one of the two wedge-shaped regions bounded by the cylinder $x^2 + y^2 = 16$ and the plane containing the x axis and oriented at an angle of $30^\circ = \pi/6$ to the xy plane. If we slice this object in the x direction, we obtain triangular cross sections with base length y and altitude $y\tan(\pi/6) = y/\sqrt{3}$. Thus



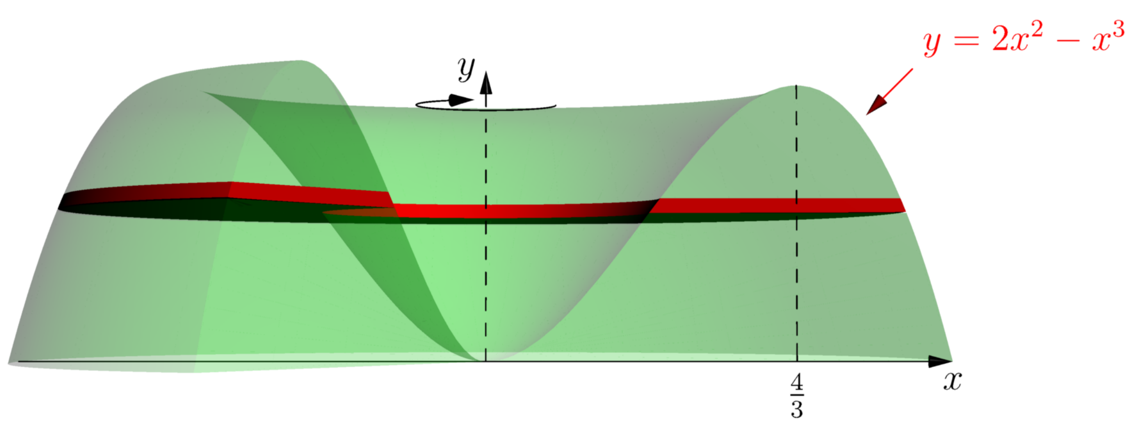
$$A(x) = \frac{1}{2}y\left(\frac{y}{\sqrt{3}}\right) = \frac{16-x^2}{2\sqrt{3}},$$

so that

$$V = \int_{-4}^{4} \frac{16-x^2}{2\sqrt{3}}\,dx = 2\int_{0}^{4} \frac{16-x^2}{2\sqrt{3}}\,dx = \frac{1}{\sqrt{3}}\left[16x - \frac{x^3}{3}\right]_0^4 = \frac{128}{3\sqrt{3}}.$$

6.3 Volumes by Shells

Suppose we wish to rotate the area under the curve $y = f(x) = 2x^2 - x^3$ about the y axis. The method of cross sections requires that we slice the y axis and express all quantities as functions of y. Finding the radii r_{in} and r_{out} amounts to inverting the equation $y = 2x^2 - x^3$ to find two distinct values of x for every y within the limits of integration.



In general, performing this kind of inversion can be a difficult problem. In this example, f has roots only at $x = 0$ and $x = 2$ and $f(x) > 0$ on $(0, 2)$. We can easily see that the maximum value of f must occur at $4/3$ since $f'(x) = 4x - 3x^2 = x(4-3x)$.

However, it is much more difficult (although in this case not impossible) to find for each y the two values r_{in} and r_{out} such that $f(r_{\text{in}}) = f(r_{\text{out}}) = y$.

For such cases, there is an easier alternative, the *method of cylindrical shells*, where one computes the volume using Riemann sums of volumes of cylindrical shells:

1. Partition an axis that is perpendicular to the rotation axis. Then compute the volume of the cylindrical shells generated by revolving the area within each subinterval around the rotation axis.

2. To find the total volume, add up the volumes of all shells and take the limit as the width of the subintervals goes to 0.

We readily see that the volume of a cylindrical shell of inner radius r_1 and outer radius r_2 and height h is given by

$$\pi r_2^2 h - \pi r_1^2 h = \pi h(r_2^2 - r_1^2) = \pi h(r_2 + r_1)(r_2 - r_1) = (2\pi r)h\Delta r,$$

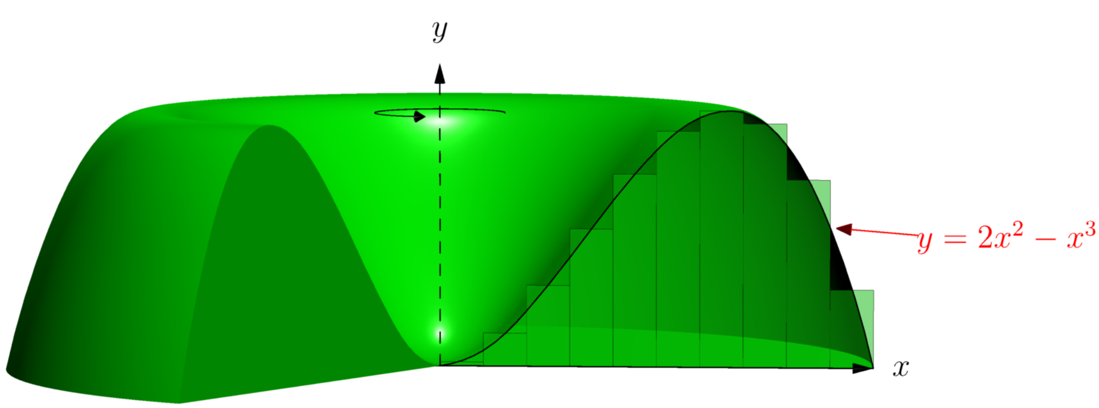
where $r \doteq (r_1 + r_2)/2$ is the mean radius and $\Delta r \doteq r_2 - r_1$ is the width of the subinterval. (We use the symbol $\doteq$ to emphasize a definition, although the notation $:=$ is more common.)

When the area under the curve $y = f(x)$ is rotated about the y axis, we can use a uniform partition to form a Riemann sum for the volume by approximating the height h of the cylindrical shell on each subinterval $[x_{i-1}, x_i]$ by the value of the function f at the mean radius $x_i^* = (x_{i-1} + x_i)/2$. Then

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*)(x_i - x_{i-1}) = 2\pi \int_a^b x f(x)\, dx.$$

Note here that $0 \le a \le b$.

- We can compute the volume formed by rotating the region under $y = f(x) = 2x^2 - x^3$ about the y axis very easily now, using the method of cylindrical shells:



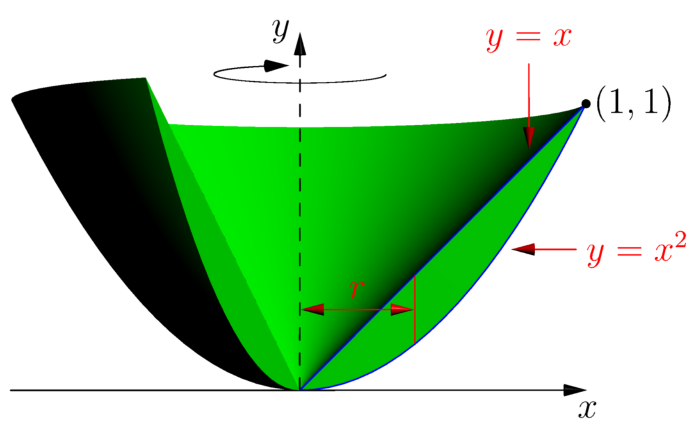
$$\begin{aligned} V &= \int_0^2 2\pi x(2x^2 - x^3)\,dx = 2\pi \int_0^2 (2x^3 - x^4)\,dx \\ &= 2\pi \left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 = 2\pi\left(8 - \frac{32}{5}\right) = \frac{16\pi}{5}. \end{aligned}$$

In general, the volume of the object generated by rotating the region bounded by the functions $f(x)$ and $g(x)$ between $x = a$ and $x = b$ about the y axis is given by

$$V = \int_a^b \underbrace{2\pi x}_{\text{circumference}} \ \underbrace{|f(x) - g(x)|}_{\text{height}} \ \underbrace{dx}_{\text{width}},$$

where we have given a geometric interpretation for each factor.

- When the region bounded by the functions $f(x) = x$ and $g(x) = x^2$ and the lines $x = 0$ and $x = 1$ is rotated about the y axis,



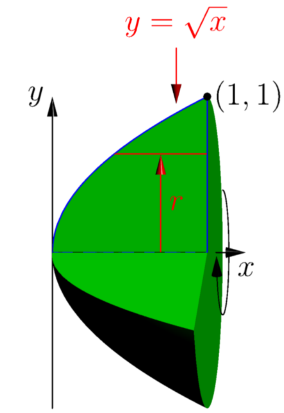
the volume generated is

$$V = 2\pi \int_0^1 x(x - x^2)\,dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}.$$

Alternatively, we could have obtained the same answer with the method of cross sections by slicing the y axis. Since $r_{\text{out}} = \sqrt{y}$ and $r_{\text{in}} = y$, we see that

$$V = \pi \int_0^1 ({r_{\text{out}}}^2 - {r_{\text{in}}}^2)\,dy = \pi \int_0^1 \left[(\sqrt{y})^2 - y^2\right] dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{6}.$$

- We can of course also rotate a region about the x axis. The region bounded by $y = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 1$

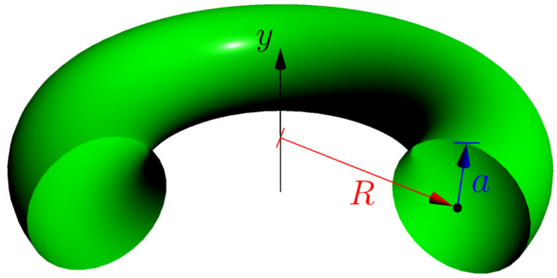


would generate the volume

$$V = \int \underbrace{2\pi y}_{\text{circumference}} \underbrace{(1-x)}_{\text{height}} \underbrace{dy}_{\text{width}} = 2\pi \int_0^1 y(1-y^2)\, dy$$
$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2},$$

in agreement with the result we previously obtained using the method of cross sections.

Problem 6.3: The region bounded by the curve $(x-R)^2 + y^2 = a^2$ is rotated about the y axis, where $R > a$ to obtain a torus of minor radius a and major radius R.



(a) Use the method of washers to find the volume of the torus.

Slice the torus in the y direction, creating washers of outer radius $R + \sqrt{a^2 - y^2}$ and inner radius $R - \sqrt{a^2 - y^2}$. The area of each washer is

$$A(y) = \pi\left[\left(R + \sqrt{a^2 - y^2}\right)^2 - \left(R - \sqrt{a^2 - y^2}\right)^2\right] = 4\pi R\sqrt{a^2 - y^2},$$

so that the volume V of the torus is given by

$$V = 4\pi R \int_{-a}^{a} \sqrt{a^2 - y^2}\, dy.$$

We recognize that $\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy$ is just half the area of a circle of radius a.

$$V = 4\pi R \frac{\pi a^2}{2} = 2\pi^2 R a^2.$$

(b) Use the method of shells to find the volume of the torus.

Slice the torus in the x direction, creating cylindrical shells of radius x and height $2\sqrt{a^2 - (x - R)^2}$.

The volume V of the object is thus, on letting $u = x - R$,

$$V = 4\pi \int_{R-a}^{R+a} x\sqrt{a^2 - (x - R)^2}\, dx = 4\pi \int_{-a}^{+a} (u + R)\sqrt{a^2 - u^2}\, du = 4\pi R \frac{\pi a^2}{2} = 2\pi^2 R a^2,$$

on exploiting the fact that $u\sqrt{a^2 - u^2}$ is an odd function of u.

Chapter 7

Techniques of Integration

7.1 Integration by parts

Recall that the substitution rule is really just an integral version of the chain rule. Another important and frequently used rule in differential calculus is the product rule.

Q. Does the product rule also have an integral version?

A. Yes, it is called integration by parts, as illustrated in Table 7.1.

$\dfrac{d}{dx}$	$\int dx$
Chain rule	Substitution rule
Product rule	Integration by parts

Table 7.1: Techniques of Integration.

Theorem 7.1 (*integration by parts*): *Suppose f' and g' are continuous functions. Then*

$$\int fg' = fg - \int f'g.$$

Proof: Then $(fg)' = f'g + fg'$, so fg is an antiderivative of $f'g + fg'$. That is,

$$\int (f'g + fg') = fg \qquad \text{to with a constant.}$$

In other words,

$$\boxed{\int f g' \, dx = f g - \int f' g \, dx.}$$

Remark: Letting $u = f(x)$, so that $du = f'(x)\,dx$, and $v = g(x)$, so that $dv = g'(x)\,dx$, we may rewrite the integration by parts formula as

$$\int u\,dv = uv - \int v\,du.$$

Remark: For definite integrals we have, by the Fundamental Theorem of Calculus,

$$\int_a^b f g' \, dx = \Big[f g \Big]_a^b - \int_a^b f' g \, dx.$$

- We can integrate $\int x \sin x \, dx$ using integration by parts:

$$\int \underbrace{x}_{f} \underbrace{\sin x}_{g'} \, dx = \underbrace{x}_{f} \underbrace{(-\cos x)}_{g} - \int \underbrace{1}_{f'} \underbrace{(-\cos x)}_{g} \, dx$$

$$\therefore \int x \sin x \, dx = -x \cos x + \sin x + C$$

Try to pick f so that f' is simple and g' has a known antiderivative. If instead we had picked

$$f = \sin x \quad (\Rightarrow f' = \cos x)$$

and

$$g' = x \quad \left(\Rightarrow g = \frac{x^2}{2} \right)$$

then the integration by parts formula leads to an even more complicated integral:

$$\int \sin x \; x \, dx = \sin x \left(\frac{x^2}{2} \right) - \int \cos x \left(\frac{x^2}{2} \right) = \ldots.$$

So this choice of f and g' was not fruitful.

- Noting that
$$\int \log x\,dx = \int 1 \cdot \log x\,dx,$$
we might be tempted to try integration by parts, setting $f = 1$ and $g' = \log x$:
$$\begin{aligned}\int \log x\,dx &= 1\int \log x\,dx - \int 0\left[\int \log x\,dx\right]dx \\ &= \int \log x\,dx.\end{aligned}$$
This doesn't help! Instead, we could try $f = \log x$ and $g' = 1$:
$$\begin{aligned}\int \underbrace{\log x}_{f} \cdot \underbrace{1}_{g'}\,dx &= \underbrace{\log x}_{f} \cdot \underbrace{x}_{g} - \int \underbrace{\frac{1}{x}}_{f'} \underbrace{x}_{g}\,dx \\ &= x\log x - x + C.\end{aligned}$$

- Similarly we can integrate $\tan^{-1}$ by parts and use the substitution $u = x^2$ to find
$$\begin{aligned}\int_0^1 \underbrace{\tan^{-1} x}_{f} \cdot \underbrace{1}_{g'}\,dx &= \Big[x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= 1\tan^{-1} 1 - 0 - \int_{0^2}^{1^2} \frac{1}{1+u}\frac{du}{2} \\ &= \frac{\pi}{4} - \frac{1}{2}[\log|1+u|]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2}\log 2.\end{aligned}$$

- In order to find
$$\int \underbrace{x^2}_{f} \underbrace{e^x}_{g'}\,dx = x^2e^x - \int 2xe^x\,dx,$$
we need to know $\int 2xe^x\,dx$. But that integral is just twice $\int xe^x\,dx$, which we can find by applying integration by parts again:
$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$
Thus
$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - 2(xe^x - e^x + C) \\ &= x^2e^x - 2xe^x + 2e^x + C_2, \qquad \text{where } C_2 = -2C.\end{aligned}$$

- We can even find integrals of the form

$$I = \int \underbrace{\sin x}_{f} \underbrace{e^x}_{g'} \, dx = \sin x \, e^x - \int \cos x \, e^x \, dx.$$

What is $\int \cos x \, e^x \, dx$?

$$\begin{aligned} \int \underbrace{\cos x}_{f} \underbrace{e^x}_{g'} \, dx &= \cos x \, e^x - \int (-\sin x) \, e^x \, dx \\ &= \cos x \, e^x + I. \end{aligned}$$

Thus $I = \sin x \, e^x - (\cos x \, e^x + I)$, from which we find $I = \frac{1}{2} \sin x \, e^x - \frac{1}{2} \cos x \, e^x + C$.

- For nonzero real numbers a and b find

$$\begin{aligned} I &= \int e^{ax} \cos bx \, dx, \\ J &= \int e^{ax} \sin bx \, dx. \end{aligned}$$

On integrating by parts, we obtain

$$\begin{aligned} I &= \int \underbrace{e^{ax}}_{g'} \underbrace{\cos bx}_{f} \, dx = \frac{1}{a} \, e^{ax} \cos bx + \frac{b}{a} \underbrace{\int e^{ax} \sin bx \, dx}_{J}, \\ J &= \int e^{ax} \sin bx \, dx = \frac{1}{a} \, e^{ax} \sin bx - \frac{b}{a} \underbrace{\int e^{ax} \cos bx \, dx}_{I}. \end{aligned}$$

We thus need to solve the system of equations

$$\begin{aligned} I &= \frac{1}{a} \, e^{ax} \cos bx + \frac{b}{a} \, J, \\ J &= \frac{1}{a} \, e^{ax} \sin bx - \frac{b}{a} \, I. \end{aligned}$$

$$\begin{aligned} \Rightarrow I &= \frac{1}{a} \, e^{ax} \cos bx + \frac{b}{a^2} \, e^{ax} \sin bx - \frac{b^2}{a^2} \, I \\ \left(1 + \frac{b^2}{a^2}\right) I &= \left(\frac{1}{a} \cos bx + \frac{b}{a^2} \sin bx\right) e^{ax}. \end{aligned}$$

$$\begin{aligned} \Rightarrow I &= \frac{a \cos bx + b \sin bx}{a^2 + b^2} \, e^{ax} + C_1, \\ J &= \frac{a \sin bx - b \cos bx}{a^2 + b^2} \, e^{ax} + C_2. \end{aligned}$$

- We now compute, for $n \geq 2$,

$$\begin{aligned} I_n &= \int \sin^n x \, dx \\ &= \int \underbrace{\sin^{n-1} x}_{f} \underbrace{\sin x}_{g'} \, dx \\ &= \sin^{n-1} x(-\cos x) - \int \underbrace{(n-1)\sin^{n-2} x(\cos x)}_{f'}(-\cos x) \, dx \end{aligned}$$

Now

$$\int \sin^{n-2} x \underbrace{\cos^2 x}_{1-\sin^2 x} \, dx = \int (\sin^{n-2} x - \sin^n x) \, dx = I_{n-2} - I_n.$$

Thus

$$\begin{aligned} I_n &= -\sin^{n-1} x \cos x + (n-1)(I_{n-2} - I_n) \\ \Rightarrow I_n &= -\sin^{n-1} x \cos x + (n-1)I_{n-2} - nI_n + I_n \\ \Rightarrow nI_n &= -\sin^{n-1} x \cos x + (n-1)I_{n-2}. \end{aligned}$$

That is,

$$\boxed{\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \ \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.}$$

This is known as a *reduction formula.*

- For $n = 2$, the reduction formula states that

$$\begin{aligned} \int \sin^2 x \, dx &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int 1 \, dx \\ &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} x + C. \end{aligned}$$

Alternatively, one can evaluate this integral using trigonometric identities:

$$\begin{aligned} \int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx &= \frac{1}{2} x - \frac{1}{4} \sin 2x + C \\ &= \frac{1}{2} x - \frac{1}{2} \sin x \cos x + C. \end{aligned}$$

- For $n = 3$,

$$\begin{aligned} \int \sin^3 x \, dx &= -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx \\ &= -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C. \end{aligned}$$

- An important integral that will soon need is, for $n \geq 1$ and $a \neq 0$,

$$\begin{aligned}
J_{n,a}(x) &= \int \frac{1}{(x^2+a^2)^n} \cdot 1\, dx \\
&= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2}{(x^2+a^2)^{n+1}}\, dx \\
&= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{n+1}}\, dx \\
&= \frac{x}{(x^2+a^2)^n} + 2n\Big(\int \frac{1}{(x^2+a^2)^n}\, dx - \int \frac{a^2}{(x^2+a^2)^{n+1}}\, dx\Big) \\
&= \frac{x}{(x^2+a^2)^n} + 2n(J_{n,a}(x) - a^2 J_{n+1,a}(x)) \\
\Rightarrow (1-2n)J_{n,a}(x) &= \frac{x}{(x^2+a^2)^n} - 2na^2 J_{n+1,a}(x).
\end{aligned}$$

The resulting reduction formula,

$$\boxed{J_{n+1,a}(x) = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2}J_{n,a}(x)} \quad (n \geq 1, a \neq 0),$$

together with the result (for $a \neq 0$)

$$J_{1,a}(x) = \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\arctan\frac{x}{a} + C,$$

allows us to compute $J_{n,a}(x)$ for any $n \geq 1$.

Problem 7.1: Find

$$\int_0^1 \arcsin x\, dx.$$

$$\int_0^1 1 \cdot \arcsin x\, dx = [x \arcsin x]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx = \arcsin 1 + \left[(1-x^2)^{1/2}\right]_0^1 = \frac{\pi}{2} - 1.$$

Problem 7.2: Let $P(x)$ be a polynomial of degree n. Prove that

$$\int P(x)e^x\, dx = e^x \sum_{k=0}^{n} (-1)^k P^{(k)}(x) + C,$$

where $P^{(k)}$ denotes the kth derivative of P. Give an explicit reason why the sum terminates at $k = n$.

This follows immediately on integrating by parts n times, using $f(x) = P(x)$ and $g(x) = e^x$. Alternatively, we can verify the result by noting that the derivative of the right-hand side is

$$\begin{aligned} e^x \sum_{k=0}^{n} (-1)^k P^{(k)}(x) + e^x \sum_{k=0}^{n} (-1)^k P^{(k+1)}(x) &= e^x \sum_{k=0}^{n} (-1)^k P^{(k)}(x) - e^x \sum_{k=1}^{n+1} (-1)^k P^{(k)}(x) \\ &= e^x P^{(0)}(x) - e^x (-1)^{n+1} P^{(n+1)}(x) = e^x P(x) \end{aligned}$$

since $P^{(n+1)}(x) = 0$.

7.2 Integrals of Trigonometric Functions

Often we encounter integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are integers. Here is an integration strategy:

Case I. If either of the integers m or n is odd, separate out one factor of $\sin x$ or $\cos x$ so that the rest of the integrand may be written entirely as a polynomial in $\cos x$ or a polynomial in $\sin x$, as the case may be. Then make the appropriate substitution. (Note: if both m and n are odd there will be two possible ways of doing this.)

Case II. If m and n are both even use the addition formulae

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \quad 2 \sin x \cos x = \sin 2x,$$

possibly repeatedly, to reduce the problem to the form of Case I.

- Find $\int \sin^3 x \cos^2 x \, dx$, using the substitution $u = \cos x$ $(du = -\sin x \, dx)$,

$$\begin{aligned} \int \sin^3 x \cos^2 x \, dx &= \int \sin x (1 - \cos^2 x) \cos^2 x \, dx \\ &= -\int (1 - u^2) u^2 \, du = -\int (u^2 - u^4) \, du \\ &= -\left(\frac{u^3}{3} - \frac{u^5}{5} \right) + C = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C. \end{aligned}$$

- Find $\int \sin^2 x \cos^3 x\,dx$, using the substitution $u = \sin x$ ($du = \cos x\,dx$),

$$\begin{aligned}\int \sin^2 x \cos^3 x\,dx &= \int \sin^2 x(1-\sin^2 x)\cos x\,dx \\ &= \int u^2(1-u^2)\,du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.\end{aligned}$$

- We can use the fact that $\sin^2 2x = (1-\cos 4x)/2$ to compute

$$\begin{aligned}\int \sin^2 x \cos^2 x\,dx &= \int \left(\frac{1}{2}\sin 2x\right)^2 dx \\ &= \frac{1}{4}\int \frac{1-\cos 4x}{2}\,dx \\ &= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + C.\end{aligned}$$

Problem 7.3: Find

$$\int \cos^7 x \sin^2 x\,dx$$

Let $u = \sin x$. The integral then evaluates to

$$\begin{aligned}\int (1-u^2)^3 u^2\,du &= \int (1-3u^2+3u^4-u^6)u^2\,du = \int (u^2-3u^4+3u^6-u^8)\,du \\ &= \frac{u^3}{3} - \frac{3u^5}{5} + \frac{3u^7}{7} - \frac{u^9}{9} + C = \frac{\sin^3 x}{3} - \frac{3\sin^5 x}{5} + \frac{3\sin^7 x}{7} - \frac{\sin^9 x}{9} + C.\end{aligned}$$

Problem 7.4: Prove that

(a)

$$\cosh^2 x = \frac{\cosh 2x + 1}{2};$$

(b)

$$\sinh^2 x = \frac{\cosh 2x - 1}{2};$$

(c)

$$2\sinh x \cosh x = \sinh 2x.$$

Remark: In view of Problem 7.4, the same technique can be used to compute $\int \cosh^m x \sinh^n x\,dx$ for integer values of m and n.

Remark: The technique of extracting out an odd factor of $\cos x$ or $\sin x$ can even be applied if one or both of the integers m and n are negative. For example, we can compute the indefinite integral of $\sec x$ by rewriting the integrand and using the substitution $u = \sin x$,

$$\begin{aligned}
\int \sec x\, dx &= \int \frac{\cos x}{\cos^2 x}\, dx = \int \frac{\cos x}{1-\sin^2 x}\, dx \\
&= \int \frac{1}{1-u^2}\, du = \int \left(\frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u} \right) dx \\
&= \frac{1}{2}\log|1+u| - \frac{1}{2}\log|1-u| + C \\
&= \frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right| + C \\
&= \frac{1}{2}\log\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C \\
&= \log\left|\sqrt{\frac{(1+\sin x)^2}{1-\sin^2 x}}\right| + C \\
&= \log\left|\frac{1+\sin x}{\cos x}\right| + C \\
&= \log|\sec x + \tan x| + C.
\end{aligned}$$

Remark: One can use a similar technique to compute certain integrals of the form

$$\int \tan^m x \sec^n x\, dx,$$

by exploiting the Pythagorean relation $\tan^2 x + 1 = \sec^2 x$, along with the derivatives

$$\frac{d}{dx}\tan x = \sec^2 x$$

and

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{+\sin x}{\cos^2 x} = \sec x \tan x.$$

For example, if m is an odd natural number, the substitution $u = \sec x$ will reduce the integrand to a polynomial. If n is an even natural number, the substitution $u = \tan x$ will work.

- Letting $u = \sec x$, we find

$$\begin{aligned}
&\int \tan^3 x \sec^3 x \, dx \\
&= \int \tan^2 x \sec^2 x \underbrace{\sec x \tan x \, dx}_{du} \\
&= \int (u^2 - 1)u^2 \, du = \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.
\end{aligned}$$

- Letting $u = \tan x$, we find

$$\begin{aligned}
&\int \tan^4 x \sec^4 x \, dx \\
&= \int \tan^4 x (1 + \tan^2 x) \underbrace{\sec^2 x \, dx}_{du} \\
&= \int u^4 (1 + u^2) \, du = \frac{u^5}{5} + \frac{u^7}{7} + C = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.
\end{aligned}$$

- An alternative method for computing the integral of $\sec x$ relies on the following trick:

$$\begin{aligned}
\int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\
&= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx \\
&= \log |\sec x + \tan x| + C.
\end{aligned}$$

- Compute

$$\int \sec^3 x \, dx = \int \underbrace{\sec x}_{f} \underbrace{\sec^2 x}_{g'} \, dx.$$

On integrating by parts, we find that

$$\begin{aligned}
\int \sec^3 x \, dx &= \sec x \tan x - \int (\sec x \tan x) \tan x \, dx \\
&= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx.
\end{aligned}$$

Thus

$$2\int \sec^3 x\,dx = \sec x\tan x + \log|\sec x + \tan x| + C$$
$$\Rightarrow \int \sec^3 x\,dx = \frac{1}{2}(\sec x\tan x + \log|\sec x + \tan x|) + C.$$

7.3 Trigonometric Substitutions

Trigonometric substitutions are often useful for evaluating integrals containing square roots of quadratic expressions. Several common trigonometric substitutions for frequently appearing quadratic expressions are listed in Table 7.2. Note that it may be necessary to complete the square of the quadratic and shift the variable of integration to put the expression into one of these forms.

Expression	x Domain	Substitution	θ or t Domain	Identity
$\sqrt{a^2-x^2}$	$[-a,a]$	$x=a\sin\theta$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$(-\infty,\infty)$	$x=a\tan\theta$ $x=a\sinh t$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ $(-\infty,\infty)$	$1+\tan^2\theta=\sec^2\theta$ $1+\sinh^2 t=\cosh^2 t$
$\sqrt{x^2-a^2}$	$(-\infty,-a]\cup[a,\infty]$	$x=a\sec\theta$ $x=a\cosh t$	$\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$ $[0,\infty)$	$\sec^2\theta-1=\tan^2\theta$ $\cosh^2 t-1=\sinh^2 t$

Table 7.2: Useful trigonometric substitutions.

- We can use the trigonometric substitution $x = a\sin\theta$ to find the area between the half-circle $y = \sqrt{a^2-x^2}$ and $y = 0$:

$$\int_{-a}^{a}\sqrt{a^2-x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos\theta\, a\cos\theta\,d\theta = a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}\,d\theta = \frac{\pi}{2}a^2.$$

- We can find the indefinite integral

$$\int \sqrt{u^2+a^2}\,du$$

with the substitution $u = a\tan\theta$ ($du = a\sec^2\theta\,d\theta$). Without loss of generality, we assume that $a > 0$. Since

$$u^2 + a^2 = a^2(\tan^2\theta + 1) = a^2\sec^2\theta,$$

we may write

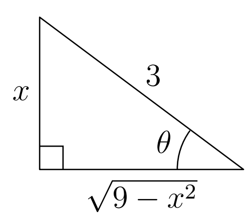
$$\begin{aligned}
\int \sqrt{u^2+a^2}\,du &= \int a\sec\theta\, a\sec^2\theta\,d\theta = a^2\int \sec^3\theta\,d\theta \\
&= \frac{a^2}{2}(\sec\theta\tan\theta + \log|\sec\theta+\tan\theta|) + C, \\
&= \frac{a^2}{2}\left(\frac{\sqrt{u^2+a^2}}{a}\cdot\frac{u}{a} + \log\left|\frac{\sqrt{u^2+a^2}}{a} + \frac{u}{a}\right|\right) + C, \\
&= \frac{1}{2}\left(u\sqrt{u^2+a^2} + a^2\log\left|\sqrt{u^2+a^2}+u\right| - a^2\log a\right) + C,
\end{aligned}$$

on making use of an integral computed previously on 137.

- For $0 < x \le 3$, the substitution $x = 3\sin\theta$ ($dx = 3\cos\theta$) and the fact that $\sqrt{9-x^2} = 3\cos\theta$ can be used to evaluate the integral

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta}3\cos\theta\,d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = \int\left(\frac{1}{\sin^2\theta} - 1\right)d\theta \\
&= -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\frac{x}{3} + C.
\end{aligned}$$

In the last line one simplifies $\cot\theta = \cot(\sin^{-1}(x/3))$ to $\sqrt{9-x^2}/x$ with the aid of the triangle below.



Problem 7.5: Find the antiderivative

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx$$

for $-3 \le x < 0$.

Problem 7.6: For $a > 0$, show that the substitution $x = a\sec\theta$ can be used to find

$$\int \frac{dx}{\sqrt{x^2 - a^2}}.$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}\,d\theta = \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta - 1}}\,d\theta = \int \frac{\sec\theta\tan\theta}{\tan\theta}\,d\theta = \int \sec\theta\,d\theta$$
$$= \log(\sec\theta + \tan\theta) + C = \log\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right) + C.$$

Remark: Alternatively, the hyperbolic substitution $x = a\cosh t$, with $dx = a\sinh t\,dt$, and the fact that $x^2 - a^2 = a^2\sinh^2 t$ can be used for $a > 0$ to evaluate

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{1}{a\sinh t}\,a\sinh t\,dt = \int dt = t + C = \cosh^{-1}\frac{x}{a} + C.$$

Show that the answer agrees with the result found in Prob. 7.6.

- An integral of the form

$$\int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,dx$$

can first be put in the form of the expressions listed in Table 7.2 with the substitution $u = 2x$, so that $4x^2 + 9 = u^2 + 9$. One could then apply the substitution $u = 3\sinh t$. In fact, both substitutions can be done in a single step by defining $x = \frac{3}{2}\sinh t$:

$$\int \left(\frac{\left(\frac{\cancel{3}}{2}\right)^3 \sinh^3 t}{\cancel{3}^3\cosh^3 t}\right)\frac{3}{2}\cosh t\,dt = \frac{3}{16}\int \frac{\sinh^3 t}{\cosh^2 t}\,dt = \frac{3}{16}\int \frac{(\cosh^2 t - 1)\sinh t}{\cosh^2 t}\,dt$$
$$= \frac{3}{16}\int\left(\sinh t - \frac{\sinh t}{\cosh^2 t}\right)dt = \frac{3}{16}\left(\cosh t + \frac{1}{\cosh t}\right) + C = \frac{1}{16}\left(\sqrt{4x^2+9} + \frac{9}{\sqrt{4x^2+9}}\right) + C,$$

on noting that

$$\cosh t = \sqrt{1 + \sinh^2 t} = \sqrt{1 + \frac{4x^2}{9}} = \frac{\sqrt{4x^2+9}}{3}.$$

The integral could have also been evaluated with the substitution $x = \frac{3}{2}\tan\theta$.

Q. How about integrals of the form $\int \sqrt{x^2+x+1}\,dx$?

A. We can first simplify the integrand somewhat by completing the square and making the substitution $u = x + 1/2$:

$$\int \sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1}\, dx = \int \sqrt{u^2 + \frac{3}{4}}\, du.$$

The resulting integral is of the form $\int \sqrt{u^2+a^2}\,du$, which we computed in section 7.3.

- The integral

$$\begin{aligned}\int \frac{x}{\sqrt{3-2x-x^2}}\,dx &= \int \frac{x}{\sqrt{-(x^2+2x-3)}}\,dx \\ &= \int \frac{x}{\sqrt{-((x+1)^2-1-3)}}\,dx \\ &= \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx\end{aligned}$$

may be evaluated with the substitution $u = x+1$, followed by $u = 2\sin\theta$, to obtain

$$\begin{aligned}\int \frac{2\sin\theta - 1}{\sqrt{4-4\sin^2\theta}}\, 2\cos\theta\, d\theta = \int (2\sin\theta - 1)\, d\theta &= -2\cos\theta - \theta + C \\ &= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C.\end{aligned}$$

Problem 7.7: Evaluate

$$\int \frac{u^3+3u^2+3u+1}{(4u^2+8u+13)^{3/2}}\, du.$$

7.4 Partial Fraction Decomposition

Recall that a *Rational functions* is a function of the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials. Consider the following techniques for integrating rational functions.

- Find

$$\int \frac{x+1}{x}\, dx = \int \left(1 + \frac{1}{x}\right) dx = x + \log|x| + C.$$

- Similarly,

$$\int \frac{x}{x+1}\,dx = \int \left(\frac{x+1}{x+1} - \frac{1}{x+1}\right) dx = x - \log|x+1| + C.$$

Q. Can these techniques be generalized for integrating any rational function?

A. Yes, using the general method of *partial fraction decomposition*:

Suppose we wish to evaluate the integral

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where P and Q are polynomials functions of x.

Step 1: If the degree of $P \geq$ degree of Q, we rewrite the integrand in *proper form*:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

such that the degree of R is less than the degree of Q.

- Suppose that $P(x) = x^3 + x$ and $Q(x) = x - 1$. We see that $\deg P = 3 \geq \deg Q = 1$, so we put $P(x)/Q(x)$ in proper form, using long division:

$$\frac{P(x)}{Q(x)} = \frac{x^3+x}{x-1} = \underbrace{x^2+x+2}_{S(x)} + \underbrace{\frac{2}{x-1}}_{\frac{R(x)}{Q(x)}}.$$

We can now go ahead and integrate $S(x)$ and, in this case, also $R(x)/Q(x)$ without doing any further work:

$$\begin{aligned}\int \frac{x^3+x}{x-1}\,dx &= \int \left(x^2+x+2+\frac{2}{x-1}\right) dx \\ &= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\log|x-1| + C.\end{aligned}$$

Remark: At this stage, we will always be able to find an antiderivative for $S(x)$ since it is just a polynomial. The following steps may be needed to integrate the remaining term $R(x)/Q(x)$.

Step 2: Factor $Q(x)$ as far as possible, into products of linear factors and irreducible quadratic factors.

- $Q(x) = x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$.

- $Q(x) = (x + 1)(x + 2)^2(x^2 + x + 3)(x^2 + x + 4)^2$.

Step 3: Suppose $Q(x)$ has the form

$$Q(x) = A(x - a)^n \ldots (x^2 + \gamma x + \lambda)^m \ldots,$$

where the *discriminant* $\gamma^2 - 4\lambda < 0$, so that $x^2 + \gamma x + \lambda$ cannot be factorized into linear factors with real coefficients. It is then possible to express $R(x)/Q(x)$, where $\deg R < \deg Q$ in the form

$$\begin{aligned}\frac{R(x)}{Q(x)} = &\left[\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_n}{(x-a)^n}\right] + \ldots \\ &+ \left[\frac{\Gamma_1 x + \Lambda_1}{x^2 + \gamma x + \lambda} + \ldots + \frac{\Gamma_m x + \Lambda_m}{(x^2 + \gamma x + \lambda)^m}\right] + \ldots .\end{aligned}$$

Step 4: Solve for the coefficients in the numerator by equating like powers of x.

- We can solve
$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1) + Bx}{x(x+1)}$$
for the coefficients A and B by equating like polynomial coefficients in the numerator. On setting
$$1 = A(x + 1) + Bx = (A + B)x + A,$$
we see that the coefficients of x^0 and x^1 are
$$\begin{aligned}x^0 &: 1 = A,\\ x^1 &: 0 = A + B.\end{aligned}$$
The unique solution to these equations is $A = 1$ and $B = -1$.

Step 5: Integrate each term separately.

- Find

$$\int \frac{x}{(x+a)(x+b)}\,dx.$$

Try to write

$$\frac{x}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} = \frac{A(x+b)+B(x+a)}{(x+a)(x+b)}.$$

Thus

$$\begin{aligned} x^1 &: \quad 1 = A + B \Rightarrow B = 1 - A, \\ x^0 &: \quad 0 = Ab + Ba. \end{aligned}$$

Solving for A and B, we find for $a \neq b$ that

$$\begin{aligned} 0 &= Ab + (1-A)a, \\ \Rightarrow A &= \frac{a}{a-b}, \\ B &= 1 - \frac{a}{a-b} = \frac{-b}{a-b}. \end{aligned}$$

$\therefore$ If $a \neq b$ then

$$\begin{aligned} \int \frac{x}{(x+a)(x+b)}\,dx &= \frac{a}{a-b}\int \frac{1}{x+a}\,dx - \frac{b}{a-b}\int \frac{1}{x+b}\,dx \\ &= \frac{a}{a-b}\log|x+a| - \frac{b}{a-b}\log|x+b| + C. \end{aligned}$$

Problem: But what if $b = a$? Then

$$\begin{aligned} 1 &= A + B \\ 0 &= Aa + Ba = (A+B)a, \end{aligned}$$

which is consistent only if $a = b = 0$.

Remedy: Write

$$\frac{x}{(x+a)^2} = \frac{A}{x+a} + \frac{B}{(x+a)^2} = \frac{A(x+a)+B}{(x+a)^2}.$$

Then

$$\begin{aligned} x^1 &: \quad 1 = A \\ x^0 &: \quad 0 = Aa + B \Rightarrow B = -a. \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{x}{(x+a)^2}\,dx &= \int \left[\frac{1}{x+a} - \frac{a}{(x+a)^2}\right] dx \\ &= \log|x+a| + \frac{a}{x+a} + C. \end{aligned}$$

- Evaluate
$$\int \frac{x^2}{(x+1)^2}\,dx.$$
Since $\deg x^2 = 2 \geq \deg(x+1)^2 = 2$, we need to rewrite
$$\frac{x^2}{(x+1)^2} = 1 - \frac{(2x+1)}{(x+1)^2}.$$
Express
$$\frac{2x+1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2};$$
this requires that $2x+1 = A(x+1)+B$. On equating like powers of x, we find
$$\begin{aligned} x^1: &\quad 2 = A,\\ x^0: &\quad 1 = A+B \Rightarrow B = -1, \end{aligned}$$
so that
$$\begin{aligned} \int \frac{x^2}{(x+1)^2}\,dx &= \int \left(1 - \left[\frac{A}{x+1} + \frac{B}{(x+1)^2}\right]\right) dx \\ &= \int \left(1 - \left[\frac{2}{x+1} + \frac{-1}{(x+1)^2}\right]\right) dx \\ &= x - 2\log|x+1| - \frac{1}{x+1} + C. \end{aligned}$$

Remark: Show that the substitution $u = x+1$ makes the previous problem much easier!

- To find
$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$$
we write
$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$
This requires that
$$\begin{aligned} -x^3+2x^2-x+1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \end{aligned}$$
Thus
$$\begin{aligned} x^4: &\quad 0 = A+B,\\ x^3: &\, -1 = C,\\ x^2: &\quad 2 = 2A+B+D,\\ x^1: &\, -1 = C+E \Rightarrow E = 0,\\ x^0: &\quad 1 = A \Rightarrow B = -1 \text{ and } D = 1, \end{aligned}$$

so that

$$\begin{aligned}\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx &= \int\left(\frac{1}{x}-\frac{x+1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx\\ &= \int\left(\frac{1}{x}-\frac{x}{x^2+1}-\frac{1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx\\ &= \log|x|-\frac{1}{2}\log(x^2+1)-\tan^{-1}x-\frac{1}{2(x^2+1)}+K.\end{aligned}$$

- Find

$$\int \frac{1}{x^3-1}\,dx.$$

Step 1: We already have $\deg P < \deg Q$.

Step 2: Noting that $Q(x) = x^3-1$ has a root at $x=1$, we factor

$$Q(x) = x^3-1 = (x-1)(x^2+x+1).$$

The quadratic factor x^2+x+1 cannot be factored into linear factors with real coefficients since it has no real roots (the discriminant $1^2-4=-3$ is negative).

Step 3: Write

$$\frac{1}{x^3-1} = \frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}.$$

Step 4: Then

$$\begin{aligned}1 &= A(x^2+x+1)+(Bx+C)(x-1)\\ &= Ax^2+Ax+A+Bx^2-Bx+Cx-C.\end{aligned}$$

We find

$$\begin{aligned}&x^2: 0 = A+B \Rightarrow B=-A,\\ &x^1: 0 = A-B+C,\\ &x^0: 1 = A-C \Rightarrow C = A-1.\end{aligned}$$

The x^1 equation then yields $0 = A+A+(A-1)$, which implies $A=\frac{1}{3}$, $B=-\frac{1}{3}$, and $C=-\frac{2}{3}$, so that

$$\int\frac{1}{x^3-1}\,dx = \int\frac{\frac{1}{3}}{x-1}\,dx+\int\frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx.$$

Step 5: On completing the square and letting $u = x + \frac{1}{2}$, we evaluate

$$\begin{aligned}
\int \frac{x+2}{x^2+x+1}\,dx &= \int \frac{x+2}{(x+\frac{1}{2})^2 - \frac{1}{4} + 1}\,dx \\
&= \int \frac{u+\frac{3}{2}}{u^2+\frac{3}{4}}\,du \\
&= \int \frac{u}{u^2+\frac{3}{4}}\,du + \frac{3}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du \\
&= \frac{1}{2}\log\left|u^2+\frac{3}{4}\right| + \frac{3}{2}\frac{1}{\sqrt{\frac{3}{4}}}\arctan\left(\frac{u}{\sqrt{\frac{3}{4}}}\right) \\
&= \frac{1}{2}\log\left|x^2+x+1\right| + \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + K.
\end{aligned}$$

Thus

$$\int \frac{1}{x^3-1}\,dx = \frac{1}{3}\log|x-1| - \frac{1}{6}\log\left|x^2+x+1\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + K.$$

Problem 7.8: Evaluate

$$\int \frac{1}{1-u^2}\,du.$$

$$\int \frac{1}{(1+u)(1-u)}\,du = \int \left(\frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u}\right)du = \frac{1}{2}\log\left|\frac{1+u}{1-u}\right| + C$$

Note: in the complex plane, the antiderivative may also be written as $\tanh^{-1} u + C$. However, in $\mathbb{R}$, the latter solution does not exist outside of $(-1, 1)$.

Problem 7.9: Compute

$$\int \frac{1}{(u^2+1)(u+1)}\,du.$$

Express

$$\frac{1}{(u^2+1)(u+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2+1}.$$

By equating coefficients of like powers in $1 = A(u^2+1) + B(u^2+u) + C(u+1)$, we obtain the system of equations

$$\begin{aligned}
u^0 &: 1 = A + C, \\
u^1 &: 0 = B + C, \\
u^2 &: 0 = A + B,
\end{aligned}$$

which has the unique solution $A = C = 1/2$, $B = -1/2$. Hence the integral becomes

$$\frac{1}{2}\int\left(\frac{1}{u+1} - \frac{u}{u^2+1} + \frac{1}{u^2+1}\right)du = \frac{1}{2}\log|u+1| - \frac{1}{4}\log(u^2+1) + \frac{1}{2}\arctan u + K,$$

where K is a constant.

Problem 7.10: Find

$$\int \frac{x^3+4x^2+7x+5}{(x+1)^2(x+2)}\,dx.$$

First, note that $(x+1)^2(x+2) = x^3+4x^2+5x+2$. The integral thus becomes

$$\int 1 + \frac{2x+3}{(x+1)^2(x+2)}\,du$$

On expressing

$$\frac{2x+3}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

and equating coefficients of like powers in $2x+3 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$, we obtain the system of equations

$$\begin{aligned} x^0 &: 3 = 2A + 2B + C, \\ x^1 &: 2 = 3A + B + 2C, \\ x^2 &: 0 = A + C, \end{aligned}$$

which has the unique solution $A = B = 1$, $C = -1$.

The integral thus evaluates to

$$x + \log\left|\frac{x+1}{x+2}\right| - \frac{1}{x+1} + K,$$

where K is an arbitrary constant.

Problem 7.11: Here is a more challenging example. Find

$$\int \frac{1}{x^2(1+x^2)^2}\,dx.$$

$$\begin{aligned}
\int \frac{1}{x^2(1+x^2)^2}\,dx = \int \frac{(1+x^2)-x^2}{x^2(1+x^2)^2}\,dx &= \int\left[\frac{1}{x^2(1+x^2)} - \frac{1}{(1+x^2)^2}\right]dx \\
&= \int\left[\frac{(1+x^2)-x^2}{x^2(1+x^2)} - \frac{1}{(1+x^2)^2}\right]dx \\
&= \int\left[\frac{1}{x^2} - \frac{1}{1+x^2} - \frac{1}{(1+x^2)^2}\right]dx = -\frac{1}{x} - \arctan x - J_{2,1}(x),
\end{aligned}$$

where

$$J_{2,a}(x) = \int \frac{dx}{(x^2+a^2)^2}.$$

Recall that the indefinite integral

$$J_{n,a}(x) = \int \frac{dx}{(x^2+a^2)^n}$$

can be evaluated using the reduction formula

$$J_{n+1,a}(x) = \frac{1}{2na^2}\frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2}J_{n,a}(x).$$

Setting $n = 1$ and $a = 1$ yields

$$J_{2,1}(x) = \frac{1}{2}\left(\frac{x}{x^2+1}\right) + \frac{1}{2}J_{1,1}(x),$$

where $J_{1,1}(x) = \arctan x + C$. Hence,

$$\begin{aligned}\int \frac{1}{x^2(1+x^2)^2}\,dx &= -\frac{1}{x} - \arctan x - \frac{1}{2}\left(\frac{x}{x^2+1}\right) - \frac{1}{2}\arctan x + C\\ &= -\frac{1}{x} - \frac{3}{2}\arctan x - \frac{1}{2}\left(\frac{x}{x^2+1}\right) + C.\end{aligned}$$

7.5 Integration of Certain Irrational Expressions

Q. How do we find integrals like

$$\int \frac{\sqrt{x+4}}{x}\,dx?$$

A. Substitute $t = \sqrt{x+4}$. Then $t^2 = x + 4 \Rightarrow 2t\,dt = dx$ and

$$\begin{aligned}\int \frac{\sqrt{x+4}}{x}\,dx &= \int \left(\frac{t}{t^2-4}\right)2t\,dt\\ &= 2\int \frac{t^2}{t^2-4}\,dt = 2\int \left[\frac{t^2-4}{t^2-4} + \frac{4}{t^2-4}\right]dt\\ &= 2t + 8\int \frac{1}{t^2-4}\,dt\\ &= 2t + 8\int \left[\frac{A}{t-2} + \frac{B}{t+2}\right]dt \qquad \begin{cases}1 = A(t+2) + B(t-2),\\ 0 = A + B \Rightarrow B = -A,\\ 1 = 2A - 2B = 4A \Rightarrow A = \frac{1}{4}.\end{cases}\\ &= 2t + 8\int \left[\frac{\frac{1}{4}}{t-2} - \frac{\frac{1}{4}}{t+2}\right]dt\\ &= 2t + 2\log\left|\frac{t-2}{t+2}\right| + C.\end{aligned}$$

Thus

$$\int \frac{\sqrt{x+4}}{x}\,dx = 2\sqrt{x+4} + 2\log\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.$$

In general, one can reduce any integral of the form

$$\int R\left(x,\ \sqrt[m]{\frac{ax+b}{cx+d}}\right) dx,$$

where R is a *birational* function

$$R(x,y) = \frac{\sum_{ij} a_{ij} x^i y^j}{\sum_{kl} b_{k\ell} x^k y^\ell},$$

of its arguments, to the integral of a rational function by using the substitution

$$t = \sqrt[m]{\frac{ax+b}{cx+d}}.$$

- We can evaluate
$$\int \frac{1}{x-\sqrt{x+2}}\, dx$$
with the substitution $t = \sqrt{x+2} \Rightarrow t^2 = x+2 \Rightarrow 2t\, dt = dx$,
$$\int \frac{1}{x-\sqrt{x+2}}\, dx = \int \left(\frac{1}{t^2-2-t}\right) 2t\, dt = 2\int \frac{t}{t^2-t-2}\, dt$$
$$= 2\int \frac{t}{(t-2)(t+1)}\, dt,$$
which can then be decomposed into partial fractions.

Remark: When more than one radical appears, it is often helpful to take m to be the least common multiple of the radical indices.

Problem 7.12: Find
$$\int \frac{1}{\sqrt[2]{x}-\sqrt[3]{x}}\, dx$$
using the substitution $t = x^{\frac{1}{2\cdot 3}} = x^{\frac{1}{6}}$.

Problem 7.13: Find
$$\int \frac{1}{\sqrt[6]{x}+\sqrt[4]{x}}\, dx.$$
using the substitution $t = x^{\frac{1}{12}}$.

7.6 Strategy for Integration

1. Simplify the integrand.

2. Look for an obvious substitution: see if you can write the integral in the form

$$\int f(g(x))g'(x)\,dx.$$

If so, try the substitution $u = g(x)$.

3. Classify the integrand.

(a) Trigonometric functions: exploit trigonometric identities to find integrals of the form

$$\left\{ \begin{array}{l} \int \sin^n x \cos^m x\,dx \\ \int \tan^n x \sec^m x\,dx \\ \int \cot^n x \csc^m x\,dx \end{array} \right\}.$$

As a last resort, use the universal substitution $t = \tan\frac{x}{2}$.

(b) Rational functions: use the Method of Partial Fractions.

(c) Polynomials (including 1) $\times$ *transcendental functions* (e.g. Trigonometric, exponential, logarithmic, and inverse functions): use integration by parts.

(d) Radicals:

(i) $\sqrt{\pm x^2 \pm a^2}$: use a trigonometric substitution

(ii) $\sqrt[n]{\frac{ax+b}{cx+d}}$: $t = \sqrt[n]{\frac{ax+b}{cx+d}}$

For $\sqrt[n]{g(x)}$: $t = \sqrt[n]{g(x)}$ sometimes helps.

4. Try again (maybe use several methods combined).

Problem 7.14: Find

$$\int \frac{x}{\sqrt{1+x^{2/3}}}\,dx.$$

Substituting first $y = x^{2/3}$ and then $t = y + 1$ we find

$$\begin{aligned}
\int \frac{x}{\sqrt{1+x^{2/3}}}\,dx &= \int \frac{y^{3/2}}{\sqrt{1+y}}\frac{3}{2}y^{1/2}\,dy = \frac{3}{2}\int \frac{y^2}{\sqrt{1+y}}\,dy = \frac{3}{2}\int \frac{(t-1)^2}{\sqrt{t}}\,dt \\
&= \frac{3}{2}\int t^{3/2} - 2t^{1/2} + t^{-1/2}\,dt = \frac{3}{2}\left(\frac{2}{5}t^{5/2} - \frac{4}{3}t^{3/2} + 2t^{1/2}\right) + C \\
&= \frac{3}{5}\left(1+x^{2/3}\right)^{5/2} - 2\left(1+x^{2/3}\right)^{3/2} + 3\left(1+x^{2/3}\right)^{1/2} + C \\
&= \frac{1}{5}\left(1+x^{2/3}\right)^{1/2}\left[3\left(1+x^{2/3}\right)^2 - 10\left(1+x^{2/3}\right) + 15\right] + C \\
&= \frac{1}{5}\left(1+x^{2/3}\right)^{1/2}\left(8 - 4x^{2/3} + 3x^{4/3}\right) + C.
\end{aligned}$$

Alternatively, substituting $y = x^{1/3}$, then $\sinh t = y$, and finally $u = \cosh t = \sqrt{1+y^2} = \sqrt{1+x^{2/3}}$ (which could be used as a more direct substitution), we find

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^{2/3}}}\,dx = \int \frac{y^3}{\sqrt{1+y^2}}\,3y^2\,dy = 3\int \frac{y^5}{\sqrt{1+y^2}}\,dy = 3\int \frac{\sinh^5 t}{\cosh t}\,\cosh t\,dt\\ &= 3\int (u^2-1)^2\,du = 3\int (u^4 - 2u^2 + 1)\,du\\ &= 3\left(\frac{u^5}{5} - 2\frac{u^3}{3} + u\right) + C\\ &= \frac{u}{5}\left(3u^4 - 10u^2 + 15\right) + C\\ &= \frac{1}{5}\left(1 + x^{2/3}\right)^{1/2}\left(8 - 4x^{2/3} + 3x^{4/3}\right) + C.\end{aligned}$$

7.7 Improper Integrals

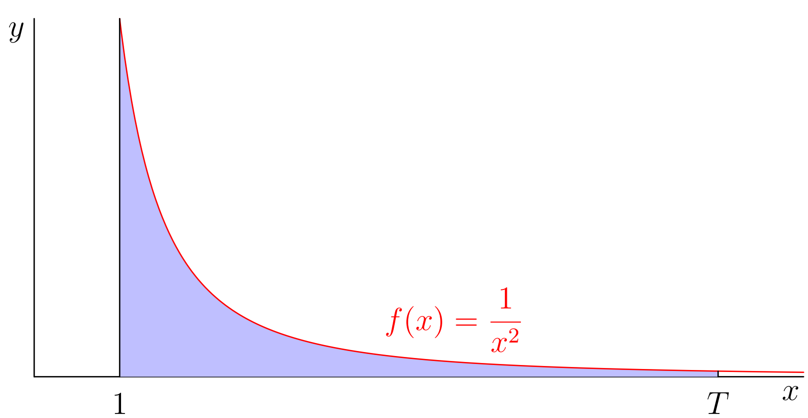
Until now, we have only defined the Riemann integral for bounded functions on closed intervals. Let us now discuss situations where these restrictions may be somewhat relaxed.

First, we extend the notion of integration to certain bounded functions on infinite intervals.

Definition: Let f be a function that is integrable on every closed subinterval $[a, T]$ of $[a, \infty)$. We define the *improper integral*

$$\int_a^\infty f(x)\,dx \doteq \lim_{T\to\infty} \int_a^T f(x)\,dx.$$

If this limit exists and is finite we say that $\int_a^\infty f(x)\,dx$ *converges*; otherwise we say that $\int_a^\infty f(x)\,dx$ *diverges*.



- For which values of p does $\displaystyle\int_1^\infty x^{-p}\,dx$ exist? To answer this question, we first compute the definite integral

$$\int_1^T \frac{dx}{x^p} = \begin{cases} \left[\dfrac{x^{1-p}}{1-p}\right]_1^T & \text{if } p \neq 1, \\ [\log|x|]_1^T & \text{if } p = 1. \end{cases}$$

But $\lim\limits_{T\to\infty} T^{1-p}$ exists only when $p \geq 1$. Also, $\lim\limits_{T\to\infty} \log T = \infty$. Thus

$$\int_1^\infty \frac{dx}{x^p} = \lim_{T\to\infty} \int_1^T \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1, \\ \infty & \text{if } p \leq 1. \end{cases}$$

Definition: Let f be a function that is integrable on every closed subinterval $[T, a]$ of $(-\infty, a]$. Define

$$\int_{-\infty}^a f(x)\,dx \doteq \lim_{T\to-\infty} \int_T^a f(x)\,dx.$$

Q. Sometimes an explicit form for the antiderivative of an integrable function f is unavailable. Are there other ways to determine whether the improper integral $\int_a^\infty f(x)\,dx$ converges?

A. Yes. The following theorem provides a test for improper integrals.

Theorem 7.2 (Comparison Test): *Suppose* $0 \leq f(x) \leq g(x)$ *and* $\int_a^T f$ *and* $\int_a^T g$ *exist for all* $T \geq a$. *Then*

(i) $\int_a^\infty g$ *converges* $\Rightarrow \int_a^\infty f$ *converges;*

(ii) $\int_a^\infty f$ *diverges* $\Rightarrow \int_a^\infty g$ *diverges.*

Proof: Note that $0 \leq \int_a^T f \leq \int_a^T g$ and both integrals are monotonic increasing functions of T. Since $\int_a^\infty g$ converges, $\int_a^T g$ must be bounded on $[a, \infty)$ and hence so is $\int_a^T f$. This means that $\int_a^\infty f$ converges.

- To decide on whether

$$\int_1^\infty \frac{1}{1+x^3}\,dx$$

converges, we could first find $\int_1^T dx/(1+x^3)$ and then check that the limit as $T \to \infty$ exists. However, it is much easier to use Theorem 7.2 (i), noting that

$$0 \leq \frac{1}{1+x^3} \leq \frac{1}{x^2}$$

for all $x \ge 1$. That is,

$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges } \Rightarrow \int_1^\infty \frac{1}{1+x^3}\,dx \text{ converges.}$$

- We may use the previous result to establish that

$$\int_0^\infty \frac{1}{1+x^3}\,dx$$

exists, even though $1/x^2$ is not bounded (and hence $\int_0^\infty x^{-2}\,dx$ does not exist). This is seen by writing

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \lim_{T\to\infty}\int_0^T \frac{1}{1+x^3}\,dx = \lim_{T\to\infty}\left(\int_0^1 \frac{1}{1+x^3}\,dx + \int_1^T \frac{1}{1+x^3}\,dx\right)$$
$$= \int_0^1 \frac{1}{1+x^3}\,dx + \int_1^\infty \frac{1}{1+x^3}\,dx.$$

Remark: When f is an integrable function,

$$\int_a^\infty f(x)\,dx \text{ converges } \Rightarrow \int_b^\infty f(x)\,dx \text{ converges}$$

for any real a and b.

- To decide on whether

$$\int_1^\infty e^{-x^2}\,dx$$

converges we note on $[1,\infty)$ that $x \le x^2$ so that $-x^2 \le -x$ and hence

$$0 \le e^{-x^2} \le e^{-x}.$$

But

$$\int_1^\infty e^{-x}\,dx = \lim_{T\to\infty}\left[-e^{-x}\right]_1^T = \lim_{T\to\infty}\left(e^{-1}-e^{-T}\right) = \frac{1}{e}.$$

Thus

$$\int_1^\infty e^{-x}\,dx \text{ converges } \Rightarrow \int_1^\infty e^{-x^2}\,dx \text{ converges.}$$

- We may use the previous result to establish that

$$\int_0^\infty e^{-x^2}\,dx$$

converges. This is seen by writing

$$\int_0^\infty e^{-x^2}\,dx = \lim_{T\to\infty}\int_0^T e^{-x^2}\,dx = \lim_{T\to\infty}\left(\int_0^1 e^{-x^2}\,dx + \int_1^T e^{-x^2}\,dx\right)$$
$$= \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx.$$

Problem 7.15: Use the fact that $\displaystyle\int_1^\infty \frac{1}{e^x}\,dx$ converges to show that $\displaystyle\int_1^\infty \frac{1}{x+e^x}\,dx$ converges.

Remark: Integration may thus be extended to bounded functions of x that converge to zero sufficiently fast as $x \to \infty$. We now see that it is even possible to extend our notion of improper Riemann integration to certain unbounded functions.

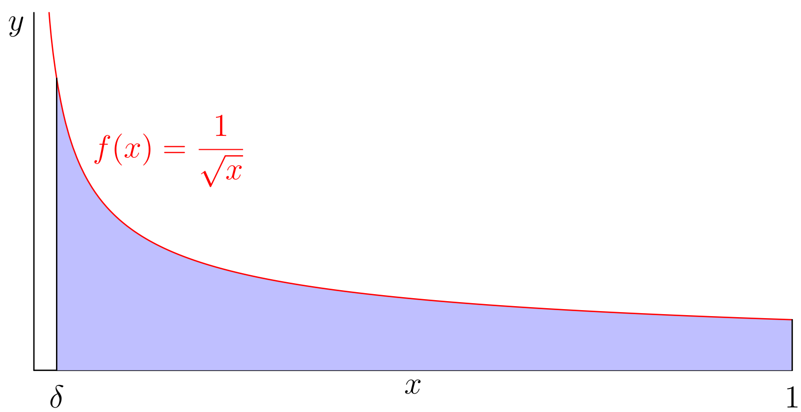
Definition: If f is integrable on $[a,t]$ for all $t \in (a,b)$ we define

$$\int_a^{b^-} f = \lim_{t\to b^-} \int_a^t f.$$

We say that $\int_a^{b^-} f$ *converges* if the limit exists; otherwise it *diverges*. Similarly, we define

$$\int_{a^+}^{b} f = \lim_{t\to a^+} \int_t^b f$$

if f is integrable on $[t,b]$ for all $t \in (a,b)$.



- Let

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{if } 0 < x \le 1, \\ 0 & \text{if } x = 0. \end{cases}$$

According to the strict definition of the Riemann integral, $\int_0^1 f$ does not exist since f is not bounded. However, the improper integral $\int_{0^+}^1 f$ does exist:

$$\int_{0^+}^1 f = \lim_{t\to 0^+} \int_t^1 f = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \left[2x^{\frac{1}{2}}\right]_t^1 = \lim_{t\to 0^+} \left(2 - 2t^{\frac{1}{2}}\right) = 2.$$

Remark: If f is Riemann integrable on $[a,b]$, then

$$\int_a^{b^-} f = \int_a^b f = \int_{a^+}^b f.$$

- For which values of p does $\int_{0^+}^{\infty} x^{-p}\,dx$ exist? Consider

$$\int_{0^+}^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \begin{cases} \left[\dfrac{x^{1-p}}{1-p}\right]_t^1 & \text{if } p \neq 1, \\ [\log|x|]_t^1 & \text{if } p = 1. \end{cases}$$

But $\lim_{t\to 0^+} t^{1-p}$ exists only when $p \le 1$. Also, $\lim_{t\to 0^+} \log t = -\infty$. Thus

$$\int_{0^+}^{\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{1-p} & \text{if } p < 1, \\ \infty & \text{if } p \ge 1. \end{cases}$$

- For which values of p is $\int_{0^+}^{\infty} \frac{1}{x^p}\,dx$ convergent?

Since

$$\int_{0^+}^1 \frac{1}{x^p}\,dx \text{ diverges for } p \ge 1, \qquad \int_1^{\infty} \frac{1}{x^p}\,dx \text{ diverges for } p \le 1,$$

we see that

$$\int_{0^+}^{\infty} \frac{1}{x^p}\,dx \text{ diverges} \qquad \text{for all } p.$$

Problem 7.16: Use the fact that $\int_{0^+}^1 \frac{1}{x}\,dx$ diverges to show that $\int_{0^+}^1 \frac{1}{x\sin^2 x}\,dx$ diverges.

Problem 7.17: Use the fact that $\int_{0^+}^1 \frac{1}{\sqrt{x}}\,dx$ converges to show that $\int_{0^+}^1 \frac{e^{-x^2}}{\sqrt{x}}\,dx$ converges.

Definition: Let f be a function that is integrable on every finite interval $[c,d]$ of $\mathbb{R}$. If the improper integrals

$$\int_{-\infty}^a f(x)\,dx \qquad \text{and} \qquad \int_a^{\infty} f(x)\,dx$$

both converge for some $a \in \mathbb{R}$, then we say that the improper interval

$$\int_{-\infty}^{\infty} f(x)\,dx \doteq \int_{-\infty}^a f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

converges.

Problem 7.18: Show that if $\int_{-\infty}^{\infty} f(x)\,dx$ exists for one $a \in \mathbb{R}$, it will exist for all $a \in \mathbb{R}$ and its value will not depend on the choice of a.

Remark: We cannot simplify the previous definition to

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{T\to\infty} \int_{-T}^{T} f(x)\,dx.$$

For example, while

$$\lim_{T\to\infty} \int_{-T}^{T} x\,dx = \lim_{T\to\infty} 0 = 0,$$

the improper integrals

$$\int_{-\infty}^{a} x\,dx \qquad \text{and} \qquad \int_{a}^{\infty} x\,dx$$

do not converge for any $a \in \mathbb{R}$. That is, $\int_{-\infty}^{\infty} x$ diverges.

Remark: However, if $\int_{-\infty}^{\infty} f$ exists then

$$\lim_{T\to\infty} \int_{-T}^{T} f = \int_{-\infty}^{\infty} f$$

since, by the properties of limits,

$$\int_{-\infty}^{\infty} f = \lim_{T\to\infty} \int_{-T}^{a} f + \lim_{T\to\infty} \int_{a}^{T} f = \lim_{T\to\infty} \left[\int_{-T}^{a} f + \int_{a}^{T} f \right] = \lim_{T\to\infty} \int_{-T}^{T} f.$$

- We thus see that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{0} e^{-x^2}\,dx + \int_{0}^{\infty} e^{-x^2}\,dx = 2\int_{0}^{\infty} e^{-x^2}\,dx$$

converges.

Problem 7.19: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Problem 7.20: Evaluate $\displaystyle\int_{-\infty}^{0} xe^{x}\,dx$.

Convergent	Divergent
$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx \quad (p>1)$	$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx \quad (p\le 1)$
$\displaystyle\int_0^\infty e^{-\alpha x}\,dx \quad (\alpha>0)$	$\displaystyle\int_0^\infty e^{-\alpha x}\,dx \quad (\alpha\le 0)$
$\displaystyle\int_{0^+}^1 \frac{1}{x^p}\,dx \quad (p<1)$	$\displaystyle\int_{0^+}^1 \frac{1}{x^p}\,dx \quad (p\ge 1)$
$\displaystyle\int_{0^+}^1 \log x\,dx$	$\displaystyle\int_0^{\pi/2^-} \tan x\,dx$

Table 7.3: Useful integrals for Comparison Test.

Definition: Let f be defined and continuous everywhere on an interval $[a,b]$ except possibly at a point $c\in[a,b]$. If f is unbounded on $[a,b]$ we know that the Riemann integral of f on $[a,b]$ does not exist. Nevertheless, it is sometimes convenient to define the *improper integral*

$$\int_a^b f \doteq \lim_{t\to c^-}\int_a^t f + \lim_{t\to c^+}\int_t^b f.$$

If both limits exist, we say that the improper Riemann integral $\int_a^b f$ converges.

Remark: Before blindly applying the Fundamental Theorem of Calculus to an integral, it is important to check first whether the integrand is bounded. If the integrand is unbounded at some point within the interval of integration, one must split the integral up into two improper integrals and check that **both** pieces convergence. For example, the integral

$$\int_0^3 \frac{1}{x-1}\,dx$$

does **not** evaluate to $[\log|x-1|]_0^3 = \log 2 - \log 1 = \log 2$ because the improper integrals $\int_0^{1-} \frac{1}{x-1}\,dx$ and $\int_{1+}^3 \frac{1}{x-1}\,dx$ do not converge.

Problem 7.21: Do the following improper integrals converge or diverge? Evaluate those that converge.

(a)

$$\int_{-2}^2 \frac{1}{(x-1)^{2/3}}\,dx.$$

(b)

$$\int_{-2}^{2} \frac{1}{(x-1)^{4/3}}\,dx.$$

Problem 7.22: Show that $\int_0^\infty \sin x\,dx$ diverges.

Problem 7.23: Use the Comparison Test to show that

$$\int_{0^+}^{\infty} \frac{2+\sin x}{\sqrt{x}}\,dx$$

diverges.

Chapter 8

Differential Equations

Definition: A differential equation is an equation that contains an unknown function and one or more of its derivatives.

8.1 Modeling with Differential Equations

- A simple model for the growth of a population of y individuals is the *first-order linear differential equation*

$$y' = ay.$$

Here $y = y(t)$ is a function of time t, a is the *growth rate*, and y' denotes the derivative of y with respect to t. One can solve this equation as follows. On rearranging $\frac{dy}{dt} = ay$ and integrating both sides we find

$$\int \frac{dy}{y} = \int a\,dt.$$

On noting that the population y is always non-negative, we see that

$$\log y = at + C.$$

Thus

$$y = e^{at+C} = Ae^{at},$$

represents a *family of solutions* to the differential equation for any constant $A = e^C$. The particular solution with *initial condition* $y(0) = y_0$ corresponds to the choice $A = y_0$.

- A better model that takes into account finite resources is the *logistic equation*

$$y' = ay\Big(1 - \frac{y}{M}\Big).$$

The additional term causes the population to decrease if it ever exceeds the *carrying capacity* M. Again, we can solve this differential equation by writing

$$\int \frac{dy}{y(1-\frac{1}{M}y)} = \int a\,dt$$

and integrating the left-hand-side using the method of partial fractions.

- Higher-order differential equations arise frequently in science and engineering. One example is the equation for the position x of a mass m attached to a spring with spring constant k:

$$m\frac{d^2x}{dt^2} = -kx.$$

Definition: An *initial-value problem* is an nth-order differential equation together with given initial conditions for $y, y', \ldots, y^{(n-1)}$.

Remark: There is unfortunately no systematic technique that enables us to solve a general first-order nonlinear differential equation of the form

$$\frac{dy}{dt} = f(t, y)$$

or a higher-order differential equation of the form

$$\frac{d^n y}{dt^n} = f(t, y, y', \ldots, y^{(n-1)}).$$

Remark: Differential equations may also be formulated using derivatives with respect to space instead of time.

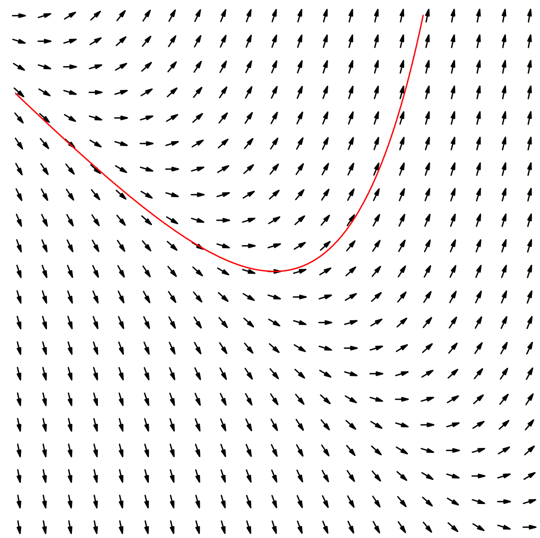
8.2 Direction Fields and Euler's Method

When an analytic solution to a differential equation is not available, it is still often possible to find use graphical or numerical techniques to understand the qualitative behaviour of the solution.

Suppose we wish to sketch the solution to the *initial-value problem*

$$\frac{dy}{dx} = x + y \qquad y(0) = 1.$$

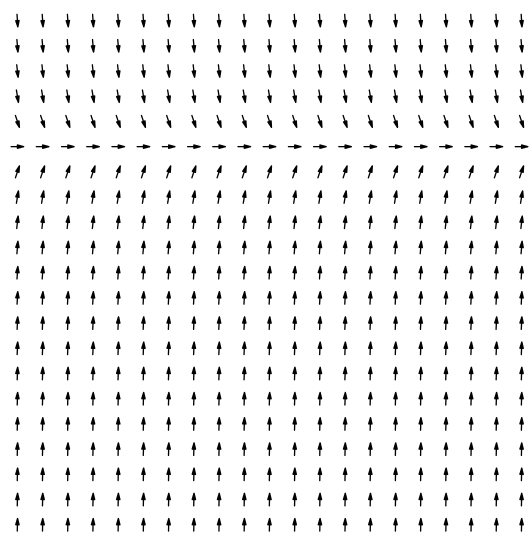
This states that the slope of the solution at a point (x, y) on the curve is given by the sum of the coordinates $x + y$. One point on the solution curve is $(0, 1)$ and it must have slope 1 there. Since y is differentiable, with a continuous derivative, the slope of the curve will be approximately 1 near $(0, 1)$ as well. That is, near $(0, 1)$ the solution looks like a line segment with slope 1.



Definition: The *direction field* for a first-order differential equation is a sketch showing line segments drawn with the slope of the corresponding solution curve at many points (x, y).

Problem 8.1: Plot the direction field for the differential equation

$$\frac{dy}{dx} = 15 - 3y.$$



Remark: The solution of a first-order differential equation through a particular initial condition will locally follow the slopes indicated in the direction field.

Remark: The direction field forms the basis of *Euler's* Method for obtaining an approximate solution (x_n, y_n) to a first-order differential equation $dy/dx = f(x, y)$:

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}),$$

where h is the *step size* and $x_n = x_0 + nh$.

8.3 Separable Differential Equations

A *separable* equation is a first-order differential equation where the expression for dy/dx can be factored as a product of a functions of x and y:

$$\frac{dy}{dx} = f(x)g(y).$$

Separable equations are easily solved by isolating each variable to a separate side of the equation and integrating:

$$\int \frac{dy}{g(y)} = \int f(x)\, dx.$$

- To solve the separable initial value problem

$$\frac{dy}{dx} = \frac{x^2}{y}, \qquad y(0) = 1,$$

we integrate both sides of the re-arranged equation:

$$\int y\, dy = \int x^2\, dx.$$

Thus

$$\frac{y^2}{2} = \frac{x^3}{3} + C,$$

so that

$$y(x) = \sqrt{\frac{2}{3}x^3 + K},$$

for some constant $K = 2C$. Here we take the positive square root since y is initially positive and hence $dy/dx \geq 0$ for all $x \geq 0$. On setting $1 = y(0) = \sqrt{\frac{2}{3}x^3 + K}$ we see that $K = 1$. Thus, the solution to the given initial value problem is $y(x) = \sqrt{\frac{2}{3}x^3 + 1}$.

8.4 Linear Differential Equations

A first-order *linear differential equation* on an interval I can be expressed in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{8.1}$$

where the given functions P and Q are continuous on I.

- The differential equation

  $$xy' + y = 2x$$

  is linear on $(0, \infty)$ since it can be expressed as

  $$y' + \frac{1}{x}y = 2.$$

  Although it is not separable, we can still solve it by rewriting it as

  $$\frac{d}{dx}(xy) = 2x,$$

  so that

  $$xy = x^2 + C.$$

  The solution is then $y = x + C/x$ for some arbitrary constant C.

- The general first-order linear differential equation Eq. (8.1) may be solved in a similar manner. We need to express it in the form

  $$I\left(\frac{dy}{dx} + Py\right) = IQ,$$

  where the *integrating factor* $I(x)$ is chosen so that

  $$I\left(\frac{dy}{dx} + Py\right) = \frac{d}{dx}(Iy) = \frac{dI}{dx}y + I\frac{dy}{dx}.$$

  This requires that

  $$IP = \frac{dI}{dx}.$$

  Although the original differential equation is not in general separable, the equation for I is:

  $$\int \frac{dI}{I} = \int P\,dx,$$

  from which we see that

  $$\log|I| + C = \int P(x)\,dx,$$

so that $I = Ae^{\int P(x)\,dx}$, where $A = e^{-C}$ is an arbitary real constant. For simplicity we choose $A = 1$. The integrating factor I allows the original equation to be expressed as

$$\frac{d}{dx}(Iy) = IQ,$$

which has the solution

$$y = \frac{1}{I}\int IQ.$$

Problem 8.2: Solve

$$\frac{dy}{dx} + 3x^2 y = 6x^2.$$

Problem 8.3: Solve

$$x^2\frac{dy}{dx} + xy = 1.$$

- In the special case where P and Q in Eq. (8.1) are constant, the integrating factor becomes $e^{\int P\,dx} = e^{Px}$. On multiplying each side of $\dfrac{dy}{dx} + Py = Q$ by e^{Px}, we find

$$\frac{d}{dx}\big(e^{Px}y\big) = e^{Px}\frac{dy}{dx} + e^{Px}Py = Qe^{Px}.$$

Thus

$$e^{Px}y = Q\int e^{Px}\,dx = \frac{Q}{P}e^{Px} + C,$$

so that

$$y = \frac{Q}{P} + Ce^{-Px}.$$

Given the initial condition $y(0) = 0$ we then find that

$$0 = \frac{Q}{P} + C,$$

so that $C = -Q/P$. The particular solution of the differential equation corresponding to this initial condition is then

$$y(x) = \frac{Q}{P}\big(1 - e^{-Px}\big).$$

It is useful to check that this solution exhibits behaviours that we can see from the initial equation. For example, for $P > 0$ we see that as $\lim_{x\to\infty} y(x) = Q/P$. That is, $y(x)$ has a horizontal asymptote of Q/P; with limiting slope $\lim_{x\to\infty} dy/dx = 0$.

This is consistent with the steady-state solution $y = Q/P$ deduced from the original differential equation

$$\frac{dy}{dx} + Py = Q.$$

For small x one can also check, on expanding $e^{-Px} = 1 - Px + \frac{1}{2}(-Px)^2 + \ldots \approx 1 - Px$, that the behaviour of the solution is

$$y(x) = \frac{Q}{P}\left(1 - e^{-Px}\right) \approx \frac{Q}{P}(1 - (1 - Px)) = Qx.$$

That, the solution for small x is approximately linear, with slope Q. To see that this is consistent with the differential equation, first note since $y(0) = 0$ and y is continuous that y is small when x is small, so that the term Py in the differential equation may be neglected:

$$\frac{dy}{dx} \approx Q.$$

Chapter 9

Infinite Sequences and Series

9.1 Infinite Series

Definition: Consider the infinite sequence $\{S_n\}_{n=1}^{\infty}$ with elements $S_n = \sum_{k=1}^{n} a_k$. If $\lim_{n\to\infty} S_n$ exists and equals a real number S, we say that the *infinite series*

$$\sum_{k=1}^{\infty} a_k$$

converges, with sum S. Otherwise, we say $\sum_{k=1}^{\infty} a_k$ is *divergent*.

Definition: The finite sum $S_n = \sum_{k=1}^{n} a_k$ is a *partial sum* of the series $\sum_{k=1}^{\infty} a_k$.

Problem 9.1: Prove that the *geometric series* $\sum_{k=0}^{\infty} r^k$ converges if and only if $|r| < 1$, with sum $1/(1-r)$. Hint: Show that

$$S_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

by considering the telescoping sum $rS_n - S_n$.

- The sequence $\{S_n\}_{n=0}^{\infty}$ of partial sums

$$\begin{aligned} S_n &= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} \\ &= \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} \\ &= 2\left(1 - \frac{1}{2^{n+1}}\right) \\ &= 2 - \frac{1}{2^n} \end{aligned}$$

is thus seen to converge to the limit 2 as $n \to \infty$. We write

- Consider

$$0.\bar{4} = 0.444\ldots = \lim_{n\to\infty} \sum_{k=1}^{n} 4\left(\frac{1}{10^k}\right) = \left(\frac{4}{10}\right) \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{1}{10}\right)^k = \left(\frac{4}{10}\right) \frac{1}{1-\frac{1}{10}} = \frac{4}{9}.$$

Problem 9.2: Find all x such that $\sum_{k=1}^{\infty} \frac{x^k}{2^k}$ converges and evaluate the sum.

This is just a geometric series with ratio $x/2$, so we expect convergence for $|x/2| < 1$ or in other words for $x \in (-2, 2)$. For such x, the series converges to

$$\sum_{k=1}^{\infty} \left(\frac{x}{2}\right)^k = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{k+1} = \left(\frac{x}{2}\right) \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = \frac{x/2}{1-x/2} = \frac{x}{2-x}.$$

Remark: It is sometimes not immediately obvious whether a series converges or diverges, but very slowly. A good example is the *harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$. On examing successive partial sums $S_{100} = 5.19$, $S_{200} = 5.87$, $S_{300} = 6.28$, ... it may at first that the series converges. In fact, the harmonic series diverges. An easy way to see this is to note for $n \geq 1$ that

$$\begin{aligned} S_{2n} - S_n &= \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n-1} + \frac{1}{2n} \\ &\geq \underbrace{\frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n} + \frac{1}{2n}}_{n \text{ terms}} = \frac{n}{2n} = \frac{1}{2}. \end{aligned}$$

Thus, if we take $n = 2^m$ for some integer m then $S_{2^{m+1}} - S_{2^m} \geq 1/2$. So every time we multiply n by 2, the corresponding partial sum grows by at least $1/2$. That is, $S_{2^m} \geq \frac{m}{2}$. This means that $\lim_{m\to\infty} S_{2^m} = \infty$.

- However,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

converges to the value 1 since

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} = 1 - \frac{1}{n+1}.$$

Problem 9.3: Let $a > 0$. Evaluate

$$\sum_{k=0}^{\infty} \frac{1}{(a+k)(a+k+1)}.$$

We can compute the partial sums using partial fraction decomposition:

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{(a+k)(a+k+1)} &= \sum_{k=0}^{n} \frac{1}{a+k} - \sum_{k=0}^{n} \frac{1}{a+k+1} \\
&= \sum_{k=0}^{n} \frac{1}{a+k} - \sum_{k=1}^{n+1} \frac{1}{a+k} \\
&= \frac{1}{a} - \frac{1}{a+n+1}.
\end{aligned}$$

As $n \to \infty$, the sum converges to $1/a$.

The following test is sometimes useful for showing that a sequence cannot possibly converge.

Theorem 9.1 (Divergence Test): *If* $\sum_{k=1}^{\infty} a_k$ *converges then* $\lim_{n\to\infty} a_n = 0$.

Proof: In terms of the convergent sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ we may express

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_{n-1} - \lim_{n\to\infty} S_n = 0.$$

- The Divergence Test shows that $\sum_{k=1}^{\infty} \frac{k}{k+1}$ and $\sum_{k=1}^{\infty} \frac{k}{\log k}$ diverge.

Remark: The contrapositive of Theorem 9.1 states

$$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

However,

$$\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges.}$$

For example, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges even though $\lim_{k\to\infty} \frac{1}{k} = 0$.

9.2 The Integral Test

The next theorem, illustrated in Fig. 9.1, sheds some light on why $\sum_{k=1}^{\infty} \frac{1}{k}$ and $\int_1^{\infty} \frac{dx}{x}$ both diverge and on why $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ and $\int_1^{\infty} \frac{dx}{x(x+1)}$ both converge.

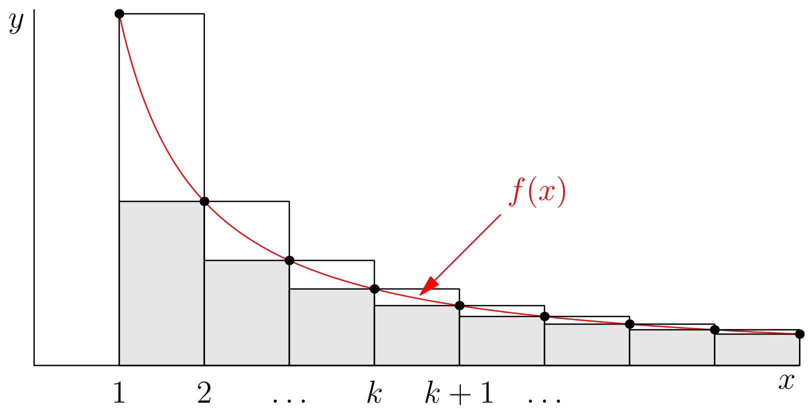


Figure 9.1: The Integral Test.

Theorem 9.2 (Integral Test): *Let f be integrable on any closed interval and decreasing and non-negative on $[1, \infty)$.*

$$\sum_{k=1}^{\infty} f(k) \text{ converges} \iff \int_1^{\infty} f \text{ converges.}$$

Proof: For $x \in [k, k+1]$ we have $f(k) \geq f(x) \geq f(k+1)$. On integrating both sides of these two inequalities we find

$$f(k) \cdot 1 \geq \int_k^{k+1} f(x)\, dx \geq f(k+1) \cdot 1.$$

We then sum this result $k = 1$ to $k = n$ to obtain

$$S_n \doteq \sum_{k=1}^{n} f(k) \geq \int_1^{n+1} f \geq \sum_{k=1}^{n} f(k+1) = \sum_{k=2}^{n+1} f(k) = S_{n+1} - f(1).$$

"$\Rightarrow$" Since f is non-negative we know that $\int_1^T f$ is an increasing function of T. Thus

$$\lim_{n\to\infty} S_n \text{ exists } \Rightarrow \lim_{n\to\infty} \int_1^{n+1} f \text{ exists } \Rightarrow \lim_{T\to\infty} \int_1^{T} f \text{ exists.}$$

"$\Leftarrow$"

$$\lim_{T\to\infty}\int_1^T f \text{ exists } \Rightarrow \lim_{n\to\infty}\int_1^{n+1} f \text{ exists } \Rightarrow \{S_{n+1}\}_{n=1}^{\infty} \text{ bounded} \Rightarrow \{S_n\}_{n=1}^{\infty} \text{ bounded.}$$

But $f(x) \geq 0 \Rightarrow \{S_n\}_{n=1}^{\infty}$ is increasing. The partial sums $S_n = \displaystyle\sum_{k=1}^{n} f(k)$ thus form a bounded increasing (and therefore convergent) sequence. That is, $\displaystyle\sum_{k=1}^{\infty} f(k)$ converges.

Problem 9.4: Use the Integral Test to show that

$$\sum_{k=1}^{\infty}\frac{1}{k^p}$$

converges if $p > 1$.

For $p > 1$, the integrable function $f(x) = 1/x^p$ is decreasing and non-negative. Moreover, we have seen that the improper integral

$$\int_1^{\infty}\frac{1}{x^p}\,dx,$$

converges when $p > 1$.

By the Integral Test, we therefore know that

$$\sum_{k=1}^{\infty}\frac{1}{k^p}$$

converges only when $p > 1$.

Problem 9.5: Use the Integral Test to show that

$$\sum_{k=2}^{\infty}\frac{1}{k\log k}$$

diverges.

We first consider the improper integral

$$\int_2^{\infty}\frac{1}{x\log x}\,dx.$$

On letting $u = \log x$, the integral becomes

$$\int_{\log 2}^{\infty}\frac{1}{u}\,du,$$

which diverges. Noting that the Riemann-integrable function $f(u) = 1/u$ is decreasing and non-negative, the Integral Test tells us that

$$\sum_{k=1}^{\infty} \frac{1}{k \log k}$$

diverges as well.

Remark: While the Integral Test is useful for establishing the convergence of a series, it does not tell us anything about its value. For example, $\int_1^\infty \frac{1}{x^2}\,dx = 1$ but it can be shown that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6$. Often a closed-form expression for a series is unavailable and one must resort to numerical computation of the partial sums up to a certain value of n. The following related theorem can be used to estimate the error in such approximations.

Theorem 9.3 (Remainder Estimate): *Let f be integrable on any closed interval and decreasing and non-negative on $[1, \infty)$. Then the remainder $\sum_{k=n+1}^{\infty} f(k)$ of $\sum_{k=1}^{\infty} f(k)$ that results on truncating the series after n terms satisfies*

$$\int_{n+1}^{\infty} f \le \sum_{k=n+1}^{\infty} f(k) \le \int_n^\infty f.$$

Proof: In the proof of the Integral Test we saw that

$$f(k+1) \le \int_k^{k+1} f$$

On summing from $k = n$ to ∞ we thus find that

$$\sum_{k=n+1}^{\infty} f(k) = \sum_{k=n}^{\infty} f(k+1) \le \sum_{k=n}^{\infty} \int_k^{k+1} f = \int_n^\infty f.$$

We also saw that

$$\int_k^{k+1} f \le f(k).$$

On summing from $k = n+1$ to ∞ we obtain

$$\int_{n+1}^{\infty} f = \sum_{k=n+1}^{\infty} \int_k^{k+1} f \le \sum_{k=n+1}^{\infty} f(k).$$

- The partial sum $S_{10} = \sum_{k=1}^{10} \frac{1}{k^2} \approx 1.5498$ underestimates $\sum_{k=1}^{\infty} \frac{1}{k^2}$ by a remainder that lies between
$$\int_{11}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{11}$$
and
$$\int_{10}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{10}.$$
Indeed, we see that the difference between the exact value $\sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6$ and S_{10} is approximately 0.095, which indeed lies between 1/11 and 1/10.

9.3 Comparison Tests

We have seen that $\sum_{k=1}^{\infty} \frac{1}{2^k}$ is convergent. What about $\sum_{k=1}^{\infty} \frac{1}{2^k+1}$? Since $0 \le \frac{1}{2^k+1} < \frac{1}{2^k}$, the partial sums of $\sum_{k=1}^{\infty} \frac{1}{2^k+1}$ form a bounded increasing sequence and hence must converge. The following theorem formalizes this idea.

Theorem 9.4 (Comparison Test): *If $0 \le a_k \le b_k$ for all $k \in \mathbb{N}$ then*

(i) $\sum_{k=1}^{\infty} b_k$ *converges* $\Rightarrow \sum_{k=1}^{\infty} a_k$ *converges;*

(ii) $\sum_{k=1}^{\infty} a_k$ *diverges* $\Rightarrow \sum_{k=1}^{\infty} b_k$ *diverges.*

Proof:

(i) Let $S_n = \sum_{k=1}^{n} a_k$ and $T_n = \sum_{k=1}^{n} b_k$. Since $0 \le S_n \le T_n$,
$$\sum_{k=1}^{\infty} b_k \text{ converges } \Rightarrow \{T_n\}_{n=1}^{\infty} \text{ bounded } \Rightarrow \{S_n\}_{n=1}^{\infty} \text{ bounded } \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges.}$$

(ii) This is just the contrapositive of (i).

Remark: The condition "$0 \le a_k$" in Theorem 9.4 cannot be dropped. Consider the counterexample given by $a_k = -1$, $b_k = 0$.

- We have seen from the Integral Test that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. Use this result to show that $\sum_{k=1}^{\infty} \frac{1}{k^2+3k+1}$ converges.

Remark: In applying the Comparison Test, we may replace the condition $k \in \mathbb{N}$ with $k \geq N$ for any given natural number N (only the long-term behaviour affects convergence).

- Show that $\sum_{k=1}^{\infty} \frac{\log k}{k}$ diverges by using the fact that $\log k > 1$ for $k \geq 3$.

Problem 9.6: Use the Comparison Test to deduce directly from the convergence of $\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$ that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

We note for $k \geq 1$ that $2k^2 \geq k^2 + k = k(k+1)$. Thus

$$\frac{1}{k^2} \leq \frac{2}{k(k+1)}.$$

The following test provides a more convenient way to establish the previous result.

Theorem 9.5 (Limit Comparison Test): *Suppose $a_k \geq 0$ and $b_k > 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} a_k/b_k = L$. Then*

(i) if $0 < L < \infty$: $$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \sum_{k=1}^{\infty} b_k \text{ converges;}$$

(ii) if $L = 0$: $$\sum_{k=1}^{\infty} b_k \text{ converges} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges.}$$

Proof:

(i) This follows from Theorem 9.4 since for all sufficiently large k,

$$0 < \frac{L}{2} < \frac{a_k}{b_k} < \frac{3L}{2} \Rightarrow 0 < \left(\frac{L}{2}\right) b_k < a_k < \left(\frac{3L}{2}\right) b_k.$$

(ii) If $L = 0$ then for sufficiently large k, $0 \leq a_k/b_k < \epsilon = 1 \Rightarrow 0 \leq a_k < b_k$. Apply Theorem 9.4.

- Since
$$\lim_{k\to\infty} \frac{k^2}{k(k+1)} = 1,$$
we see immediately (without using the Integral Test) that
$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \text{ converges } \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges.}$$

Remark: When $\lim\limits_{k\to\infty} a_k/b_k = 0$, it is possible that $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} b_k$ diverges. Consider
$$a_k = \frac{1}{k(k+1)}, \qquad b_k = \frac{1}{k}, \qquad \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

- Use the Limit Comparison Test to show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k - 1}$ converges.

9.4 Alternating Series

So far we have only considered series with positive terms. Let us now discuss an important class of series consisting of terms that alternate in sign.

Definition: An *alternating series* is of the form
$$\sum_{k=1}^{\infty} (-1)^k a_k,$$
where $a_k \geq 0$ for $k \in N$.

- The sum $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is an alternating series that, in contrast to the harmonic series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, converges. The changes in sign prevent the slow accumulation exhibited by the nonalternating version and instead leads to oscillation of the partial sums. Because the magnitude of each term tends to zero, these oscillations are eventually damped out. The following theorem formalizes this notion.

Theorem 9.6 (Leibniz Alternating Series Test): *The alternating series* $\displaystyle\sum_{k=1}^{\infty} (-1)^k a_k$ *is convergent if the sequence* $\{a_k\}_{k=1}^{\infty}$ *decreases monotonically to* 0.

Proof: Consider the even partial sums

$$S_{2n} = -a_1 + (a_2 - a_3) + \ldots + (a_{2n-2} - a_{2n-1}) + a_{2n} \geq -a_1,$$

noting that $a_n - a_{n+1} \geq 0$ for all $n \in \mathbb{N}$. Notice also that

$$S_{2n+2} - S_{2n} = a_{2n+2} - a_{2n+1} \leq 0.$$

Hence $\{S_{2n}\}_{n=1}^{\infty}$ is a bounded decreasing, and therefore convergent, sequence. In contrast, $\{S_{2n-1}\}_{n=1}^{\infty}$ is a bounded increasing sequence since

$$S_{2n+1} - S_{2n-1} = -a_{2n+1} + a_{2n} \geq 0.$$

Finally,

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} (S_{2n-1} + a_{2n}) = \lim_{n\to\infty} S_{2n-1} + \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} S_{2n-1};$$

that is, the subsequences of even and odd partial sums converge to the same limit. The full sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ therefore converges to this limit.

- The alternating series $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{k^2}{k^3+1}$ converges since the function $f(x) = \dfrac{x^2}{x^3+1}$ decreases monotonically to zero on $[2, \infty)$.

- The alternating series $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{3k}{4k-1}$ does not converge by the Divergence Test since $\displaystyle\lim_{k\to\infty} \frac{(-1)^k 3k}{4k-1} = \frac{3}{4} \neq 0$.

Remark: The next theorem establishes that if the magnitude of the terms of an alternating series decreases monotonically to zero, the truncation error is no greater than the magnitude of the very next term!

Theorem 9.7 (Alternating Series Remainder Estimate): *Let $\{a_k\}_{k=1}^{\infty}$ be a monotonically decreasing sequence that converges to* 0*. Then*

$$\left|\sum_{k=n+1}^{\infty} (-1)^k a_k\right| \leq a_{n+1}.$$

Proof: Let $S_n = \sum_{k=1}^{n}(-1)^k a_k$ and $S = \sum_{k=1}^{\infty}(-1)^k a_k$. Since $\{S_{2n-1}\}_{n=1}^{\infty}$ is increasing and $\{S_{2n}\}_{n=1}^{\infty}$ is decreasing, we see that $S_{2n-1} \leq S \leq S_{2n}$ for all $n \geq 1$. Since $a_n - a_{n+1} \geq 0$, we then find that

$$0 \leq S - S_{2n-1} = \sum_{k=2n}^{\infty} (-1)^k a_k = a_{2n} - (a_{2n+1} - a_{2n+2}) + \ldots \leq a_{2n}$$

and

$$0 \le S_{2n} - S = -\sum_{k=2n+1}^{\infty} (-1)^k a_k = a_{2n+1} - (a_{2n+2} - a_{2n+3}) + \ldots \le a_{2n+1}.$$

For either odd or even n the error in approximating S by S_n is seen to be less than the magnitude of the first neglected term:

$$\left| \sum_{k=n+1}^{\infty} (-1)^k a_k \right| = |S - S_n| \le a_{n+1}.$$

Problem 9.7: Evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k!}$ to within 0.001.

If we sum up the first 6 terms we obtain the estimate

$$-1 + 1/2 - 1/6 + 1/24 - 1/120 + 1/720 = -0.6319\bar{4},$$

which overestimates the sum by at most $1/5040 < 0.0002$ (since the next term can only reduce the sum). So the infinite sum evaluates to approximately -0.632.

9.5 Absolute Convergence

Q. What happens when some of the terms a_k are negative but the series isn't in the form of an alternating series?

A. In these situations, the following concept is sometimes helpful.

Definition: A series $\sum_{k=1}^{\infty} a_k$ is *absolutely convergent* if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

- The sequence $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ is absolutely convergent by the Comparison Test since $|\sin k| \le 1$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent. That is, $\displaystyle\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$ is convergent. The following theorem establishes that the original series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ is itself convergent.

Theorem 9.8 (Absolute Convergence): *An absolutely convergent series is convergent.*

Proof: Suppose $\sum_{k=1}^{\infty} |a_k|$ is convergent. Since each term a_k in the series $\sum_{k=1}^{\infty} a_k$ is either $|a_k|$ or $-|a_k|$, we find

$$0 \le a_k + |a_k| \le 2|a_k|$$

On applying the Comparison Test we then see that $\sum_{k=1}^{\infty}(a_k + |a_k|)$ is convergent. The difference $\sum_{k=1}^{\infty} a_k$ between the two convergent series $\sum_{k=1}^{\infty}(a_k + |a_k|)$ and $\sum_{k=1}^{\infty} |a_k|$ is therefore also convergent.

Remark: The converse of Theorem 9.8 need not be true: the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is convergent but not absolutely convergent: the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Definition: A series is *conditionally convergent* if is convergent but not absolutely convergent.

- The alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is conditionally convergent.

Problem 9.8: Show that the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ is conditionally convergent.

Problem 9.9: Show that the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k \log k}{\sqrt{k}}$ is conditionally convergent.

The following three tests are often useful for establishing absolute convergence.

Theorem 9.9 (Ratio Comparison Test): *If $a_k > 0$ and $b_k > 0$ and*

$$\frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k}$$

for all $k \in \mathbb{N}$, then

(i) $\sum_{k=1}^{\infty} b_k$ *converges* $\Rightarrow \sum_{k=1}^{\infty} a_k$ *converges;*

(ii) $\sum_{k=1}^{\infty} a_k$ *diverges* $\Rightarrow \sum_{k=1}^{\infty} b_k$ *diverges.*

Proof: For $k \in \mathbb{N}$ we have

$$\frac{a_{k+1}}{a_k} \le \frac{b_{k+1}}{b_k} \Rightarrow \frac{a_{k+1}}{b_{k+1}} \le \frac{a_k}{b_k} \Rightarrow \frac{a_k}{b_k} \le \frac{a_1}{b_1} \doteq M.$$

Thus $0 < a_k \le M b_k$ for all $k \in \mathbb{N}$ and the result follows from Theorem 9.4.

Theorem 9.10 (Ratio Test): *Suppose* $a_k > 0$.

(i) If there exists a number $r < 1$ *such that* $\frac{a_{k+1}}{a_k} \le r$ *for all* k*, then* $\sum_{k=1}^{\infty} a_k$ *converges;*

(ii) If $\frac{a_{k+1}}{a_k} \ge 1$ *for all* k*, then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Proof: Let $b_k = r^k$. Then $\frac{b_{k+1}}{b_k} = r$ and

$$\begin{cases} \displaystyle\sum_{k=1}^{\infty} b_k \text{ converges} & \text{if } |r| < 1, \\ \displaystyle\sum_{k=1}^{\infty} b_k \text{ diverges} & \text{if } |r| \ge 1. \end{cases}$$

Apply Theorem 9.9.

Theorem 9.11 (Limit Ratio Test): *Suppose* $a_k > 0$ *for all* $k \in \mathbb{N}$ *and*

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r.$$

Then

(i) $r < 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *converges;*

(ii) $r > 1 \Rightarrow \sum_{k=1}^{\infty} a_k$ *diverges;*

(iii) $r = 1 \Rightarrow ?$

Proof: Choose $\epsilon = (1 - r)/2$. Then for sufficiently large k, we know that

$$\frac{a_{k+1}}{a_k} < r + \epsilon = \frac{1+r}{2} < 1.$$

Apply the Ratio Test.

Problem 9.10: Find examples corresponding to each of the three cases in Theorem 9.11.

Theorem 9.12 (Root Test): *Suppose $a_k \geq 0$ for all $k \in \mathbb{N}$. Then*

(i) If there exists a number $r < 1$ such that $\sqrt[k]{a_k} \leq r$ for all k, then $\sum_{k=1}^{\infty} a_k$ converges;

(ii) If $\sqrt[k]{a_k} \geq 1$ for all k, then $\sum_{k=1}^{\infty} a_k$diverges.

Proof: Let $b_k = r^k$. Then

$$\begin{cases} \displaystyle\sum_{k=1}^{\infty} b_k \text{ converges} & \text{if } |r| < 1, \\ \displaystyle\sum_{k=1}^{\infty} b_k \text{ diverges} & \text{if } |r| \geq 1. \end{cases}$$

Note that $\sqrt[k]{a_k} \leq r$ implies $a_k \leq r^k = b_k$ and apply Theorem 9.4.

Theorem 9.13 (Limit Root Test): *Suppose $a_k \geq 0$ for all $k \in \mathbb{N}$ and*

$$\lim_{k\to\infty} \sqrt[k]{a_k} = r.$$

Then

(i) $r < 1 \Rightarrow \displaystyle\sum_{k=1}^{\infty} a_k$ *converges*;

(ii) $r > 1 \Rightarrow \displaystyle\sum_{k=1}^{\infty} a_k$ *diverges*;

(iii) $r = 1 \Rightarrow\ ?$

Proof:

(i) Choose $\epsilon = (1-r)/2$. Then for sufficiently large k, we know that

$$\sqrt[k]{a}_k < r + \epsilon = \frac{1+r}{2} < 1.$$

Apply the Root Test.

(ii) If $\lim_{k\to\infty} \sqrt[k]{a}_k > 1$ then for sufficiently large k we know that $a_k > 1$. The Divergence Test then implies that the series diverges.

Problem 9.11: Find examples corresponding to each of the three cases in Theorem 9.13.

(i) The series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges.

(ii) The series $\sum_{k=1}^{\infty} 2^k$ diverges.

(iii) For the series $\sum_{k=1}^{\infty} k^p$ we see with the help of L'Hôpital's Rule that

$$\lim_{k\to\infty} \sqrt[k]{k^p} = \lim_{k\to\infty} k^{p/k} = \lim_{k\to\infty} k^{p/k} = \lim_{k\to\infty} e^{\frac{p}{k}\log k} = \exp\left(p \lim_{k\to\infty} \frac{\log k}{k}\right) = \exp\left(p \lim_{k\to\infty} \frac{\frac{1}{k}}{1}\right) = \exp 0 = 1.$$

We recall that this series converges for $p = -2$ but diverges $p = -1$.

Problem 9.12: Does $\sum_{k=2}^{\infty} \frac{1}{(\log k)^k}$ converge or diverge?

Since $\lim_{k\to\infty} \frac{1}{\log k} = 0 < 1$, we know from the Root Test that the series converges.

9.6 Strategy

To determine which test to use for a given series, it sometimes helps to consider the behaviour of the terms a_k for large k.

9.7 Power Series

Definition: A *power series* about the *expansion point* a is an infinite series of the form $\sum_{k=0}^{\infty} c_k(x-a)^k$, where the coefficients c_k are independent of x.

Remark: By definition, $(x-a)^0 = 1$ for all x, even at $x = a$.

Remark: When dealing with power series, it is often convenient to shift the variable x so that the expansion point $a = 0$. The power series then simplifies to $\sum_{k=0}^{\infty} c_k x^k$.

Remark: A power series always converges at its expansion point.

- We have seen that the geometric series

$$\sum_{k=0}^{\infty} x^k$$

converges for any $x \in (-1, 1)$ to the function $\dfrac{1}{1-x}$.

- If we apply the Limit Ratio Test to the series

$$\sum_{k=0}^{\infty} k!\, x^k,$$

we see for $x \neq 0$ that

$$\lim_{k\to\infty} \frac{\left|(k+1)!\, x^{k+1}\right|}{\left|k!\, x^k\right|} = \lim_{k\to\infty} (k+1)\,|x| = \infty.$$

Thus, the series converges absolutely only at $x = 0$.

- In contrast, the Limit Ratio Test tells us that the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges absolutely for all real x:

$$\lim_{k\to\infty} \frac{|x|^{k+1}}{(k+1)!} \cdot \frac{k!}{|x|^k} = \lim_{k\to\infty} \frac{|x|}{(k+1)} = 0 < 1.$$

Remark: The following theorem tells us that every power series *converges absolutely strictly inside* some closed interval (which could be a point or all of $\mathbb{R}$) and *diverges strictly outside* that closed interval. It does not say anything about what happens at the endpoints themselves.

Theorem 9.14 (Radius of Convergence): *For each power series $\sum_{k=0}^{\infty} c_k x^k$ there exists a number R, called the radius of convergence, with $0 \le R \le \infty$, such that*

$$\sum_{k=0}^{\infty} c_k x^k \begin{cases} \textit{converges absolutely} & \textit{if } |x| < R, \\ ? & \textit{if } |x| = R, \\ \textit{diverges} & \textit{if } |x| > R. \end{cases}$$

- We have seen that the series $\sum_{k=0}^{\infty} x^k/k!$ has an infinite radius of convergence ($R = \infty$).

Problem 9.13: For what values of x does the series $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k}$ converge?

The Limit Ratio Test tells us that the series converges absolutely if

$$\lim_{k\to\infty} \frac{|x-1|^{k+1}}{(k+1)} \cdot \frac{k}{|x-1|^k} = \lim_{k\to\infty} \frac{k}{(k+1)} |x-1| = |x-1| < 1,$$

that is, when $-1 < x - 1 < 1$ or in other words, $0 < x < 2$. Furthermore, we see that the series converges at $x = 0$ (where it reduces to the alternating harmonic series and diverges at $x = 2$ (where it reduces to the harmonic series). The Limit Ratio Test tell us that the series does not converge absolutely for $|x - 1| > 1$; that is, when $x > 2$ or $x < 0$. The interval of convergence is thus $[0, 2)$ and the radius of convergence is 1.

- Find the radius of convergence R of

$$\sum_{k=2}^{\infty} \frac{x^k}{\log k}.$$

The ratio of consecutive terms has limit

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{\log(k+1)} \cdot \frac{\log k}{x^k} \right| = |x| \lim_{k\to\infty} \frac{\log k}{\log(k+1)} = |x| \lim_{u\to\infty} \frac{\frac{1}{u}}{\frac{1}{u+1}} = |x|\,,$$

using L'Hôpital's Rule, where $u \in \mathbb{R}$. The Limit Ratio Test then implies that $R = 1$.

Remark: To determine the actual interval of convergence, we need to determine R and then test for convergence at $x = a + R$ and $x = a - R$ by other means.

Problem 9.14: Determine the radius of convergence, interval of convergence, and expansion point for the power series

$$\sum_{k=0}^{\infty} \frac{(3x+4)^k}{5^k}.$$

From the ratio test we see that the series converges whenever $\left|\frac{3x+4}{5}\right| < 1$; that is, when $-5 < 3x + 4 < 5$. This corresponds to the interval $(-3, 1/3)$, a radius of convergence of $5/3$, and an expansion point of $-4/3$. Note that the series diverges at both $x = -3$ and $x = 1/3$.

Problem 9.15: Consider the power series $\sum_{k=0}^{\infty} c_k x^k$.

(a) Suppose that $\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right|$ exists. Use the Limit Ratio Test to show that the radius of convergence of the power series is given by

$$R = \frac{1}{\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right|}.$$

If the limit of the ratio of successive terms $\left|\frac{c_{k+1}}{c_k}x\right|$ is less than 1 (i.e. if $|x| < R$) the series $\sum_{k=0}^{\infty} \left|c_k x^k\right|$ converges and if it is bigger than 1 (i.e. if $|x| > R$) the series diverges. Hence R is indeed the radius of convergence.

(b) Suppose that $\lim_{k\to\infty} \sqrt[k]{|c_k|}$ exists. Use the Root Test to show that

$$R = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|c_k|}}$$

is another expression for the radius of convergence.

If $\lim_{k\to\infty} \sqrt[k]{|c_k x^k|}$ is less than 1 (i.e. if $|x| < R$), the series $\sum_{k=0}^{\infty} \left|c_k x^k\right|$ converges, and if it is bigger than 1 (i.e. if $|x| > R$), the series diverges. Hence R is indeed the radius of convergence.

9.8 Representation of Functions as Power Series

The closed form sum of a geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \qquad |x| < 1$$

can be used to sum up other power series.

- On substituting $-x^2$ for x, we find

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \qquad |x| < 1.$$

- On substituting $-x/2$ for x, we find

$$\frac{1}{2+x} = \frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right) = \frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{x}{2}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} x^k, \qquad |x| < 2.$$

- On multiplying $\dfrac{1}{2+x}$ by x^2 we find

$$\frac{x^2}{2+x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} x^{k+2}, \qquad |x| < 2.$$

Theorem 9.15 (Derivative and Integral of a Power Series): *The power series*

$$(i) \qquad \sum_{k=0}^{\infty} c_k x^k,$$

$$(ii) \qquad \sum_{k=0}^{\infty} k c_k x^{k-1},$$

$$(iii) \qquad \sum_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}$$

all have the same radius of convergence.

Proof: For $k \geq |x|$, note that

$$\left|c_k x^k\right| = |x| \left|c_k x^{k-1}\right| \leq \left|k c_k x^{k-1}\right|.$$

We thus see from the Comparison Test that if (ii) converges absolutely, so does (i). On the other hand, suppose (i) converges absolutely at some $x_0 \neq 0$. The Divergence Test implies that $\left|c_k x_0^k\right|$ is bounded by some positive number M for all k. Thus

$$\left|k c_k x^{k-1}\right| = \left|c_k x_0^k\right| \frac{k}{|x_0|} \left|\frac{x}{x_0}\right|^{k-1} \leq \frac{M}{|x_0|} k \left|\frac{x}{x_0}\right|^{k-1}.$$

We know from the Limit Ratio Test that $\displaystyle\sum_{k=0}^{\infty} k \left|\frac{x}{x_0}\right|^{k-1}$ is convergent for $|x| < |x_0|$. The absolute convergence of (ii) then follows from the Comparison Test. On noting that (i) is just the result of formally differentiating (iii), we see that (i) and (iii) also have the same radius of convergence.

9.9 Taylor Series

Let us now discuss a general method for developing power series for any infinitely differentiable function. We first recall the following special case of the Mean Value Theorem:

Theorem 9.16 (Rolle's Theorem): *Suppose*

(i) *f is continuous on $[a, b]$,*

(ii) *f' exists on (a, b),*

(iii) *$f(a) = f(b)$.*

Then there exists a number $c \in (a, b)$ for which $f'(c) = 0$.

We now use this theorem to prove a key result.

Theorem 9.17 (Taylor's Theorem): *Let $n \in \mathbb{N}$. Suppose*

(i) *$f^{(n-1)}$ exists and is continuous on $[a, b]$,*

(ii) *$f^{(n)}$ exists on (a, b).*

Then there exists a number $c \in (a, b)$ such that

$$f(b) = \underbrace{\sum_{k=0}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a)}_{\textit{Taylor polynomial}} + \underbrace{\frac{(b-a)^n}{n!} f^{(n)}(c)}_{R_n}.$$

That is,

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2} f''(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n.$$

Remark: This is known as the *Taylor expansion* of f at b about a to n terms. The term R_n is known as the *remainder* after n terms.

- For $n = 1$:
$$f(b) = \frac{(b-a)^0}{0!} f(a) + \frac{(b-a)^1}{1!} f'(c)$$
i.e. $f(b) = f(a) + (b-a)f'(c)$ (MVT).

- For $n = 2$:
$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2} f''(c).$$

Proof (of Taylor's Theorem): We will apply Rolle's Theorem to

$$\varphi(x) = f(x) + \sum_{k=1}^{n-1} \frac{(b-x)^k}{k!} f^{(k)}(x) + M(b-x)^n,$$

where M is a constant. Noting that $\varphi(b) = f(b)$, we choose M so that $\varphi(a) = f(b)$ also:

$$f(b) = \varphi(a) = f(a) + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a) + M(b-a)^n. \tag{9.1}$$

That is, we choose

$$M = \frac{1}{(b-a)^n}\left[f(b) - f(a) - \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a)\right].$$

Note that $\varphi(x)$ is continuous on $[a,b]$. Using the Chain Rule, we find that

$$\begin{aligned}
\varphi'(x) &= f'(x) + \sum_{k=1}^{n-1}\left[-\frac{(b-x)^{k-1}}{(k-1)!} f^{(k)}(x) + \frac{(b-x)^k}{k!} f^{(k+1)}(x)\right] - n(b-x)^{n-1}M \\
&= f'(x) - \sum_{k=\boxed{1}}^{n-1} \frac{(b-x)^{k-1}}{(k-1)!} f^{(k)}(x) + \sum_{k=2}^{\boxed{n}} \frac{(b-x)^{k-1}}{(k-1)!} f^{(k)}(x) - n(b-x)^{n-1}M \\
&= f'(x) - f'(x) + \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x) - n(b-x)^{n-1}M
\end{aligned}$$

exists for all $x \in (a,b)$. We then apply Rolle's Theorem to deduce that there exists a number $c \in (a,b)$ such that

$$\begin{aligned}
0 = \varphi'(c) &= \frac{(b-c)^{n-1}}{(n-1)!} f^{(n)}(c) - n(b-c)^{n-1}M \\
\Rightarrow M &= \frac{1}{n!} f^{(n)}(c).
\end{aligned}$$

Upon substituting this result into Eq. (9.1), we obtain Taylor's Theorem:

$$f(b) = f(a) + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(c).$$

Remark: If $\left|f^{(n)}(c)\right| \le M$ for all c between b and a, then $|R_n| \le M\,|b-a|^n/n!$ on this same interval. This is sometimes called *Taylor's inequality*. For example, if $|f^{(n)}|$ is an increasing function on $[a,b]$, then Taylor's inequality holds with $M = |f^{(n)}(b)|$, whereas if $|f^{(n)}|$ is a decreasing function on $[a,b]$ one can choose $M = |f^{(n)}(a)|$.

- Compute the first three digits of $\sin 1$ after the decimal point and determine its value correctly rounded to two digits.

Step 1: Let $f(x) = \sin x$. Choose a value a reasonably close to 1 at which the value of f and its derivatives are known, such as $a = 0$.

Step 2: Write down the Taylor expansion to enough terms so that $|R_n|$ is less than or equal to the allowed error. Set $b = x$.

$$\begin{aligned}\sin x &= \sin 0 + (x-0)\cos 0 - \frac{(x-0)^2}{2!}\sin 0 - \frac{(x-0)^3}{3!}\cos 0 + \frac{(x-0)^4}{4!}\sin 0\\ &\quad + \frac{(x-0)^5}{5!}\cos 0 - \frac{(x-0)^6}{6!}\sin 0 + R_7,\\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + R_7,\end{aligned}$$

where $R_7 = -\frac{1}{7!}(x-0)^7\cos c$ for some $c \in (0,x)$. For $x = 1$ we know that $|\cos c| < 1$, so

$$|R_7| < \frac{1}{7!} = \frac{1}{5040} < 0.0002$$

and

$$\sin 1 \approx 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} = 0.841\bar{6}.$$

Hence $\sin 1 \approx 0.841\bar{6} \pm 0.0002$, so the first three digits of $\sin 1$ are 0.841. If we round this result to two digits after the decimal place, we obtain $\sin 1 \approx 0.84$.

Remark: If the magnitude of the terms of an alternating Taylor series decreases monotonically to zero, it is much easier to use the Alternating Series Remainder Estimate rather than explicitly estimating the remainder using Taylor's Theorem: the error is simply less than the magnitude of the very next term!

Definition: If $\lim_{n\to\infty} R_n = 0$ in the Taylor expansion of f at b about a then

$$f(b) = \sum_{k=0}^{\infty} \frac{(b-a)^k}{k!} f^{(k)}(a)$$

This is known as the *Taylor Series* of f at b about a.

Definition: The special case of a Taylor Series about $a = 0$ is sometimes called a *Maclaurin Series*.

Problem 9.16: Suppose that a function f is differentiable on $\mathbb{R}$ and $f'(x) = f(x)$ for all $x \in \mathbb{R}$.

(a) Use induction to prove that the n-th derivative $f^{(n)}(x) = f(x)$ for all $n \in \mathbb{N}$.
We are told that the desired result holds for $n = 1$. Assume that it holds for n. Then

$$f^{(n+1)}(x) = (f^n(x))' = f'(x) = f(x).$$

Hence the result holds for all $n \in \mathbb{N}$.

(b) Let $x \in \mathbb{R}$. Show for any $n \in \mathbb{N}$ that the value of f at a point $x \neq 0$ can be expressed in terms of $f(0)$ and a remainder term R_n,

$$f(x) = f(0)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!}\right) + R_n,$$

where

$$R_n = \frac{f^{(n)}(c_n)}{n!}x^n$$

for some point $c_n \in (0, x)$. (If $x < 0$ then by $(0, x)$ we mean the interval $(x, 0)$. Note that c_n depends on n.)

This is just the Taylor expansion of f at x about the point $a = 0$.

(c) Show that f is bounded on $[0, x]$.

Since f is differentiable on $\mathbb{R}$, it must be continuous on $\mathbb{R}$. Hence f is bounded on any closed interval (and in particular on $[0, x]$).

(d) Using part (c) and the fact that the infinite sum $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all x to establish that $\lim_{n\to\infty} R_n = 0$.

The convergence of $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for any fixed x, which follows from the Limit Ratio Test, implies that $\lim_{n\to 0} \frac{x^n}{n!} = 0$. From part (c) we know that there exists a number M such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. From part (a) we have that

$$|R_n| \leq \left|\frac{f(c_n)}{n!}x^n\right| \leq M\frac{|x|^n}{n!} \underset{n\to\infty}{\to} 0.$$

(e) If $f(0) = 1$ show that

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This is the power series for the exponential function $\exp(x)$.

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{x^k}{k!} = \lim_{n\to\infty} f(x) - \lim_{n\to\infty} R_n = f(x).$$

Problem 9.17: Let $f(x) = \sqrt[3]{1+x}$. (a) Determine the first three terms of the Taylor expansion of $f(x)$ about the point $a = 0$,

$$f(x) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!}x^k + R_3.$$

When $a = 0$, the remainder term R_3 is simply

$$R_3 = \frac{f^{(3)}(c)}{3!}x^3.$$

We find

$$f^{(1)}(x) = \left(\frac{1}{3}\right)(1+x)^{-2/3},$$

$$f^{(2)}(x) = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(1+x)^{-5/3},$$

and

$$f^{(3)}(c) = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(1+c)^{-8/3}.$$

Hence

$$f(x) = 1 + \left(\frac{1}{3}\right)\frac{x}{1!} - \left(\frac{2}{9}\right)\frac{x^2}{2!} + R_3 = 1 + \frac{x}{3} - \frac{x^2}{9} + R_3.$$

(b) Use part (a) to find a lower bound for $\sqrt[3]{\frac{3}{2}}$ and show that your result approximates the exact value to within 1%. (You may leave your answer as a fraction.)

$$\sqrt[3]{\frac{3}{2}} = f\left(\frac{1}{2}\right) = 1 + \left(\frac{1}{3}\right)\frac{(\frac{1}{2})}{1!} - \left(\frac{2}{9}\right)\frac{(\frac{1}{2})^2}{2!} + R_3 = 1 + \frac{1}{6} - \frac{1}{36} + R_3 = \frac{41}{36} + R_3,$$

where

$$R_3 = \frac{(\frac{1}{2})^3}{3!} f^{(3)}(c).$$

for some number $c \in (0, \frac{1}{2})$. The third derivative of f at c can be easily bounded:

$$0 \le f^{(3)}(c) \le \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{5}{3}\right) = \frac{10}{27},$$

so

$$0 \le R_3 \le \frac{(\frac{1}{2})^3}{3!}\left(\frac{10}{27}\right) = \frac{5}{24 \times 27} < \frac{1}{24 \times 5} < \frac{1}{100} < \frac{1}{100}\sqrt[3]{\frac{3}{2}}.$$

Thus $\sqrt[3]{\frac{3}{2}}$ lies in the interval

$$\left[\frac{41}{36}, \frac{41}{36} + \frac{1}{100}\right].$$

Problem 9.18: Find the Taylor series for $f(x) = \sin^2 x$ about $x = 0$. Hint: after computing the first derivative, simplify the result before proceeding to take further derivatives.

Remark: Suppose that the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ converges to $f(x)$ for $|x| < R$. Theorem 9.15 tells us that the term-by-term differentiated series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} k x^{k-1} = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{(k-1)!} x^{k-1} = \sum_{k=0}^{\infty} \frac{f^{(k+1)}(0)}{k!} x^k,$$

which we note is just the Taylor series for f', has the same radius of convergence as the Taylor series for f. This means that we may differentiate (or integrate) a power series term by term within its radius of convergence: if $\sum_{k=0}^{\infty} c_k x^k$ converges to $f(x)$ for $|x| < R$, then $\sum_{k=1}^{\infty} k c_k x^{k-1}$ converges to $f'(x)$ for $|x| < R$.

- For $|x| < 1$, we may differentiate the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

term by term to find that

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \ldots, \qquad |x| < 1.$$

- For $|x| < 1$, we may integrate the geometric series

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k = 1 - x + x^2 - x^3 + \ldots$$

term by term to find

$$\log(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \qquad |x| < 1,$$

where we see that the constant of integration vanishes since both sides evaluate to zero when $x = 0$. While both series converge for $|x| < 1$, notice that the Leibniz Alternating Series Test guarantees that the differentiated series also converges at $x = 1$. That is, the interval of convergence of the differentiated series is $(-1, 1]$. On taking the limit as $x \to 1$, we see from the above closed-form expression that the alternating harmonic series converges to $-\log 2$.

- For $|x| < 1$, we may integrate the geometric series

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-x^2)^k = 1 - x^2 + x^4 - x^6 \ldots$$

term by term to find

$$\tan^{-1} x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots.$$

Again, the constant of integration is seen to vanish (for the principal branch of the arctangent).

- We can use power series to integrate functions that we cannot integrate by elementary means:

$$\int_0^t e^{-x^2}\,dx = \int_0^t \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)k!} = t - \frac{t^3}{3} + \frac{t^5}{10} + \ldots,$$

where we see the constant of integration is seen to be zero.

Remark: If $\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} b_k x^k$ whenever $|x| < R$, on setting $x = 0$ we see that $a_0 = b_0$, so that $\sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} b_k x^k$. On differentiating each side with respect to x and again setting $x = 0$, we see that $a_1 = b_1$. On repeating this procedure, we deduce that $a_k = b_k$ for $k = 0, 1, 2, \ldots$. That is, the coefficients of a power series are unique, just like the coefficients of a polynomial.

Remark: If $\sum_{k=0}^{\infty} c_k x^k$ converges to $f(x)$ for $|x| < R$, the uniqueness of power series guarantees that $\sum_{k=0}^{\infty} c_k x^k$ is the Taylor series for f; that is $c_k = f^{(k)}(0)/k!$.

Remark: Within their radii of convergence, power series can be added, subtracted, multiplied, divided, differentiated, and integrated just like polynomials.

- For $|x| < 1$ we may expand

$$\begin{aligned}
f(x) &= \frac{e^x}{1+x} \\
&= \left(1 + x + \frac{x^2}{2} + \ldots\right)(1 - x + x^2 + \ldots) \\
&= (1 - x + x^2 + \ldots) + x(1 - x + x^2 + \ldots) + \frac{x^2}{2}(1 - x + x^2 + \ldots) \\
&= (1 - x + x^2 + \ldots) + (x - x^2 + \ldots) + \frac{x^2}{2} + \ldots \\
&= 1 + \frac{x^2}{2} + \ldots.
\end{aligned}$$

From Taylor's theorem, we immediately see that $f(0) = 1$, $f'(0) = 0$, and $f''(0) = 1$.

Problem 9.19: Using long division, show that

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots.$$

Problem 9.20: Consider the Taylor series for the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

about the point $a = 0$. Show using L'Hôpital's Rule, that $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. That is, the Taylor series converges to zero for all $x \in \mathbb{R}$ (it has an infinite radius of convergence), even though $f(x) \neq 0$ for nonzero x. This example emphasizes that the Taylor series for an infinitely differentiable function f does not necessarily converge to f, even within its radius of convergence!

- Another important series is the *Binomial Series*: for $|x| < 1$ and any real number n, the Taylor Series for the function $f(x) = (1 + x)^n$ evaluates to (see Problem 9.17)

$$\sum_{k=0}^{\infty} \binom{n}{k} x^k,$$

where

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0, \\ \dfrac{n(n-1)\ldots(n-k+1)}{k!} & \text{if } k \geq 1. \end{cases}$$

The Limit Ratio Test tell us that the series converges when

$$\lim_{k\to\infty} \frac{|n-k|}{k+1} |x| = |x| < 1.$$

To see that the series actually converges to $f(x)$ define

$$g(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

and consider

$$h(x) = (1 + x)^{-n} g(x).$$

Note that

$$h'(x) = -n(1+x)^{-n-1}g(x) + (1+x)^{-n}g'(x) = (1+x)^{-n-1}[-ng(x) + (1+x)g'(x)].$$

On using the identity

$$k\binom{n}{k} = k\frac{n(n-1)\dots(n-k+1)}{k!} = n\frac{(n-1)\dots(n-k+1)}{(k-1)!} = n\binom{n-1}{k-1},$$

along with Pascal's Triangle Law,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},$$

we find that

$$\begin{aligned}
(1+x)g'(x) &= (1+x)\sum_{k=1}^{\infty}\binom{n}{k}kx^{k-1} \\
&= (1+x)n\sum_{k=1}^{\infty}\binom{n-1}{k-1}x^{k-1} \\
&= n\sum_{k=0}^{\infty}\binom{n-1}{k}x^k + n\sum_{k=1}^{\infty}\binom{n-1}{k-1}x^k \\
&= n + n\sum_{k=1}^{\infty}\left[\binom{n-1}{k} + \binom{n-1}{k-1}\right]x^k \\
&= n + n\sum_{k=1}^{\infty}\binom{n}{k}x^k \\
&= ng(x).
\end{aligned}$$

Thus $h'(x) = 0$ and since $h(0) = g(0) = 1$, we see that $h(x) = 1$ for all $x \in (-1, 1)$. Thus for $|x| < 1$ we find

$$(1+x)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^k = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots.$$

- For $n = 1/2$ and $|x| < 1$, we find that

$$\begin{aligned}
\sqrt{1+x} &= \sum_{k=0}^{\infty}\binom{1/2}{k}x^k \\
&= 1 + \sum_{k=1}^{\infty}\frac{1/2(1/2-1)(1/2-2)\dots(1/2-k+1)}{k!}x^k \\
&= 1 + \sum_{k=1}^{\infty}\frac{1(1-2)(1-4)\dots(1-2k+2)}{2^k k!}x^k \\
&= 1 + \frac{x}{2} + \sum_{k=2}^{\infty}\frac{(-1)(-3)\dots(3-2k)}{2^k k!}x^k \\
&= 1 + \frac{x}{2} + \sum_{k=2}^{\infty}(-1)^{k-1}\frac{1\cdot 3\cdot\dots\cdot(2k-3)}{2^k k!}x^k.
\end{aligned}$$

Chapter 10

Parametrization

10.1 Parametric Equations

Parametric equations provide a convenient way of describing general relations that cannot be represented as functions.

- The relation that exists between the ordered pairs (x, y) on the unit circle cannot be expressed as a function $y = f(x)$ since this curve fails to satisfy the vertical line test. However, we can parametrize each point (x, y) on the unit circle by the angle θ that the line from the origin to (x, y) makes with the positive x axis:

$$x(\theta) = \cos\theta,$$
$$y(\theta) = \sin\theta.$$

  Here the parameter θ lies in the interval $[0, 2\pi]$.

- The functional representation $y = mx + b$ of a line segment, with $x \in (x_1, x_2)$, fails for vertical lines. However, the *parametric form*

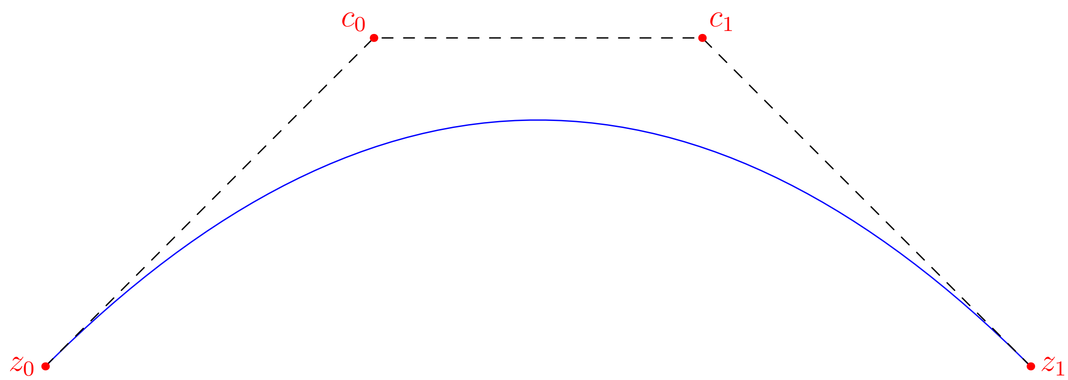
$$x(t) = (1 - t)p_x + tq_x$$
$$y(t) = (1 - t)p_y + tq_y,$$

  where the parameter $t \in [0, 1]$, remains valid for the line segment between *any* two points (p_x, p_y) and (q_x, q_y) in the plane.

- A *cubic Bézier curve* between the *node* $z_0 = (x_0, y_0)$, with *postcontrol point* $c_0 = (p_x, p_y)$, and the node $z_1 = (x_1, y_1)$, with *precontrol point* $c_1 = (q_x, q_y)$, is defined by the parametric equations

$$x(t) = (1 - t)^3 x_0 + 3t(1 - t)^2 p_x + 3t^2(1 - t)q_x + t^3 x_1,$$
$$y(t) = (1 - t)^3 y_0 + 3t(1 - t)^2 p_y + 3t^2(1 - t)q_y + t^3 y_1, \qquad 0 \le t \le 1.$$

  Bézier curves are widely used in computer graphics for drawing smooth curves between a given set of nodes.



10.2 Polar Coordinates

Polar coordinates (r, θ) are related to the usual *Cartesian coordinates* (x, y) by

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

Remark: Polar coordinates are not unique:

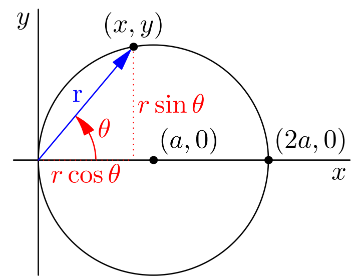
$$\begin{aligned} x &= r\cos(\theta + 2m\pi), \\ y &= r\sin(\theta + 2m\pi), \end{aligned}$$

specify the same point for all integers m. Also, the points $(r, \theta + \pi)$ and $(-r, \theta)$ are identical and $(0, \theta)$ denotes the origin for all θ.

- Describe the circle $(x - a)^2 + y^2 = a^2$ in polar coordinates.

$$\begin{aligned} x^2 + y^2 - 2ax &= 0 \\ \Rightarrow r^2 - 2ra\cos\theta &= 0 \\ \Rightarrow r(r - 2a\cos\theta) &= 0 \\ \Rightarrow r = 0 \text{ or } r &= 2a\cos\theta. \end{aligned}$$

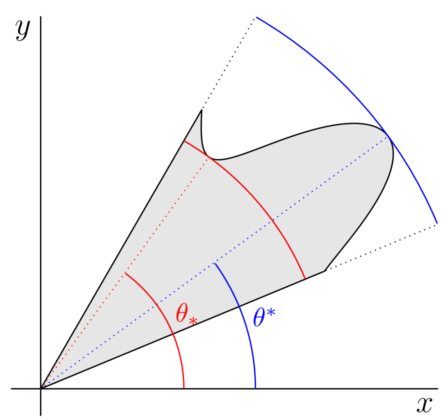
Remark: Thus, a point on the circle $(x - a)^2 + y^2 = a^2$ is either the origin $(r = 0)$ or else it lies on the curve $r = 2a\cos\theta$. In fact, since the origin is already contained in the second solution $r = 2a\cos\theta$ (at $\theta = \pi/2$), this equation alone generates the entire curve.



Remark: Notice that θ varies from 0 to 2π, the point (r, θ) moves **twice** around the circle (this corresponds to an elementary result from geometry that the angle subtended by an arc measured at the center of a circle is twice that measured on the circumference).

Q. Can we compute the area of a region bounded by a continuous curve, say $r = f(\theta) \geq 0$ for $\theta \in [a, b]$, in polar coordinates?

A. Yes. Let P be a partition of $[a, b]$. If f is continuous, there exists points θ_* and θ^* where f takes on its minimum and maximum values, respectively, in each subinterval of P.



The area contribution Δ_A from each subinterval of width Δ_θ must lie between the areas $r^2\Delta_\theta/2$ bounded by the circular arcs $r = f(\theta_*)$ and $r = f(\theta^*)$:

$$f^2(\theta_*)\frac{\Delta_\theta}{2} \leq \Delta_A \leq f^2(\theta^*)\frac{\Delta_\theta}{2}$$
$$\Rightarrow \lim_{\Delta_\theta \to 0} \frac{f^2(\theta_*)}{2} \leq \lim_{\Delta_\theta \to 0} \frac{\Delta_A}{\Delta_\theta} \leq \lim_{\Delta_\theta \to 0} \frac{f^2(\theta^*)}{2}.$$

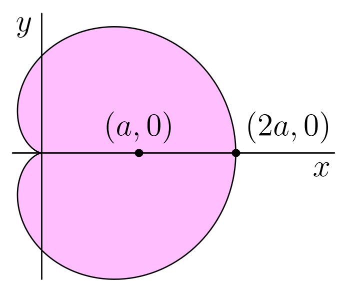
Since $\lim_{\Delta_\theta \to 0} f^2(\theta_*) = \lim_{\Delta_\theta \to 0} f^2(\theta^*) = f^2(\theta)$, we see from the squeeze principle that

$$\frac{dA}{d\theta} = \frac{1}{2} f^2(\theta) \Rightarrow A = \frac{1}{2} \int f^2(\theta)\, d\theta.$$

- The area enclosed by the circle $r = a$ is
$$\frac{1}{2} \int_0^{2\pi} a^2\, d\theta = \pi a^2.$$

- To find the area enclosed by the circle $r = 2a\cos\theta$, we must be careful to restrict θ to $[0, \pi]$ (one loop around the curve):
$$\frac{1}{2} \int_0^{\pi} 4a^2 \cos^2\theta\, d\theta = 2a^2 \int_0^{\pi} \cos^2\theta\, d\theta = 2a^2 \int_0^{\pi} \frac{1+\cos 2\theta}{2}\, d\theta = a^2 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = \pi a^2;$$
this is consistent with our earlier observation that $r = 2a\cos\theta$ describes a circle of radius a.

- The area enclosed by the *cardioid* $r = a(1 + \cos\theta)$



is given by
$$\frac{1}{2} \int_0^{2\pi} a^2(1+\cos\theta)^2\, d\theta = \frac{a^2}{2} \int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta = \frac{a^2}{2}(2\pi + \pi) = \frac{3\pi a^2}{2}.$$

Problem 10.1: Find the area enclosed by the circle $r = 3a\cos\theta$ that lies outside the cardioid $r = a(1 + \cos\theta)$.

The two curves intersect when $3\cos\theta = 1+\cos\theta$; that is when $\cos\theta = 1/2$. This happens at the angles $\theta = \pi/3$ and $\theta = -\pi/3$. The desired area is thus

$$\frac{1}{2} \int_{-\pi/3}^{\pi/3} \left[9a^2\cos^2\theta - a^2(1+\cos\theta)^2\right] d\theta = a^2 \int_0^{\pi/3} \left[9\cos^2\theta - (1 + 2\cos\theta + \cos^2\theta)\right] d\theta$$
$$= a^2 \int_0^{\pi/3} \left(8\cos^2\theta - 1 - 2\cos\theta\right) d\theta = a^2 \int_0^{\pi/3} [4(1+\cos 2\theta) - 1 - 2\cos\theta]\, d\theta$$
$$= a^2 \int_0^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta = a^2[3\theta + 2\sin 2\theta - 2\sin\theta]_0^{\pi/3} = \pi a^2.$$

Remark: To compute the first derivative of a curve parametrized by t, one needs to use the Chain Rule to convert the derivative with respect to t to a derivative with respect to x:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Problem 10.2: Find the slope of the tangent to the cardioid $r(\theta) = 1 + \sin\theta$ at $\theta = \pi/3$.

Since $x(\theta) = r(\theta)\cos\theta$ and $y(\theta) = r(\theta)\sin\theta$ we find at $\theta = \pi/3$ that

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\\ &= \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}\\ &= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}\\ &= \frac{\cos\theta(1+2\sin\theta)}{1-\sin^2\theta - (1+\sin\theta)\sin\theta}\\ &= \left.\frac{\cos\theta(1+2\sin\theta)}{(1-2\sin\theta)(1+\sin\theta)}\right|_{\pi/3}\\ &= -1.\end{aligned}$$

Problem 10.3: A plane curve is given by the polar equation

$$r(\theta) = \sqrt{7 + 4\sin\theta}.$$

(a) Find a Cartesian equation for the tangent line to the curve at the point $r(\pi/6)$.
The slope of the tangent is equal to

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}.$$

Here, if $\theta = \pi/6$,

$$r(\pi/6) = \sqrt{7 + 4\sin(\pi/6)} = 3$$

and

$$r'(\pi/6) = \left.\frac{4\cos\theta}{2\sqrt{7+4\sin\theta}}\right|_{\theta=\pi/6} = \frac{\sqrt{3}}{3}.$$

So the slope of the tangent when $\theta = \pi/6$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{\frac{\sqrt{3}}{3}\cdot\frac{1}{2} + 3\cdot\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2} - 3\cdot\frac{1}{2}} = -\frac{5\sqrt{3}}{3}$$

and the point on the curve at $\theta = \pi/6$ is $(r\cos\theta, r\sin\theta) = \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Therefore the tangent line is

$$y - \frac{3}{2} = -\frac{5\sqrt{3}}{3}\left(x - \frac{3\sqrt{3}}{2}\right), \quad \text{i.e.} \quad y = -\frac{5\sqrt{3}}{3}x + 9.$$

(b) Find the area A of the region that lies outside the above curve and inside the cardioid $r(\theta) = 2(1+\sin\theta)$.

First, we need to find the interval for θ satisfying

$$2(1+\sin\theta) \geq \sqrt{7 + 4\sin\theta}.$$

On squaring this inequality, we find

$$4(1+\sin\theta)^2 \geq 7 + 4\sin\theta \Rightarrow 0 \leq 4\sin^2\theta + 4\sin\theta - 3 = (2\sin\theta - 1)(2\sin\theta + 3),$$

which implies that $\sin\theta \geq 1/2$, i.e., $\pi/6 \leq \theta \leq 5\pi/6$. Thus, the area of the region is

$$\begin{aligned} A &= \int_{\pi/6}^{5\pi/6} \frac{1}{2}[2(1+\sin\theta)]^2\,d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(7 + 4\sin\theta)\,d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4\sin^2\theta + 4\sin\theta - 3)\,d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (-2\cos(2\theta) + 4\sin\theta - 1)\,d\theta \\ &= \frac{1}{2}\Big[-\sin(2\theta) - 4\cos\theta - \theta\Big]_{\pi/6}^{5\pi/6} = \frac{5\sqrt{3}}{2} - \frac{\pi}{3} \end{aligned}$$

Remark: To compute the second derivative of a curve parametrized by t, we need to use the Chain Rule twice:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{dt}{dx}\cdot\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right).$$

Remark: To find the area between the curve $(x(t), y(t))$ and the x axis, we can use the fact that $dx = x'(t)\,dt$ to express the area as an integral over t:

$$\int y\,dx = \int y(t)x'(t)\,dt.$$

- For example, for $x(t) = e^t$ and $y(t) = t^2$, the area between the curve $(x(t), y(t))$ and the x axis from $t = 0$ to $t = 1$ is

$$\int y\,dx = \int_0^1 y(t)x'(t)\,dt = \int_0^1 2te^t\,dt = \left[2te^t\right]_0^1 - \int_0^1 2e^t\,dt = 2e-2\left[e^t\right]_0^1 = 2e-2(e-1) = 2.$$

10.3 Cylindrical Coordinates

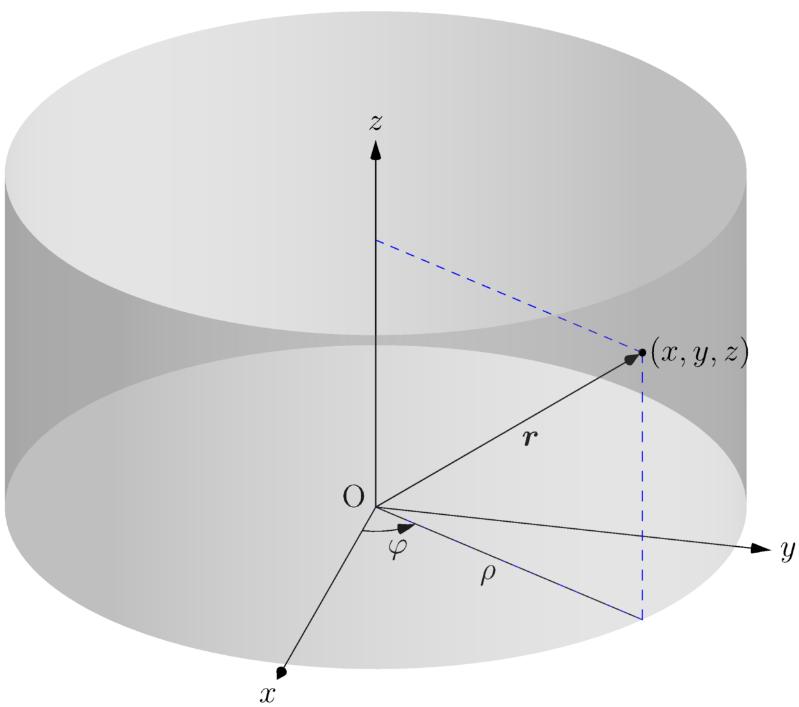
Objects that are symmetric about an axis (aligned with the z axis) can be conveniently described using polar coordinates on the xy plane with an added z coordinate that describes height above the xy plane. In ISO standard 31–11, these *cylindrical coordinates* are written (ρ, φ, z), using

$$x = \rho\cos\varphi,$$

$$y = \rho\sin\varphi,$$

$$z = z,$$

where $\varphi \in [0, 2\pi]$.



- The right circular cylinder $x^2 + y^2 = a^2$ can be simply described in cylindrical coordinates as $\rho = a$.

- The right circular cone $x^2 + y^2 = z^2$ can be simply described in cylindrical coordinates as $\rho = z$.

10.4 Spherical Polar Coordinates

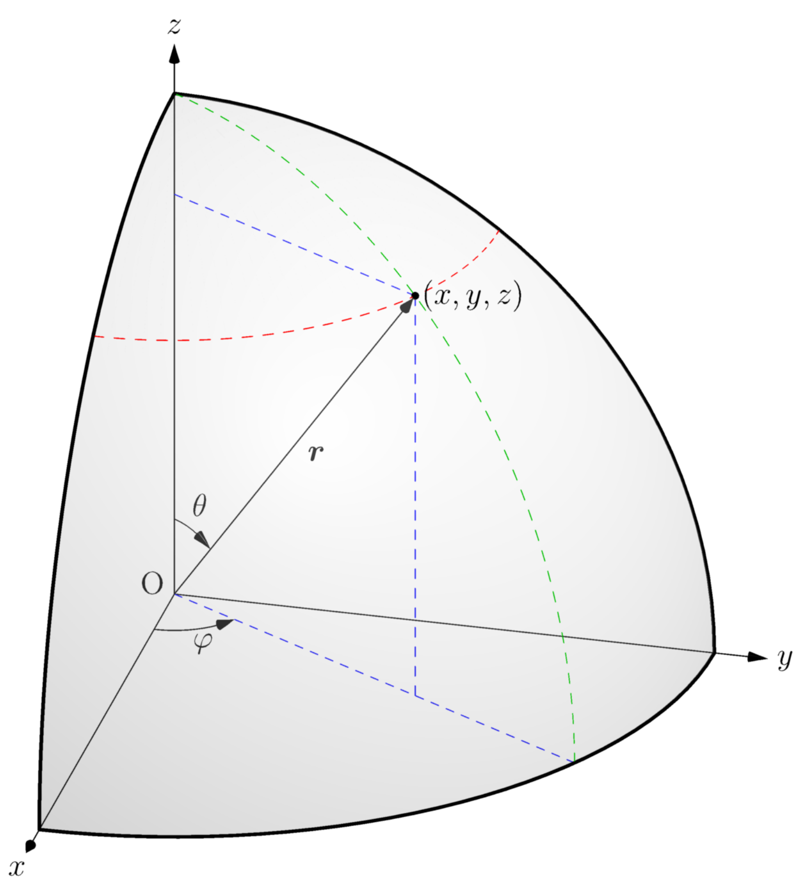
Objects that exhibit spherical symmetry are more conveniently described with *spherical polar coordinates*, written in ISO standard 31–11 as (r, θ, φ). Here r is the length of the vector $\boldsymbol{r} = (x, y, z)$ and the co-latitude θ is the angle between $\boldsymbol{r}$ and the z axis, so that $z = r\cos\theta$. The angle φ is the angle in the xy plane relative to the positive x axis of the projection, of length $\rho = r\sin\theta$, of $\boldsymbol{r}$ onto the xy plane. Since $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$, we find that

$$x = r\sin\theta\cos\varphi,$$

$$y = r\sin\theta\sin\varphi,$$

$$z = r\cos\theta,$$

where $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$.



Problem 10.4: Find the Cartesian coordinates (x, y, z) for the point described by the spherical coordinates $(r, \theta, \varphi) = (2, \pi/3, \pi/4)$.

$$(\sqrt{3/2}, \sqrt{3/2}, 1).$$

Problem 10.5: Find the spherical coordinates (r, θ, φ) for the point described by the Cartesian coordinates $(x, y, z) = (0, 2\sqrt{3}, -2)$.

$$(4, 2\pi/3, \pi/2).$$

Problem 10.6: Find the spherical coordinates (r, θ, φ) of the point with cylindrical coordinates $(\rho, \varphi, z) = \left(3, \frac{5\pi}{6}, \sqrt{3}\right)$.

Since $x^2 + y^2 = \rho^2 = 9$, we see that $r = \sqrt{x^2 + y^2 + z^2} = 2\sqrt{3}$. Then $z = r\cos\theta$ implies that $\cos\theta = z/r = \frac{1}{2}$, so that $\theta = \pi/3$. Thus $(r, \theta, \varphi) = \left(2\sqrt{3}, \dfrac{\pi}{3}, \dfrac{5\pi}{6}\right)$.

Surfaces in spherical coordinates can be described by a function $r = r(\theta, \varphi)$, just as curves in plane polar coordinates can be described by a function $r = r(\theta)$.

- The surface $r(\theta, \varphi) = \sin\theta\cos\varphi$ can be expressed in Cartesian coordinates by considering
$$r = \sin\theta\cos\varphi = \frac{x}{r},$$
so that
$$x^2 + y^2 + z^2 = r^2 = x.$$
On completing the square, we obtain a sphere of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0, 0)$:
$$\left(x - \frac{1}{2}\right)^2 + y^2 + z^2 = \frac{1}{4}.$$

10.5 Vector Functions and Space Curves

While parametric equations in three dimensions can be expressed in component form, in terms of three functions $x = x(t)$, $y = y(t)$, $z = z(t)$, it is often more convenient at each t to bundle all three components into a vector $(x(t), y(t), z(t))$.

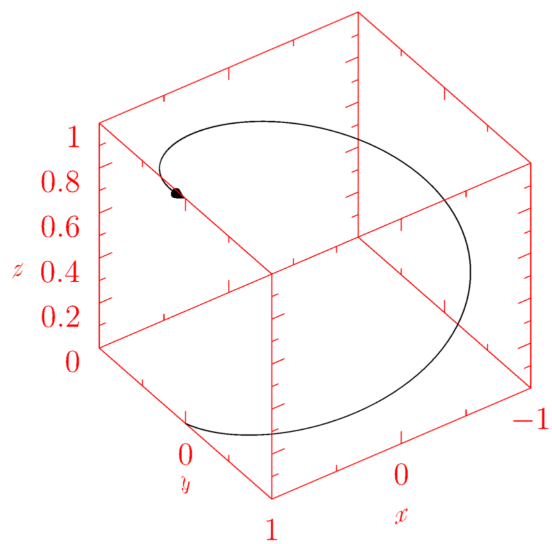
Definition: A *vector-valued function* or *vector function* is a function from a domain to a set of vectors.

Definition: The special case of a vector function that maps values from $\mathbb{R}$ to $\mathbb{R}^3$ is called a *space curve*.

- The vector function $\boldsymbol{r}(t) = (x(t), y(t), z(t))$ maps each real value t to a point in $(x(t), y(t), z(t))$ in $\mathbb{R}^3$.

- The vector function $\boldsymbol{r}(t) = (t, 2 + 3t, 4 + 5t)$ for $t \in \mathbb{R}$ describes the line in three dimensions that passes through $(0, 2, 4)$ and is parallel to the vector $(1, 3, 5)$.

- The vector function $(\cos 2\pi t, \sin 2\pi t, t)$ for $t \in [0, 1]$ describes a helix:



- The vector function that describes the curve of intersection of the circular cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$ can be easily found using cylindrical polar coordinates $(\rho\cos\varphi, \rho\sin\varphi, z)$, with $\rho = 1$ and $\varphi \in [0, 2\pi]$. The equation of the plane becomes $z = 2 - y = 2 - \sin\varphi$, so the parametric equation for the curve of intersection is the ellipse $(\cos\varphi, \sin\varphi, 2 - \sin\varphi)$ for $\varphi \in [0, 2\pi]$.

Remark: To do calculus on vector functions, we first need to extend the notion of a limit to vector values.

Definition: If $\boldsymbol{r}(t) = (x(t), y(t), z(t))$, then $\lim_{t\to a} \boldsymbol{r}(t) = (\lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t))$.

Definition: If $\boldsymbol{r}(t) = (x(t), y(t), z(t))$, then $\boldsymbol{r}'(t) = (x'(t), y'(t), z'(t))$.

Definition: If $\boldsymbol{r}(t) = (x(t), y(t), z(t))$, then $\int \boldsymbol{r}(t)\, dt = (\int x(t)\, dt, \int y(t)\, dt, \int z(t)\, dt)$.

Definition: A curve described by the parametrization $\boldsymbol{r}(t)$ is called *smooth* if $\boldsymbol{r}'(t)$ is continuous and $\boldsymbol{r}'(t) \neq \mathbf{0}$.

Definition: A *cusp* of the curve $\boldsymbol{r}(t)$ is a point where the vector $\boldsymbol{r}'(t)$ is zero and at least one of its components changes sign.

Definition: A smooth curve exhibits no sharp turns, reversals, or cusps.

Remark: The direction of the tangent line to a smooth curve $\boldsymbol{r}(t)$ is given by the vector $\boldsymbol{r}'(t)$. The *unit tangent vector* $\boldsymbol{T}$ is therefore

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}.$$

The unit tangent vector $\boldsymbol{T}$ changes slowly when the curve is straight and rapidly when the curve bends or twists.

Remark: Since $\boldsymbol{T}$ is a unit vector, its magnitude is constant. This means that

$$0 = \frac{d}{dt}\,|\boldsymbol{T}|^2 = \frac{d}{dt}(\boldsymbol{T}\cdot\boldsymbol{T}) = \boldsymbol{T}'\cdot\boldsymbol{T} + \boldsymbol{T}\cdot\boldsymbol{T}' = 2\boldsymbol{T}'\cdot\boldsymbol{T}.$$

Thus, at every point of a curve, the derivative $\boldsymbol{T}'$ of the unit tangent vector is always perpendicular to $\boldsymbol{T}$.

Chapter 11

The Geometry of Space

11.1 Lines and Planes

- The *parametric equation of a line* through $\boldsymbol{p}$ and $\boldsymbol{q}$ in $\mathbb{R}^n$ is

$$\boldsymbol{v} = (1-t)\boldsymbol{p} + t\boldsymbol{q},$$

where t is a real parameter.

- If we express the above parametric form as

$$\boldsymbol{v} = \boldsymbol{p} + t(\boldsymbol{q} - \boldsymbol{p})$$

and denote $\boldsymbol{p} = (x_0, y_0, z_0)$ and $\boldsymbol{q} - \boldsymbol{p} = (a, b, c)$, we obtain the *point-direction parametric equation of a line* through the point (x_0, y_0, z_0) in the direction (a, b, c):

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c).$$

- To obtain the symmetric equation of a line we eliminate t by solving for and equating the expression for t in terms of each component:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Remark: In two dimensions, lines either intersect or are parallel. In three dimensions, there is a third possibility, as illustrated in Figure 11.1.

- two lines can *intersect*; e.g. the x axis and the y axis.
- two lines can be *parallel*; e.g. the line $(0, t, 1)$ for $t \in \mathbb{R}$ and the y axis.
- two lines can be *skew*; e.g. the line $(0, t, 1)$ for $t \in \mathbb{R}$ and the x axis.

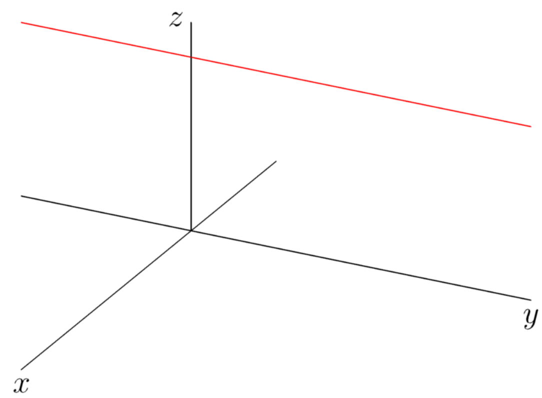


Figure 11.1: The x and y axes intersect; the red line is parallel to the y axis; the red line and the x axis are skew.

- A *plane* is determined by a point (x_0, y_0, z_0), through which it passes, and a vector (a, b, c), called the *normal*, that is perpendicular to every vector $(x, y, z)-(x_0, y_0, z_0)$ in the plane:
$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$
- If we let $d = ax_0 + by_0 + cz_0$, then the *equation of a plane* becomes
$$ax + by + cz = d.$$

Problem 11.1: Find the point of intersection of the plane $x + 2y - z = 6$ and the line through two points $(1, 0, 1)$ and $(2, -1, 3)$.

Let (x, y, z) be the intersection point. Then
$$(x, y, z) = (1, 0, 1) + t[(2, -1, 3) - (1, 0, 1)] = (1 + t, -t, 1 + 2t).$$
Since (x, y, z) also lies on the plane $x + 2y - z = 6$, we find from
$$(1 + t) + 2(-t) - (1 + 2t) = 6$$
that $t = -2$. Thus $(x, y, z) = (-1, 2, -3)$.

Remark: In general, the (perpendicular) *distance* of a point (X, Y, Z) from a plane
$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$
passing through (x_0, y_0, z_0) and with unit normal $(a, b, c)/\sqrt{a^2 + b^2 + c^2}$ is given by
$$D = \left| \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}} \cdot (X - x_0, Y - y_0, Z - z_0) \right| = \left| \frac{(a, b, c) \cdot (X, Y, Z) - d}{\sqrt{a^2 + b^2 + c^2}} \right|,$$
where $d = ax_0 + by_0 + cz_0$. In particular, on setting $(X, Y, Z) = (0, 0, 0)$, we see that $|d| / \sqrt{a^2 + b^2 + c^2}$ is the distance of the plane from the origin.

Problem 11.2: Determine the distance between the point $(0,1,1)$ and the plane containing $(1,0,0)$, $(-1,1,0)$, and $(1,-1,1)$.

The normal to the plane must be perpendicular to $(-1,1,0)-(1,0,0)=(-2,1,0)$ and $(1,-1,1)-(1,0,0)=(0,-1,1)$. It therefore lies in the direction

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = (1,2,2).$$

The distance D of the point $(0,1,1)$ from the plane is then given by the length of the projection of the vector $(0,1,1)-(1,0,0)=(-1,1,1)$ in the direction $(1,2,2)$:

$$D = \left| \frac{(-1,1,1)\cdot(1,2,2)}{\sqrt{1^2+2^2+2^2}} \right| = \frac{3}{3} = 1.$$

Alternatively, D can be computed as the distance of the point $(0,1,1)$ from the plane

$$1(x-1)+2y+2z=0,$$

or equivalently,

$$x+2y+2z=1.$$

One then finds

$$D = \left| \frac{1\cdot 0+2\cdot 1+2\cdot 1-1}{\sqrt{1^2+2^2+2^2}} \right| = \frac{3}{3} = 1.$$

Problem 11.3: Find the distance between the lines $(7t+1,-4t,2t+1)$ and $(5s-1,-s+1,7s-1)$.

Since the lines are not parallel, they either intersect or are skew. In either case, the lines belong to two parallel (possibly identical) planes. Our strategy is to first find parallel planes containing these lines and then find the distance between the two planes. The normal vector $\boldsymbol{n}$ to these planes must be perpendicular to the line directions $(7,-4,2)$ and $(5,-1,7)$:

$$\boldsymbol{n} = (7,-4,2)\times(5,-1,7) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 7 & -4 & 2 \\ 5 & -1 & 7 \end{vmatrix} = (-26,-39,13),$$

which is parallel to $(2,3,-1)$.

The distance between the planes is then given by the length of the projection of the vector $(1,0,1)-(-1,1,-1)=(2,-1,2)$ in the direction $(2,3,-1)$:

$$D = \left| \frac{(2,-1,2)\cdot(2,3,-1)}{\sqrt{2^2+3^2+1^2}} \right| = \frac{1}{\sqrt{14}}.$$

Alternatively, the distance between the planes can be computed as the distance between the point $(1,0,1)$ on the first line and the plane

$$0 = 2(x+1)+3(y-1)-1(z+1) = 2x+3y-z-2$$

containing the second line:

$$\left| \frac{2\cdot 1+3\cdot 0-1\cdot 1-2}{\sqrt{2^2+3^2+1^2}} \right| = \frac{1}{\sqrt{14}}.$$

- The *parametric equation of a plane* spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$ through $\boldsymbol{r}_0$ is given by

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{u} + s\boldsymbol{v},$$

where t and s are real parameters.

11.2 Cylinders

Definition: A *cylinder* is composed of all lines in a given direction that pass through a given plane curve.

- The set of points (x, y, z) such that $x^2+y^2 = a^2$ represents a cylinder of radius a (and infinite height). The intersection of this cylinder with every plane $z =$ constant is a circle of radius a.

- The set of points (x, y, z) such that $y = x^2$ is a *parabolic cylinder*, composed of infinitely many vertically shifted copies of the parabola $\{(x, x^2, 0) : x \in \mathbb{R}\}$.

Problem 11.4: Show that the surface

$$y^2 + 4z^2 = 4$$

is an elliptic cylinder parallel to the x axis, with major radius 2 (in the y direction) and minor radius 1 (in the z direction).

Problem 11.5: Show that the intersection of the sphere $r = 2$ with the plane $x-y = 0$ lies on two elliptic cylinders. Determine the major and minor radii of these elliptic cylinders.

In spherical polar coordinates, the plane $x = y$ simplifies to $\varphi = \pi/4$. Since $\cos\varphi = \sin\varphi = 1/\sqrt{2}$, any point (x, y, z) on this intersection satisfies

$$x = \sqrt{2}\sin\theta$$

$$y = \sqrt{2}\sin\theta$$

$$z = 2\cos\theta,$$

where $\theta \in [0, \pi]$. We then see that $2x^2+z^2 = 4\sin^2\theta+4\cos^2\theta = 4$ and similarly $2y^2+z^2 = 4$. From the equations

$$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$$

and

$$\left(\frac{y}{\sqrt{2}}\right)^2 + \left(\frac{z}{2}\right)^2 = 1,$$

we see that the elliptic cylinders have minor radii of $\sqrt{2}$ and major radii of 2.

11.3 Quadric Surfaces

Definition: A *quadric surface* is the generalization of a conic section to three dimensional space. The form of a general quadric surface,

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy = Iz + J,$$

may be reduced to one of the two standard forms

$$Ax^2 + By^2 + Cz^2 = J$$

or

$$Ax^2 + By^2 = Iz,$$

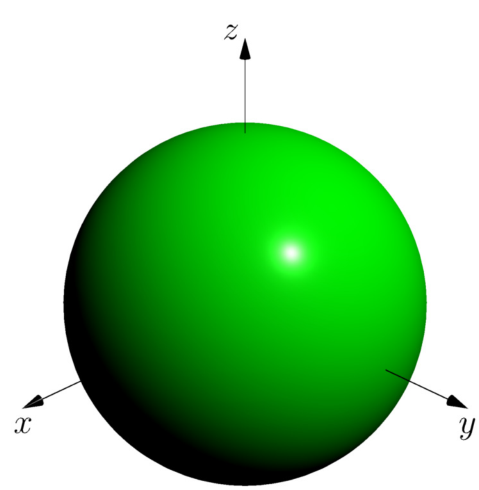
by appropriate translation and rotation of the coordinate axes.

Remark: To visualize such a three dimensional surface, it helps to draw the various cross sections or *traces* obtained when one of the coordinates is held fixed.

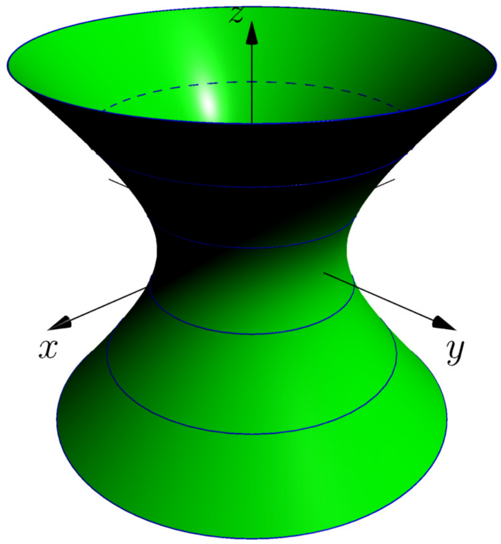
- The cross sections of an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ obtained by holding z fixed (thereby slicing the object with a plane parallel to the xy plane) are the ellipses
$$\frac{x^2}{(ka)^2} + \frac{y^2}{(kb)^2} = 1$$
with $k^2 = 1 - z^2/c^2$.

Remark: By stretching the axes by positive constants it is possible to put each standard form for a quadric surface into one of the following canonical forms:

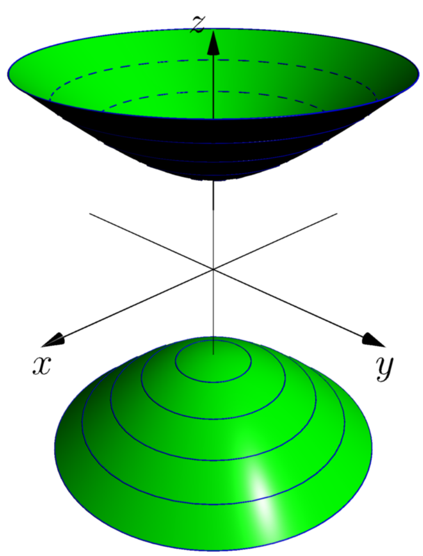
- With appropriate stretching of the coordinate axes, the *ellipsoid* takes on the form of a sphere $x^2 + y^2 + z^2 = 1$:



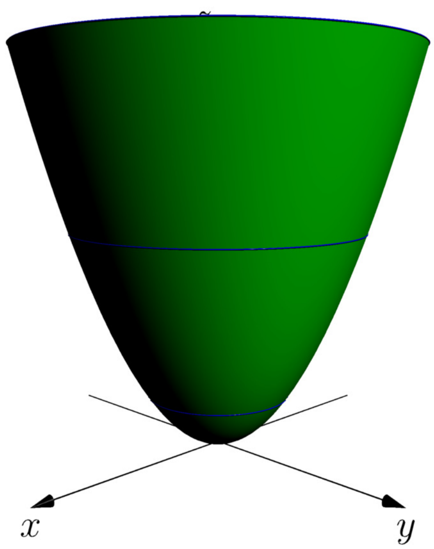
- The *hyperboloid of one sheet* has the generic form $x^2 + y^2 - z^2 = 1$:



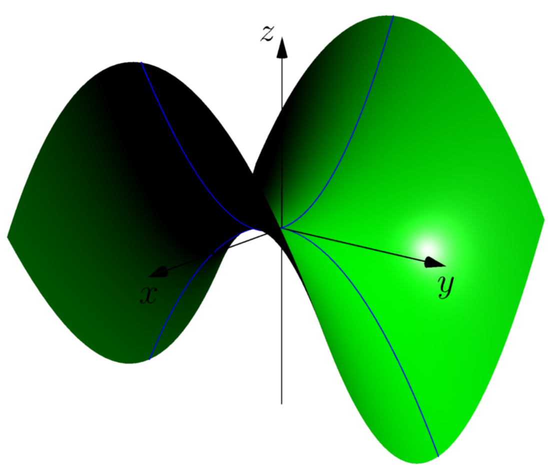
- The *hyperboloid of two sheets* has the generic form $-x^2 - y^2 + z^2 = 1$:



- The *elliptic paraboloid* has the generic form $z = x^2 + y^2$:



- The *hyperbolic paraboloid* has the generic form $z = x^2 - y^2$:



Problem 11.6: Find an equation for the surface consisting of all points (x, y, z) that are twice as far from the point $(0, 0, 4)$ as they are from the plane $z = 1$. Sketch and identify the surface.

The distance between (x, y, z) and $(0, 0, 4)$ is equal to $\sqrt{x^2 + y^2 + (z-4)^2}$ and the distance of (x, y, z) from the plane $z = 1$ is $|z - 1|$. We thus want

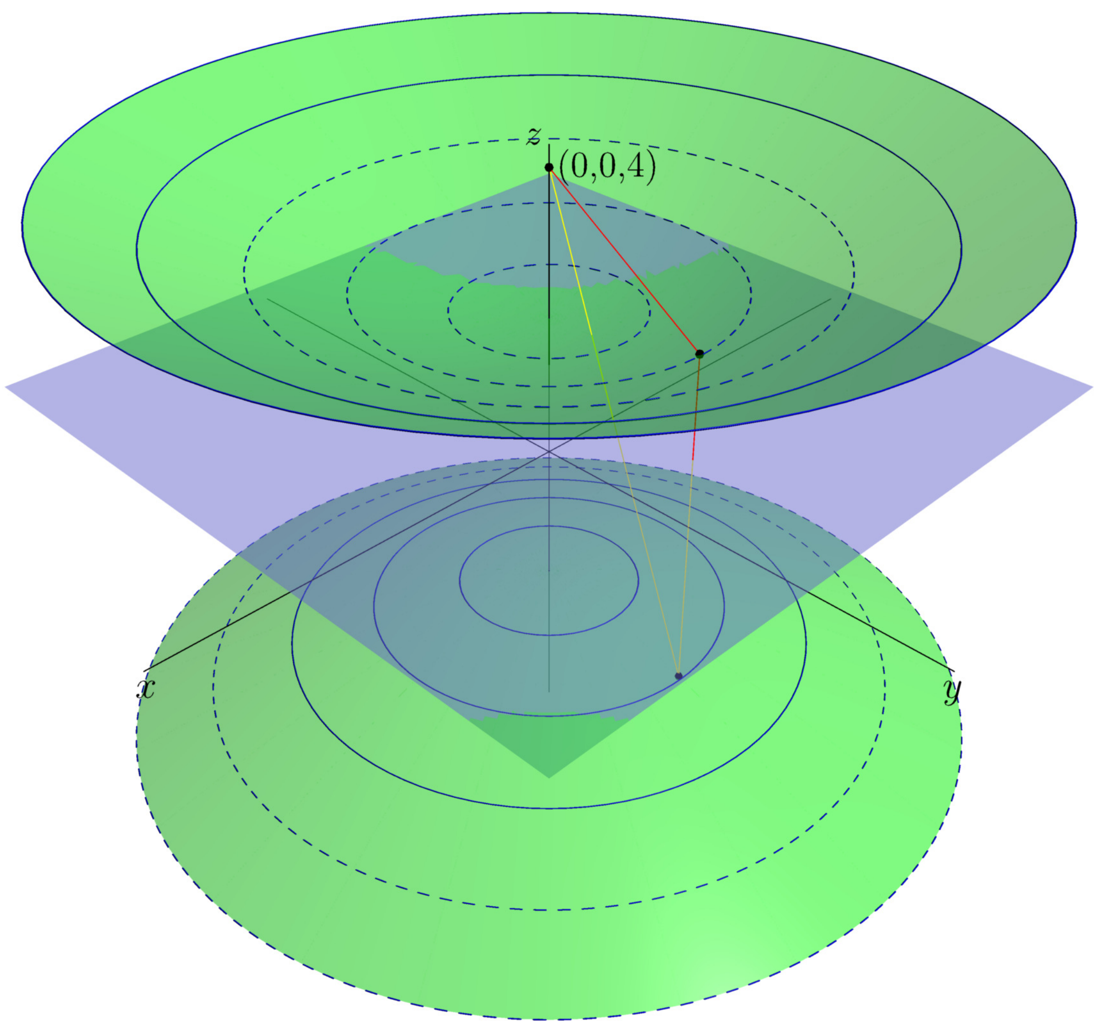
$$x^2 + y^2 + (z-4)^2 = [2(z-1)]^2.$$

This expands to

$$x^2 + y^2 + z^2 - 8z + 16 = 4z^2 - 8z + 4,$$

which reduces to the hyperboloid of two sheets

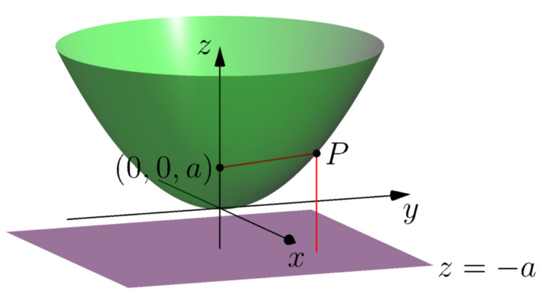
$$3z^2 - x^2 - y^2 = 12.$$



- The equation
$$x^2 - y^2 + z^2 = 1.$$
describes a hyperboloid of one sheet about the y axis. On constant x surfaces the *trace* is a hyperbola
$$z^2 - y^2 = 1 - x^2.$$
On constant z surfaces the trace is also a hyperbola:
$$x^2 - y^2 = 1 - z^2.$$
However, on constant y surfaces the trace is a circle of radius $\sqrt{1+y^2}$:
$$x^2 + z^2 = 1 + y^2.$$

- The *right circular cone* $x^2 + y^2 = z^2$ is a degenerate limit of the hyperboloid of one sheet.

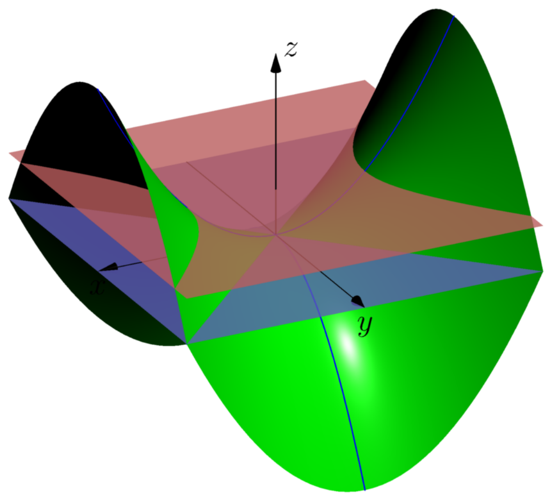
Remark: Recall that the parabola is the equation of all points equidistant from a *focus* point and a line called the *directrix*. Similarly, the *elliptic paraboloid* is the equation of all points equidistant from a focus point and a given plane. For example, if we take the focus to be the point $(0, 0, a)$ and the plane given by $z = -a$, we find the distance from $P = (x, y, z)$ to the focus to be $\sqrt{x^2 + y^2 + (z-a)^2}$. If we equate this to the distance $z + a$ of P from the plane $z = -a$, we obtain
$$x^2 + y^2 + (z-a)^2 = (z+a)^2,$$
which simplifies to the paraboloid
$$x^2 + y^2 = 4az.$$



Problem 11.7: Find an equation for the surface consisting of all points that are equidistant from the point $(0, 1, 0)$ and the plane $y = -1$. Sketch and identify the surface.

Let (x, y, z) be a point on the surface. Then the distance between (x, y, z) and $(0, 1, 0)$ is equal to $\sqrt{x^2 + (y-1)^2 + z^2}$ and the distance from the plane $y = -1$ is $|y + 1|$. So
$$x^2 + (y-1)^2 + z^2 = (y+1)^2.$$
On simplifying, we obtain the paraboloid $4y = x^2 + z^2$.

Remark: The traces of the hyperbolic paraboloid $z = x^2 - y^2$ are a hyperbola on constant z surfaces and parabolas on constant x and y surfaces, as seen in the following figure.



Chapter 12

Arc Length, Surface Area, and Curvature

12.1 Arc Length

Suppose $x(t)$ and $y(t)$ are functions on $[a, b]$ with continuous derivatives. We have seen that the equations

$$x = x(t), \qquad y = y(t)$$

provide a *parametric representation* of a *smooth curve* $(x(t), y(t))$ in $\mathbb{R}^2$ in terms of the *parameter* t.

As a special case, we could take $x(t) = t$ and $y(t) = f(t)$. The points $(t, f(t))$ describe the familiar *graph* of the function $f(t)$. However, the parametric representation allows us to describe relations, such as circles, that are not the graph of a single function.

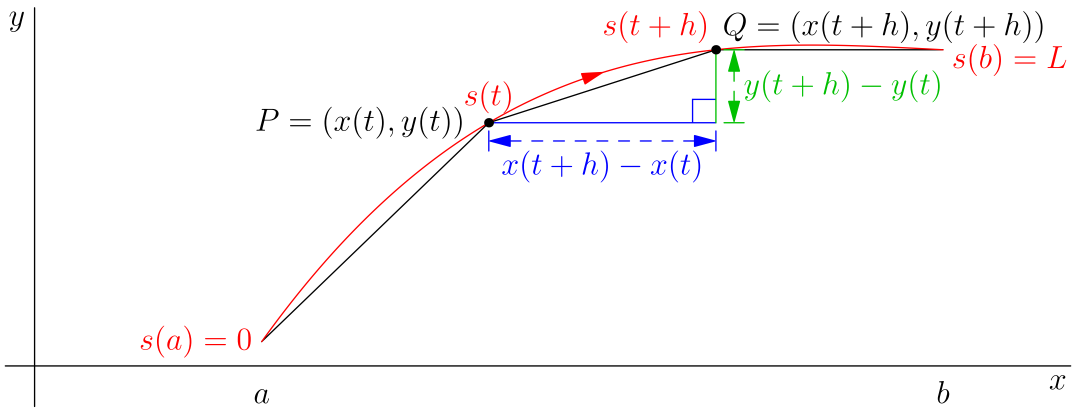
Q. What is the length of such a curve?

A. To answer this question, we must first define the notion of what we mean by the "length" of a smooth curve. What we seek is an extension of Pythagoras' Theorem, which allows us to calculate the length of line segments in terms of their endpoints, to general curves.

Definition: The *arc length* or *path length* $s(t)$ of a smooth curve $(x(t), y(t))$ on $[a, b]$ is the unique differentiable function $s(t)$ that satisfies $s(a) = 0$ and the property that

$$\lim_{h \to 0^+} \frac{s(t+h) - s(t)}{\sqrt{[x(t+h) - x(t)]^2 + [y(t+h) - y(t)]^2}} = 1 \tag{12.1}$$

for all $t \in [a, b)$. That is, the difference between the path lengths $s(t+h)$ and $s(t)$ to any points $P = (x(t), y(t))$ and $Q = (x(t+h), y(t+h))$ on the curve, respectively, should reduce to the length of the straight line segment joining P and Q in the limit $h \to 0$ (in which case $Q \to P$).



Upon dividing the numerator and denominator on the left-hand side of Eq. (12.1) by h we see that

$$\frac{\displaystyle\lim_{h\to 0^+} \frac{s(t+h)-s(t)}{h}}{\sqrt{\displaystyle\lim_{h\to 0^+} \left[\frac{x(t+h)-x(t)}{h}\right]^2 + \lim_{h\to 0^+} \left[\frac{y(t+h)-y(t)}{h}\right]^2}} = 1.$$

This gives us a formula for the derivative of $s(t)$ for every $t \in [a, b]$,

$$s'(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}. \tag{12.2}$$

Upon integrating this result from a to b, we find an expression for the arc length $L = s(b)$ of a curve $(x(t), y(t))$ on $[a, b]$. Since $s(a) = 0$, we have

$$s(b) = s(b) - s(a) = \int_a^b s'(t)\,dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt.$$

Remark: One can think of each point on the curve $(x(t), y(t))$ as the position of a point in $\mathbb{R}^2$ at each time t. The integrand $\sqrt{[x'(t)]^2 + [y'(t)]^2}$ is just the magnitude $|\boldsymbol{v}|$ of the *velocity* vector $\boldsymbol{v} = (x'(t), y'(t))$. The arc length, being the integral of the *speed* $|\boldsymbol{v}|$ with respect to time, is then seen to be the distance travelled by the point $(x(t), y(t))$ over the time interval $[a, b]$.

Remark: An easy way to remember the arc-length formula is to multiply Eq. (12.2) formally by dt and square the result:

$$ds^2 = dx^2 + dy^2.$$

This can be thought of as a statement of Pythagoras' Theorem for differentials. The arc length can then be computed by integrating ds between $t = a$ and $t = b$, remembering that $dx = x'(t)\,dt$ and $dy = y'(t)\,dt$.

- A circle of radius $r \geq 0$ centered on the origin can be described either by the equation $x^2 + y^2 = r^2$ or in parametric form as $(r\cos t, r\sin t)$ for $t \in [0, 2\pi]$. The *circumference* of the circle is then given by

$$\int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\, dt = \int_0^{2\pi} r\, dt = 2\pi r.$$

Remark: If the curve $(x(t), y(t))$ for $t \in [a, b]$ can be described by a differentiable function $y = f(x)$, then

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{x(a)}^{x(b)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x(a)}^{x(b)} \sqrt{1 + [f'(x)]^2}\, dx.$$

- We could also compute the circumference of a circle as twice the arc length of the function $f(x) = \sqrt{r^2 - x^2}$ on $[-r, r]$:

$$\begin{aligned}
2\int_{-r}^{r} \sqrt{1 + [f'(x)]^2}\, dx &= 2\int_{-r}^{r} \sqrt{1 + \left(\frac{-2x}{2\sqrt{r^2 - x^2}}\right)^2}\, dx = 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\
&= 4\int_0^r \sqrt{\frac{r^2}{r^2 - x^2}}\, dx = 4r\int_0^r \frac{1}{\sqrt{r^2 - x^2}}\, dx \\
&= 4r\int_0^1 \frac{1}{\sqrt{1 - u^2}}\, du = 4r[\arcsin u]_0^1 = 2\pi r,
\end{aligned}$$

where we have used the substitution $u = x/r$.

Remark: Of course, we can also express arc length as an integral in y:

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{y(a)}^{y(b)} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy.$$

- Find the arc length L of the parabola $y^2 = x$ between $(0, 0)$ and $(1, 1)$.

Since $dx/dy = 2y$, we know that $L = \int_0^1 \sqrt{(2y)^2 + 1}\, dy$. Let $2y = \tan\theta$, so that $2\, dy = \sec^2\theta\, d\theta$. Then

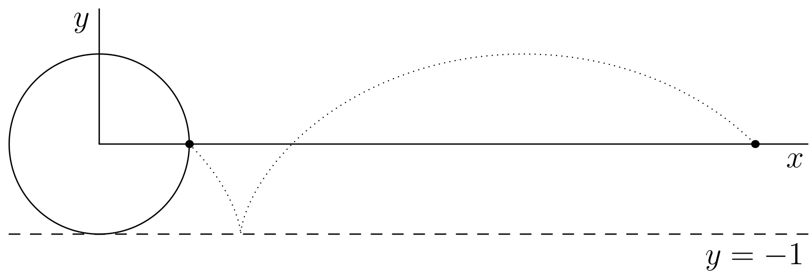
$$\int \sqrt{4y^2 + 1}\, dy = \frac{1}{2}\int \sec^3\theta\, d\theta = \frac{1}{4}(\sec\theta\tan\theta + \log|\sec\theta + \tan\theta|) + C,$$

using a result from page 137. Thus

$$L = \frac{1}{4}\left[2y\sqrt{4y^2 + 1} + \log\left|\sqrt{4y^2 + 1} + 2y\right|\right]_0^1 = \frac{1}{4}\left[2\sqrt{5} + \log\left(\sqrt{5} + 2\right)\right].$$

Problem 12.1: A quadratic Bezier approximation to a quarter unit circle is described by the curve $x(t) = 1 - t^2$ and $y(t) = 2t - t^2$, with $t \in [0, 1]$. Show that the arc length of this curve is $1 + \log(1+\sqrt{2})/\sqrt{2}$. Hint: Recall that $\sinh^{-1} 1 = \log(1+\sqrt{2})$.

Problem 12.2: A wheel of radius 1 is initially centered at $(0, 0)$. The point on its surface at $(1, 0)$ is marked with a dot. The wheel then rolls along the line $y = -1$ one complete rotation.



The position of the marked point at any time $t \in [0, 2\pi]$ is given by the parametric equations $x(t) = t + \cos t$, $y(t) = -\sin t$. Find the arc length L of the path traced out by the point (indicated above by the dotted curve).

Hint: Try multiplying the integrand by $\frac{\sqrt{1+\sin t}}{\sqrt{1+\sin t}}$. Be careful about signs when simplifying square roots!

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{x'^2(t) + y'^2(t)}\, dt = \int_0^{2\pi} \sqrt{(1-\sin t)^2 + \cos^2 t}\, dt \\
&= \int_0^{2\pi} \sqrt{1 - 2\sin t + \sin^2 t + \cos^2 t}\, dt = \sqrt{2}\int_0^{2\pi} \sqrt{1-\sin t}\, dt = \sqrt{2}\int_0^{2\pi} \frac{\sqrt{1-\sin t}\sqrt{1+\sin t}}{\sqrt{1+\sin t}}\, dt \\
&= \sqrt{2}\int_0^{2\pi} \frac{\sqrt{1-\sin^2 t}}{\sqrt{1+\sin t}}\, dt = \sqrt{2}\int_0^{2\pi} \frac{|\cos t|}{\sqrt{1+\sin t}}\, dt = \sqrt{2}\int_{-\pi/2}^{3\pi/2} \frac{|\cos t|}{\sqrt{1+\sin t}}\, dt \\
&= \sqrt{2}\int_{-\pi/2}^{\pi/2} \frac{\cos t}{\sqrt{1+\sin t}}\, dt + \sqrt{2}\int_{\pi/2}^{3\pi/2} \frac{-\cos t}{\sqrt{1+\sin t}}\, dt \\
&= 2\sqrt{2}\Big[\sqrt{1+\sin t}\Big]_{-\pi/2}^{\pi/2} - 2\sqrt{2}\Big[\sqrt{1+\sin t}\Big]_{\pi/2}^{3\pi/2} \\
&= 2\sqrt{2}\Big[\sqrt{2} - 0\Big] - 2\sqrt{2}\Big[0 - \sqrt{2}\Big] = 8.
\end{aligned}
$$

12.2 Arc Length in Polar Coordinates

Q. How can we, using polar coordinates, find the arc length of a curve $r = r(\theta)$ for $\theta \in [a, b]$?

A. Use the fact that $x(\theta) = r(\theta)\cos\theta$ and $y(\theta) = r(\theta)\sin\theta$:

$$\int ds = \int \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$
$$= \int_a^b \sqrt{[r'(\theta)\cos\theta - r(\theta)\sin\theta]^2 + [r'(\theta)\sin\theta + r(\theta)\cos\theta]^2}\, d\theta$$
$$= \int_a^b \sqrt{r'^2 + r^2}\, d\theta.$$

- The circumference of a circle described in polar coordinates as $r(\theta) = a$ is thus seen to be $\int_0^{2\pi} \sqrt{a^2}\, d\theta = 2\pi a$.
- Consider the circle of radius a centered at $(a, 0)$ described in polar coordinates as $r(\theta) = 2a\cos\theta$. Recall that as θ varies from 0 to 2π, the point (r, θ) moves **twice** around the circle. Therefore, in order to compute the arc length of this curve, we should only integrate from $\theta = 0$ to $\theta = \pi$:

$$\int_0^\pi \sqrt{4a^2\sin^2\theta + 4a^2\cos^2\theta}\, d\theta = \pi(2a) = 2\pi a.$$

Problem 12.3: Find the length of the cardioid $r(\theta) = 1 + \cos\theta$.

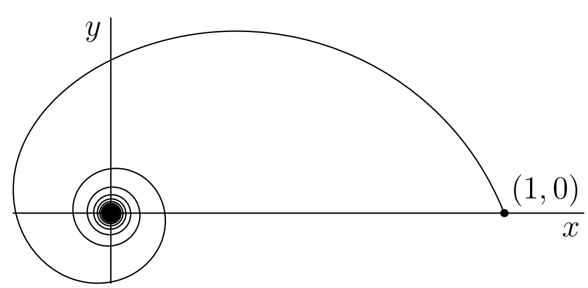
Problem 12.4: Using the polar coordinates $(r\cos\theta, r\sin\theta)$, consider the curve $r = r(\theta) = e^{-a\theta}$ for $\theta \in [0, \infty)$, where $a > 0$.

(a) Sketch the graph of this curve for $a = \frac{1}{2\pi}$.

(b) Show that the total arc length $L = \int_0^\infty \sqrt{r'^2 + r^2}\, d\theta$ of this curve is finite. Evaluate L in terms of a.

Problem 12.5: In polar coordinates, $(x, y) = (r\cos\theta, r\sin\theta)$, consider the curve $r(\theta) = \dfrac{1}{1+\theta}$ for $\theta \in [0, \infty)$.

(a) Sketch this curve on an xy graph.



(b) Express the arc length of this curve as an improper integral.
Since $r'(\theta) = -1/(1+\theta)^2$,

$$\int_0^\infty \sqrt{\frac{1}{(1+\theta)^4} + \frac{1}{(1+\theta)^2}}\, d\theta = \int_1^\infty \sqrt{\frac{1}{u^4} + \frac{1}{u^2}}\, du = \int_1^\infty \frac{1}{u}\sqrt{\frac{1}{u^2} + 1}\, du.$$

(c) Does this curve have finite length? Justify your answer.
No, the integral diverges:

$$0 \le \frac{1}{u} \le \frac{1}{u}\sqrt{\frac{1}{u^2} + 1}$$

and $\displaystyle\int_1^\infty \frac{1}{u}\, du$ diverges, so we know from the Comparison Test that

$$\int_1^\infty \frac{1}{u}\sqrt{\frac{1}{u^2} + 1}\, du$$

also diverges.

12.3 Arc Length in Three Dimensions

In three dimensions, the arc length formula generalizes to

$$\int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt,$$

which can be thought of as the time integral of the speed of the point $\boldsymbol{r}(t)$:

$$\int_a^b |\boldsymbol{r}'(t)|\, dt.$$

- The arc length of the helix $(\cos t, \sin t, t)$ for $t \in [0, 2\pi]$ is given by

$$\int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

Remark: It is often convenient to *reparametrize* a curve by its arc length, which is a natural property of the curve that does not depend on a particular coordinate system. In the above example, we see that the arc length s at parameter value t is given by

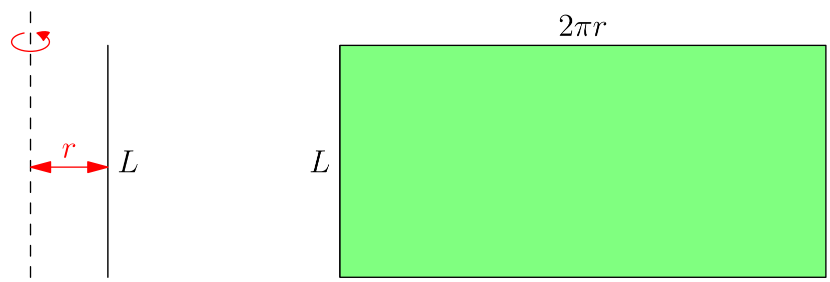
$$s = \int_0^t \sqrt{2}\, d\bar{t} = \sqrt{2}t.$$

If we change variables from t to s, we can then express the helix as $\left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ for $s \in [0, 2\sqrt{2}\pi]$.

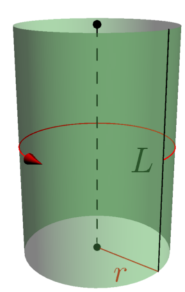
12.4 Surface Area of Revolution

We have already discussed methods for finding the volume of an object that results when we rotate a smooth curve about an axis. We now consider how to find the surface area of such an object.

- If we revolve a line segment L about an axis parallel to itself, we obtain a cylinder. If we cut this cylinder along L and unfold it, we see immediately that its surface area is given by the product of its circumference $2\pi r$ and length L: $A = 2\pi r L$.

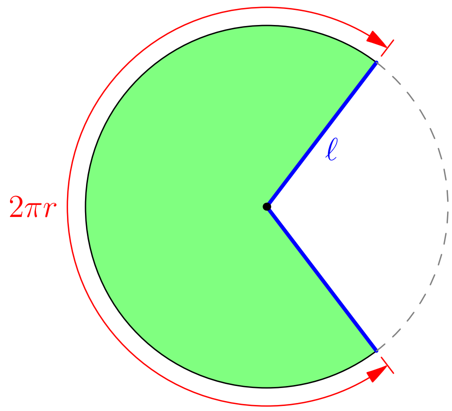


In three dimensions, the green shaded region can be wrapped into a cylinder:

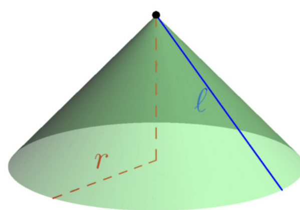


- Similarly, the surface area of a cone of slant height ℓ and radius r is given by

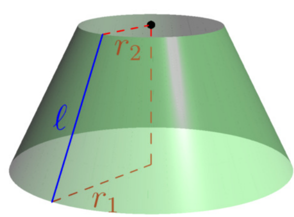
$$A = \underbrace{\left(\frac{2\pi r}{2\pi \ell}\right)}_{\text{fraction of circle}} \underbrace{\pi \ell^2}_{\text{area of circle of radius } \ell} = \pi r \ell.$$

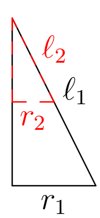


In three dimensions, the two blue lines in the above figure can be joined by wrapping the green shaded region into a cone:



- A *conical band* (*frustum*) is obtained by removing from a large cone of radius r_1 and slant height ℓ_1 a smaller cone of radius $r_2 < r_1$ and slant height ℓ_2 with the same axis of symmetry,





such that (by similar triangles)

$$\frac{\ell_1}{r_1} = \frac{\ell_2}{r_2}.$$

The surface area of a conical band may be computed as the difference of the respective surface areas A_1 and A_2 of the large and small cones. We may express this area as

$$A = A_1 - A_2 = \pi r_1 \ell_1 - \pi r_2 \ell_2 = \pi(r_1\ell_1 - r_2\ell_2) = \pi(r_1 + r_2)(\ell_1 - \ell_2) \doteq 2\pi r \ell$$

since $r_1\ell_2 - r_2\ell_1 = 0$, where $r \doteq (r_1 + r_2)/2$ is the mean radius and $\ell \doteq \ell_1 - \ell_2$ is the length of the sloped edge of the band.

Remark: Thus, the surface area generated by rotating a straight line segment of length ℓ about an axis a mean distance r away is just $2\pi r \ell$.

We can now calculate the surface area of the object formed by rotating any smooth curve about an axis.

Definition: The *surface area* of the object formed by rotating the smooth curve $(x(t), y(t))$ on $[a, b]$ is the unique differentiable function $A(t)$ that satisfies $A(a) = 0$ and the property that

$$\lim_{h\to 0^+} \frac{A(t+h) - A(t)}{2\pi\left(\dfrac{r(t) + r(t+h)}{2}\right)\sqrt{[x(t+h) - x(t)]^2 + [y(t+h) - y(t)]^2}} = 1,$$

for all $t \in [a, b)$, where $r(t)$ is the distance of $(x(t), y(t))$ from the axis of rotation.

Hence

$$A'(t) = 2\pi r(t)\sqrt{[x'(t)]^2 + [y'(t)]^2},$$

so that

$$A(b) = A(b) - A(a) = 2\pi \int_a^b r(t)\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

Remark: An easy way to remember this result is to integrate the product of the circumference $2\pi r$ (associated with a complete revolution of a point on the curve about the axis) and the infinitesimal arc length $ds = \sqrt{dx^2 + dy^2}$.

- The area generated by revolving the curve $y = f(x)$ for $x \in [a, b]$ about the x axis is

$$2\pi \int |y|\, ds = 2\pi \int_a^b |f(x)|\sqrt{1 + [f'(x)]^2}\, dx.$$

- The area generated by revolving the curve $y = f(x)$ for $x \in [a, b]$ about the y axis is

$$2\pi \int |x|\, ds = 2\pi \int_a^b |x|\sqrt{1 + [f'(x)]^2}\, dx.$$

- The surface area of a sphere of radius a can be computed by revolving the curve $y = \sqrt{a^2 - x^2}$ for $x \in [-a, a]$ about the x axis. Since $dy/dx = -x/\sqrt{a^2 - x^2}$, the surface area is seen to be

$$2\pi \int y\, ds = 2\pi \int_{-a}^{a} y\sqrt{1 + \frac{x^2}{a^2 - x^2}}\, dx = 2\pi \int_{-a}^{a} \sqrt{a^2 - x^2}\left(\frac{a}{\sqrt{a^2 - x^2}}\right) dx = 4\pi a^2.$$

- Alternatively, the surface area of a sphere of radius a can be computed using the parametric representation $(a\cos t, a\sin t)$ of a half circle, with $t \in [0, \pi]$. If we rotate this curve about the x axis, the surface area is seen to be

$$2\pi \int y\, ds = 2\pi \int_0^{\pi} a\sin t\sqrt{a^2\sin^2 t + a^2\cos^2 t}\, dt = 2\pi a^2 \int_0^{\pi} \sin t\, dt = 2\pi a^2[-\cos t]_0^{\pi} = 4\pi a^2.$$

- The surface area generated by rotating the section of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ about the y axis can be computed from the formula $2\pi \int x\, ds = 2\pi \int x\sqrt{dx^2 + dy^2}$:

$$\begin{aligned} 2\pi \int_1^2 x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= 2\pi \int_1^2 x\sqrt{1 + 4x^2}\, dx \\ &= 2\pi \left[\frac{2}{3}\left(1 + 4x^2\right)^{\frac{3}{2}}\frac{1}{8}\right]_1^2 = \frac{\pi}{6}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right). \end{aligned}$$

12.5 Curvature

Remark: If we parametrize a curve with respect to arc length s, the derivative $\boldsymbol{T}'(s)$ provides us with a convenient measure, independent of parametrization, of how quickly the curve changes direction.

Definition: The *curvature* κ of a curve with unit tangent vector T parametrized by arc length s is given by

$$\kappa = \left|\frac{d\boldsymbol{T}}{ds}\right|.$$

Remark: Since the arc length up to a point $\boldsymbol{r}(t)$ on a curve is given by

$$s = \int_0^t |\boldsymbol{r}'(\bar{t})|\, d\bar{t},$$

we see that $ds/dt = |\boldsymbol{r}'(t)|$. This allows us to express curvature in terms of an arbitrary parametrization $\boldsymbol{r}(t)$:

$$\kappa = \left|\frac{d\boldsymbol{T}/dt}{ds/dt}\right| = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|}.$$

- The curvature of a straight line is zero (since the tangent vector is constant).

- If we parametrize a circle of radius a as $\boldsymbol{r}(t) = a\cos t\boldsymbol{i} + a\sin t\boldsymbol{j}$, we find $\boldsymbol{r}'(t) = -a\sin t\boldsymbol{i} + a\cos t\boldsymbol{j}$, so that

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \frac{-a\sin t\boldsymbol{i} + a\cos t\boldsymbol{j}}{a} = -\sin t\boldsymbol{i} + \cos t\boldsymbol{j}.$$

The rate of change in the tangent vector,

$$\boldsymbol{T}'(t) = -\cos t\boldsymbol{i} - \sin t\boldsymbol{j},$$

can then be used to find the curvature:

$$\kappa = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{1}{a}.$$

We thus see that small circles have large curvature and large circles have small curvature.

Remark: A more convenient expression for curvature can be obtained by noting that

$$\boldsymbol{r}' = |\boldsymbol{r}'|\,\boldsymbol{T} = \frac{ds}{dt}\boldsymbol{T},$$

so that

$$\boldsymbol{r}'' = \frac{d^2s}{dt^2}\boldsymbol{T} + \frac{ds}{dt}\boldsymbol{T}'.$$

Then since $\boldsymbol{T}\times\boldsymbol{T} = \boldsymbol{0}$, $\boldsymbol{T}\cdot\boldsymbol{T}' = 0$, and $|\boldsymbol{T}| = 1$, we find

$$|\boldsymbol{r}'\times\boldsymbol{r}''| = \left|\frac{ds}{dt}\boldsymbol{T}\times\left(\frac{d^2s}{dt^2}\boldsymbol{T} + \frac{ds}{dt}\boldsymbol{T}'\right)\right| = \left|\left(\frac{ds}{dt}\right)^2\boldsymbol{T}\times\boldsymbol{T}'\right| = |\boldsymbol{r}'|^2\,|\boldsymbol{T}'|\,.$$

Thus

$$\kappa = \frac{|\boldsymbol{T}'|}{|\boldsymbol{r}'|} = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3}.$$

- To find the curvature of the twisted cubic $\boldsymbol{r}(t) = (t, t^2, t^3)$, first compute

$$\boldsymbol{r}'(t) = (1, 2t, 3t^2),$$

and

$$\boldsymbol{r}''(t) = (0, 2, 6t),$$

from which we see that $|\boldsymbol{r}'(t)| = \sqrt{1+4t^2+9t^4}$. Also,

$$\boldsymbol{r}'(t)\times\boldsymbol{r}''(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\boldsymbol{i} - 6t\boldsymbol{j} + 2\boldsymbol{k},$$

so that

$$|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)| = \sqrt{36t^4+36t^2+4} = 2\sqrt{9t^4+9t^2+1}.$$

Thus

$$\kappa(t) = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{2\sqrt{9t^4+9t^2+1}}{(1+4t^2+9t^4)^{3/2}}.$$

Problem 12.6:

(a) Show that the curve

$$r(t) = \left(2+\sqrt{2}\cos t, 1-\sin t, 3+\sin t\right), \qquad t\in\mathbb{R}$$

lies at the intersection of a sphere and a plane.

Let $x = 2+\sqrt{2}\cos t$, $y = 1-\sin t$, $z = 3+\sin t$. Since

$$(x-2)^2+(y-1)^2+(z-3)^2 = 2\cos^2 t+\sin^2 t+\sin^2 t = 2$$

we conclude that the curve lies on the sphere of radius $\sqrt{2}$ centered at $(2,1,3)$. Also, since $y+z = 1-\sin t+3+\sin t = 4$ we conclude that the curve lies in the plane $y+z=4$. The curve therefore lies at the intersection of a sphere and a plane, which is a circle. Notice that the center of the sphere is on this plane and hence the curve is actually a *great circle* on the sphere, having the same center and radius.

(b) Find the curvature at an arbitrary point on the curve.

Since the curvature at every point of a circle of radius R is $1/R$, we conclude that the curvature of the curve is $1/\sqrt{2}$.

Alternatively, we can find the curvature of the curve directly from the formula

$$\kappa = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3}.$$

Since

$$\boldsymbol{r}'(t) = \left(-\sqrt{2}\sin t, -\cos t, \cos t\right)$$

and

$$\boldsymbol{r}''(t) = \left(-\sqrt{2}\cos t, \sin t, -\sin t\right),$$

we find that

$$\boldsymbol{r}'\times\boldsymbol{r}'' = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\sqrt{2}\sin t & -\cos t & \cos t \\ -\sqrt{2}\cos t & \sin t & -\sin t \end{vmatrix} = 0\boldsymbol{i} - \sqrt{2}\boldsymbol{j} - \sqrt{2}\boldsymbol{k}.$$

Hence

$$\kappa = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{2}{\left(\sqrt{2}\right)^3} = \frac{1}{\sqrt{2}}.$$

12.6 The Normal and Binormal Vectors

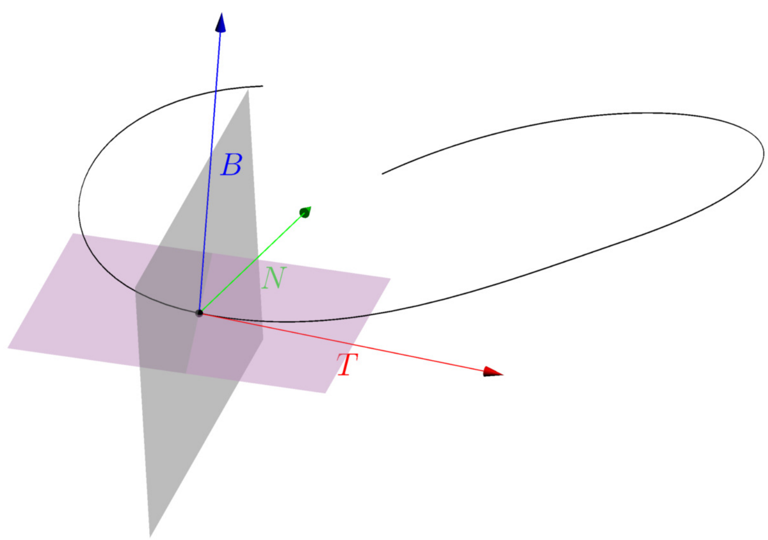
In computer graphics, it is often convenient to represent a curve using a local coordinate system that is aligned with and twists with the curve. One obvious candidate for one of the component vectors of such a coordinate system is the unit tangent vector $\boldsymbol{T} = \dfrac{\boldsymbol{r}'}{|\boldsymbol{r}'|}$. We have already seen that the vector $\boldsymbol{T}'$ is perpendicular to $\boldsymbol{T}$. A good choice for the second component vector is therefore the *unit normal*

$$\boldsymbol{N} = \frac{\boldsymbol{T}'}{|\boldsymbol{T}'|}.$$

To find a third such vector, we simply take the cross product of $\boldsymbol{T}$ and $\boldsymbol{N}$. This is known as the *binormal vector*

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}.$$

Notice that since $\boldsymbol{T}$ and $\boldsymbol{N}$ are unit vectors that are perpendicular to each other, $\boldsymbol{B}$ is already a unit vector. It is perpendicular to both $\boldsymbol{T}$ and $\boldsymbol{N}$.



Remark: The plane spanned by $\boldsymbol{T}$ and $\boldsymbol{N}$ is called the *osculating plane*; this is the plane that comes closest to containing the portion of the curve near P. The circle of radius $1/\kappa$ that lies in the osculating plane, has the same tangent as the curve at P, and lies in the direction of $\boldsymbol{N}$ is called the *osculating circle* or *circle of curvature* to the curve at P. It is the circle that at P has the same tangent, normal, and curvature as the curve itself. The radius $1/\kappa$ is called the *radius of curvature*.

Remark: The plane spanned by $\boldsymbol{N}$ and $\boldsymbol{B}$ at a point P on a curve is called the *normal plane*; this is the set of all vectors that are perpendicular to the tangent vector $\boldsymbol{T}$.

Problem 12.7: Find the equations of the osculating and normal planes of the helix $r(t) = (\cos t, \sin t, t)$ at the point $(0, 1, \pi/2)$.

First note that the point $(0, 1, \pi/2)$ corresponds to the parameter value $t = \pi/2$. Since

$$\boldsymbol{r}'(t) = (-\sin t, \cos t, 1)$$

has magnitude $\sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$, the tangent vector is

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1).$$

Then

$$\boldsymbol{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t, -\sin t, 0).$$

At $t = \pi/2$ we thus find that $\boldsymbol{T}(\pi/2) = \frac{1}{\sqrt{2}}(-1, 0, 1)$ and $\boldsymbol{T}'(\pi/2) = \frac{1}{\sqrt{2}}(0, -1, 0)$. The unit normal vector at $t = \pi/2$ is thus

$$\boldsymbol{N}(\pi/2) = \frac{\boldsymbol{T}'(\pi/2)}{|\boldsymbol{T}'(\pi/2)|} = (0, -1, 0).$$

We can then compute the binormal vector

$$\boldsymbol{B}(\pi/2) = \boldsymbol{T}(\pi/2) \times \boldsymbol{N}(\pi/2) = \frac{1}{\sqrt{2}} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}(1, 0, 1).$$

The osculating plane spanned by $\boldsymbol{T}$ and $\boldsymbol{N}$ has normal $\boldsymbol{B}$, which is parallel to $(1, 0, 1)$, and contains $(0, 1, \pi/2)$:

$$x + (z - \pi/2) = 0.$$

The normal plane spanned by $\boldsymbol{N}$ and $\boldsymbol{B}$ has normal $\boldsymbol{T}$, which is parallel to $(-1, 0, 1)$, and contains $(0, 1, \pi/2)$:

$$-x + (z - \pi/2) = 0.$$

Remark: It is possible to find $\boldsymbol{B}$ directly from $\boldsymbol{r}$ and its derivatives, without first computing $\boldsymbol{T}$ and $\boldsymbol{N}$. From

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|},$$

we see that

$$\boldsymbol{T}'(t) = \frac{\boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|} + \boldsymbol{r}'(t)\frac{d}{dt}\frac{1}{|\boldsymbol{r}'(t)|}.$$

Since

$$\boldsymbol{N} = \frac{\boldsymbol{T}'}{|\boldsymbol{T}'|},$$

we see that the direction of $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$ is therefore the same as the direction of

$$\boldsymbol{r}' \times \boldsymbol{r}'',$$

on noting that $\boldsymbol{r}' \times \boldsymbol{r}' = \boldsymbol{0}$.

- In Problem 12.7, the direction of $\boldsymbol{B}$ can alternatively be computed by evaluating $\boldsymbol{r}'(t) = (-\sin t, \cos t, 1)$ and $\boldsymbol{r}''(t) = (-\cos t, -\sin t, 0)$ at $t = \pi/2$. We find

$$\boldsymbol{r}'(\pi/2) \times \boldsymbol{r}''(\pi/2) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} = (1, 0, 1).$$

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