Extension of Minkowski's polarization result to quasi-concave functions and related geometric inequalities

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Minkowski's theorem

Denote by $\mathcal{K}_c^n$ the family of compact, convex sets in $\mathbb{R}^n$

Theorem (Minkowski)

Fix $K_1, K_2, \ldots, K_m$ in $\mathcal{K}_c^n$. Then the function

$$F(\lambda_1, \lambda_2, \cdots, \lambda_m) = Vol(\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m)$$

is a homogeneous polynomial of degree n, with non-negative coefficients

Here $+$ is the Minkowski addition:

$$A + B = \{a + b : \; a \in A, \; b \in B\}$$

Minkowski's theorem - Contd.

Theorem (Minkowski, second version)

There exists a form $V : (\mathcal{K}_c^n)^n \to [0, \infty)$ *which:*

- *is symmetric in its arguments:* $V(K_1, K_2, \ldots, K_n) = V(K_{\sigma(1)}, K_{\sigma(2)}, \ldots, K_{\sigma(n)})$.
- *is multilinear (linear in each argument).*
- *satisfies* $V(K, K, \ldots, K) = Vol(K)$.

We say that V is the *polarization* of the volume w.r.t the Minkowski addition.

$V(K_1, K_2, \ldots, K_n)$ is called the *mixed volume* of the bodies.

Minkowski's discovery was actually about the Minkowski sum and volume.

The Steiner polynomial was about something else: it is on polynomiality of the volume of an "ε-tube" around a convex body.

Quasi-concave functions

We will work in the following class:

Definition

A function $f : \mathbb{R}^n \to [0, \infty)$ is quasi-concave if

$$f\left(\lambda x + (1-\lambda)y\right) \geq \min\left\{f(x), f(y)\right\}$$

for $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$.

For this talk, also assume that f is upper semicontinuous, $\max f = 1$ and $f(x) \to 0$ as $x \to \infty$.

Denote the class of these functions by $\mathrm{QC}\left(\mathbb{R}^n\right)$.

Operations on quasi-concave functions

Definition

The sum of functions $f, g \in \mathrm{QC}\left(\mathbb{R}^n\right)$ is

$$\left(f \oplus g\right)(x) = \sup_{y+z=x} \min\left\{f(y), g(z)\right\}.$$

Similarly, for $\lambda > 0$ we define a compatible product via

$$\left(\lambda \odot f\right)(x) = f\left(\frac{x}{\lambda}\right).$$

A geometric interpretation of our addition

Fact

A function $f : \mathbb{R}^n \to [0,1]$ *is in* $\mathrm{QC}(\mathbb{R}^n)$ *if and only if its level sets*

$$\overline{K}_t(f) := K_t(f) = \{x \in \mathbb{R}^n : \ f(x) \geq t\}$$

are convex and compact for $0 < t \leq 1$.

Fact

For $f, g \in \mathrm{QC}(\mathbb{R}^n)$ *and* $\lambda > 0$

$$K_t(f \oplus (\lambda \odot g)) = K_t(f) + \lambda K_t(g)$$

In particular, for convex bodies $K, T \in \mathcal{K}_c^n$

$$\mathbf{1}_K \oplus (\lambda \odot \mathbf{1}_T) = \mathbf{1}_{K+\lambda T}$$

A functional Minkowski theorem

Theorem (M.-Rotem)

Fix $f_1, f_2, \ldots, f_m \in \mathrm{QC}\left(\mathbb{R}^n\right)$. *Then the function*

$$F(\lambda_1, \lambda_2, \cdots, \lambda_m) = \int \left((\lambda_1 \odot f_1) \oplus (\lambda_2 \odot f_2) \oplus \cdots \oplus (\lambda_m \odot f_m)\right)$$

is a homogeneous polynomial of degree n, with non-negative coefficients.

Hence, there exists a polarization of the integral w.r.t $\oplus$. i.e. there exists $V : \mathrm{QC}\left(\mathbb{R}^n\right)^n \to [0, \infty]$ which is symmetric, multilinear (w.r.t. our operations) and $V(f, f, \cdots, f) = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$.

We call $V(f_1, f_2, \ldots, f_n)$ the *mixed integral* of $f_1, f_2, \ldots, f_n$.

Properties of the addition

It is obvious that the class $\mathrm{QC}\left(\mathbb{R}^n\right)$ is closed under $\oplus$.

Surprisingly, the same is true for log-concave functions (and also more generally α-concave functions, for any α).

Definition

Fix $-\infty \leq \alpha \leq \infty$. A function $f : \mathbb{R}^n \to [0, \infty)$ is α-concave if for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$ and every $0 \leq \lambda \leq 1$ we have

$$f\left(\lambda x + (1-\lambda)y\right) \geq \left[\lambda f(x)^\alpha + (1-\lambda) f(y)^\alpha\right]^{\frac{1}{\alpha}}.$$

f is log-concave if it is 0-concave, i.e. it satisfies

$$f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}.$$

Like before, assume also that f is upper semicontinuous, $\max f = 1$ and $f(x) \to 0$ as $x \to \infty$.

Addition on convex functions

The operation $\oplus$ on log (and α)-concave functions is induced by an operation $\boxplus$ on convex functions, defined by

$$(\varphi \boxplus \psi)(x) = \inf_{y+z=x} \max\{\varphi(y), \psi(z)\}.$$

Equivalently,

$$\underline{K}_t(\varphi \boxplus \psi) = \underline{K}_t(\varphi) + \underline{K}_t(\psi),$$

where

$$\underline{K}_t(\varphi) = \{x \in \mathbb{R}^n : \ \varphi(x) \leq t\}.$$

Also, $\boxplus$ preserves convexity (!)

Another addition on convex functions

Definition

If $\varphi, \psi : \mathbb{R}^n \to [0, \infty]$ are convex functions, their inf-convolution is defined by
$$(f \square g)(x) = \inf_{y+z=x} \left(\varphi(y) + \psi(z)\right).$$

($\square$ is the Legendre image of the sum of convex functions)

This allows us to define the sum $f \star g$ of log-concave functions, by
$$(f \star g)(x) = \sup_{y+z=x} f(y)g(z)$$

Notice that $\square$ and $\boxplus$ are actually close. For geometric convex functions we have
$$\frac{1}{2}(\varphi \square \psi) \leq \varphi \boxplus \psi \leq \varphi \square \psi$$

Convex sets with measures

As a special case of the general construction, one can define a Minkowski type addition for the class of pairs (K, μ) where K is a convex set and μ a measure (say, log-concave) supported on K.

Indeed, we can identify such a measure μ with its density f and define

$$(K, f) + (T, g) = (K + T, f \oplus g),$$

where, as usual

$$\{x : \ (f \oplus g)(x) \geq t\} = \{x : \ f(x) \geq t\} + \{x : \ g(x) \geq t\}$$

Uniqueness

Question

Assume that $\boxplus$ is an addition operation on $\mathrm{QC}\left(\mathbb{R}^n\right)$ which polarizes the Lebesgue integral. Is it necessarily true that $\boxplus = \oplus$?

In fact, even the corresponding question for $\mathcal{K}_c^n$ is unsolved.

Quermassintegrals

A very important particular case of mixed volumes is the quermassintegrals of K:

$$W_k(K) = V(\underbrace{K, K, \ldots, K}_{n-k \text{ times}}, \underbrace{D, D, \ldots, D}_{k \text{ times}})$$

They are the coefficients of the Steiner polynomial $\operatorname{Vol}(K + \varepsilon D)$.

In our functional generalization,

$$W_k(f) = V(\underbrace{f, f, \ldots, f}_{n-k \text{ times}}, \underbrace{\mathbf{1}_D, \mathbf{1}_D, \ldots, \mathbf{1}_D}_{k \text{ times}})$$

will be called the quermassintegrals of f. These are the coefficients of the generalized Steiner polynomial

$$\int (f \oplus (\varepsilon \odot \mathbf{1}_D)) = \int \sup_{|y|<\varepsilon} f(x+y)dx.$$

This notion was independently introduced by Bobkov, Colesanti and Fragalà (using a different approach).

Inequalities

Remember that a log-concave function $f : \mathbb{R}^n \to [0, \infty)$ is called geometric if $\max_{x \in \mathbb{R}^n} f(x) = f(0) = 1$.

There are well known Alexandrov inequalities for "volume" quermassintegrals. We have the corresponding Alexandrov type inequalities for geometric log-concave functions:

Theorem (M.-Rotem)

Define $g(x) = e^{-|x|}$. For every geometric log-concave function f and every integers $0 \le k < m < n$ we have

$$\left(\frac{W_k(f)}{W_k(g)}\right)^{\frac{1}{n-k}} \le \left(\frac{W_m(f)}{W_m(g)}\right)^{\frac{1}{n-m}},$$

with equality if and only if $f(x) = e^{-c|x|}$ for some $c > 0$.

In particular, this theorem implies an isoperimetric type and an Urysohn type inequalities.

Inequalities for quasi-concave functions

Let us return to our case of quasi-concave functions with $\oplus$ as sum.

Definition

For $f \in \mathrm{QC}(\mathbb{R}^n)$ define its symmetric decreasing rearrangement f^* by the relation

$$K_t(f^*) = K_t(f)^*.$$

Here K^* is the Euclidean ball centered at 0 with the same volume as K.

Facts (M.-Rotem)

Assume $f_i \in \mathrm{QC}(\mathbb{R}^n)$. Then

- $S(f) \geq S(f^*)$, where $S(f) = nV(f, f, \ldots, f, \mathbf{1}_D)$.
- $V(f_1, f_2, \ldots, f_n) \geq V(f_1^*, f_2^*, \ldots, f_n^*)$. If $f_i = \mathbf{1}_{K_i}$ this reduces to $V(K_1, K_2, \ldots, K_n) \geq \left(\prod_{i=1}^n |K_i|\right)^{\frac{1}{n}}$.
- Brunn-Minkowski inequality: $(f_1 \oplus f_2)^* \geq f_1^* \oplus f_2^*$.

Inequalities for quasi-concave functions

More delicate results require a generalized notion of rearrangement. For example:

Definition

For $K \in \mathcal{K}_c^n$ and $0 \leq i \leq n-1$ let K^{W_i} be the Euclidean ball centered around 0 with the same i-th quermassintegral as K.
If $f \in \mathrm{QC}\left(\mathbb{R}^n\right)$ we define f^{W_i} by the relation

$$K_t\left(f^{W_i}\right) = K_t(f)^{W_i}$$

Now the following is a proper generalization of the Alexandrov inequalities:

Proposition

If $0 \leq i < j < n$ then $f^{W_j} \geq f^{W_i}$ for every $f \in \mathrm{QC}\left(\mathbb{R}^n\right)$.

Inequalities (not connected with polynomiality)

Whenever we have a cone structure on a set X and a "size" function $|\cdot|$, we can try and bound the first variation functional

$$V(A, B) = \liminf_{\lambda \to 0^+} \frac{|A \boxplus (\lambda \boxdot B)| - |A|}{\lambda}$$

in terms of $|A|$ and $|B|$.

Polynomiality is NOT a necessary condition for such lower bounds to be meaningful. Several examples include:

Inequalities (not connected with polynomiality)

- Convex bodies containing the origin, with standard volume and the operation of p-sum:

$$h^p_{K+T}(x) = h^p_K(x) + h^p_T(x)$$

We have a Brunn-Minkowski type inequality, which implies an inequality of Minkowski type (Lutwak):

$$V(K,T) \geq \frac{n}{p} |K|^{1-\frac{p}{n}} |T|^{\frac{p}{n}}.$$

In particular, the Urysohn inequality in this case reads

$$\int_{S^{n-1}} h^p_K(\theta) d\sigma(\theta) = \frac{p}{n|D|} \cdot V(D,K) \geq \left(\frac{|K|}{|D|}\right)^{\frac{p}{n}}.$$

Inequalities (not connected with polynomiality)

- Convex bodies containing the origin, with standard volume and the operation of polar sum

$$K \boxplus T = \left(K^\circ + T^\circ\right)^\circ .$$

Again, we have a Brunn-Minkowski type inequality (Firey, Lutwak):

$$\left|K \boxplus T\right|^{-\frac{1}{n}} \geq \left|K\right|^{-\frac{1}{n}} + \left|T\right|^{-\frac{1}{n}} .$$

The corresponding Urysohn inequality, for example, will read

$$\frac{1}{w\left(K^\circ\right)} \leq \left(\frac{|K|}{|D|}\right)^{\frac{1}{n}} ,$$

which is well known and has a lot of meaning in Asymptotic Theory.

Inequalities (not connected with polynomiality)

- Convex bodies, with standard volume and the operation of Blaschke sum.
  The Brunn-Minkowski inequality (Kneser–Süss) is
  $$\left|K\#T\right|^{\frac{n-1}{n}} \geq \left|K\right|^{\frac{n-1}{n}} + \left|T\right|^{\frac{n-1}{n}},$$
  which leads to the bound
  $$V\left(K,T\right) \geq \frac{n}{n-1}\left|K\right|^{\frac{1}{n}}\left|T\right|^{1-\frac{1}{n}}$$

Inequalities (not connected with polynomiality)

- Log-concave functions with Lebesgue integral as volume, and sup-convolution as addition.
  There exists Brunn-Minkowski inequalities (Prèkopa–Leindler), which again yields first variation bounds (Klartag-M., Rotem, Colesanti-Fragalà).
- More generally, α-concave functions with Lebesgue integral as volume, and $\star_\alpha$ as addition.
  The Brunn-Minkowski type inequality is due to Borell, Brascamp and Lieb, and the resulting Urysohn inequality is due to Rotem.

In almost all results, it is possible to replace the volume (or integral) with general quermassintegrals. In the case of α-concave functions, this is due to Bobkov, Colesanti, and Fragalà.