Divergence for log concave functions

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Joint work with C. Schütt and E. Werner

Outline

- K is a convex body in $\mathbb{R}^n$

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- **Affine surface area** $$as_1(K) = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu(x)$$

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- **L_p affine surface area** $$as_p(K) = \int_{\partial K} \frac{\kappa^{\frac{p}{n+p}}}{\langle x, N(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu$$

(Blaschke, Leichtweiss, Lutwak, Meyer-Werner, Schütt-Werner ...)

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Theorem 1 (Artstein-Klartag-Schütt-Werner, '12)

Let $\varphi : \mathbb{R}^n \to [0, \infty)$ *be a log concave function such that*

$\int \varphi dx = 1$.

$$\int_{supp(\varphi)} \varphi \ln \Big(\det \left(Hess \left(-\ln \varphi \right) \right) \Big) dx \le 2 \big[\mathrm{Ent}(\varphi) - \mathrm{Ent}(g) \big]$$

where $\quad g(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{2}}$.

Note that $\quad \mathrm{Ent}(\varphi) = \int \varphi \ln \varphi dx$ and $\mathrm{Ent}(g) = -\ln(2\pi e)^{\frac{n}{2}}$.

Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be a log concave function.

The polar (Artstein-Klartag-Milman) of φ is

$$\varphi^\circ(x) = \inf_y \frac{e^{-\langle x,y \rangle}}{\varphi(y)}$$

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Functional Blaschke Santaló inequality (Ball;

Artstein-Klartag-Milman)

$$\left(\int \varphi dx\right)\left(\int \varphi^\circ dx\right) \leq (2\pi)^n$$

Theorem 2 (C.-Schütt-Werner, '13)

Let $f:(0,\infty)\to\mathbb{R}$ *be a convex function. Let* $\varphi:\mathbb{R}^n\to[0,\infty)$ *be a log concave function.*

$$\int \varphi\, f\left(\frac{e^{\frac{\langle\nabla\varphi,x\rangle}{\varphi}}}{\varphi^2}\det\left(Hess\left(-\ln\varphi\right)\right)\right)dx \geq \left(\int \varphi dx\right)\ f\left(\frac{\int \varphi^{\circ} dx}{\int \varphi dx}\right)$$

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If f *is concave, the inequality is reversed. If* f *is linear, there is equality.*

If f *is strictly convex or concave, there is equality for* $\varphi(x) = Ce^{-\langle Ax, x \rangle}$, *where* $C > 0$ *and* A *is an* $n \times n$ *positive-definite matrix.*

Theorem 2 $\Longrightarrow$ Theorem 1

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Put $f(t) = \ln t$ in Theorem 2:

$$\int \varphi \left[\frac{\langle \nabla \varphi , x\rangle}{\varphi} - 2 \ln \varphi + \ln\left(\det\left(Hess\left(-\ln \varphi\right)\right)\right)\right]$$

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Definition 3 (Csiszar, Morimoto, Ali-Silvery)

(X, μ) is a measure space.

$P = p\mu$ and $Q = q\mu$ are measures on X that are absolutely continuous with respect to the measure μ.

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$f : (0, \infty) \to \mathbb{R}$ is a convex or a concave function.

The f-divergence $D_f(P, Q)$ of P and Q is

$$D_f(P, Q) = \int_X f\left(\frac{p}{q}\right) q d\mu$$

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- **Examples**

1. $f(t) = \ln t$ gives the Kullback-Leibler divergence or relative entropy from P to Q

$$D_{KL}(P\|Q) = \int_X q \ln \frac{p}{q} d\mu.$$

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1. $f(t) = \ln t$ gives the Kullback-Leibler divergence or relative entropy from P to Q

$$D_{KL}(P\|Q) = \int_X q \ln \frac{p}{q} d\mu.$$

2. The convex or concave functions $f(t) = t^\alpha$ leads to the Rényi entropy of order α

$$D_\alpha(P\|Q) = \frac{1}{\alpha - 1} \ln\left(\int_X p^\alpha q^{1-\alpha} d\mu\right)$$

The case $\alpha = 1$ is the relative entropy $D_{KL}(P\|Q)$.

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- Q_K and P_K are the cone measures of K and K°.

- L_p**-affine surface areas** are α-**Rényi entropy powers** **(Werner, '12)**

$$as_p(K) = e^{\frac{n}{n+p} D_{\frac{p}{n+p}}(P_K || Q_K)}$$

with $\alpha = \frac{p}{n+p}$.

The f-divergence for a convex body K in $\mathbb{R}^n$ with respect to the (cone) measures P_K and Q_K was defined by Werner

$$\begin{aligned} D_f(P_K, Q_K) &= \int_{\partial K} f\left(\frac{p_K}{q_K}\right) q_K \, d\mu_K \\ &= \int_{\partial K} f\left(\frac{\kappa(x)}{\langle x, N(x)\rangle^{n+1}}\right) \langle x, N(x)\rangle \, d\mu_K \end{aligned}$$

Definition 4

Let $s \in \mathbb{R}$. $\varphi : \mathbb{R}^n \to [0, \infty)$ is s-concave if for all $\lambda \in (0, 1)$, for all x, y

$$\varphi\left((1-\lambda)x + \lambda y\right) \geq \left[(1-\lambda)\varphi(x)^s + \lambda\varphi(y)^s\right]^{\frac{1}{s}}$$

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- relation to log concave functions:

Let φ be a log concave function. Then for all s

$$\varphi_s = (1 + s \ln \varphi)_+^{\frac{1}{s}}$$

is s-concave and $\varphi_s \to \varphi$ as $s \to 0$

Definition 5 (f-divergences for s-concave functions)

$$q_\varphi^{(s)} = \varphi \left(1 - s\frac{\langle \nabla\varphi, x\rangle}{\varphi}\right)$$

$$p_\varphi^{(s)} = \frac{\det\left[\mathrm{Hess}\,(-\ln\varphi) + s\frac{\nabla\varphi\otimes\nabla\varphi}{\varphi^2}\right]}{\varphi\left(1 - s\frac{\langle\nabla\varphi, x\rangle}{\varphi}\right)^{n+\frac{1}{s}}}$$

$$D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}) =$$

$$\int f\left(\frac{\det\left[\mathrm{Hess}\,(-\ln\varphi) + s\frac{\nabla\varphi\otimes\nabla\varphi}{\varphi^2}\right]}{\varphi^2\left(1 - s\frac{\langle\nabla\varphi, x\rangle}{\varphi}\right)^{n+1+\frac{1}{s}}}\right)\varphi\left(1 - s\frac{\langle\nabla\varphi, x\rangle}{\varphi}\right)dx$$

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for log concave φ with the corresponding $q_\varphi = \varphi$ and

$$p_\varphi = \varphi^{-1}e^{\frac{\langle\nabla\varphi,x\rangle}{\varphi}}\text{det}\left[\text{Hess}\left(-\ln\varphi\right)\right].$$

Theorem 2

$$D_f(P_\varphi, Q_\varphi) \geq \left(\int \varphi dx \right) \, f \, \left(\frac{\int \varphi^\circ dx}{\int \varphi dx} \right)$$

If f is concave, the inequality is reversed. If f is linear, there is equality.

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Theorem 1 Let $\int \varphi dx = 1$.

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- relation to convex bodies

$\varphi : \mathbb{R}^n \to [0, \infty)$ s-concave. $s > 0$ such that $\frac{1}{s} \in \mathbb{N}$.

For such φ we consider the convex bodies introduced by

Artstein-Klartag-Milman

$$K_s(\varphi) = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^{\frac{1}{s}} : \frac{x}{\sqrt{s}} \in \mathrm{supp}(f), \|y\| \le \varphi^s(\frac{x}{\sqrt{s}})\right\}$$

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Proposition 6

$$D_f^{(s)}(P_\varphi^{(s)}, Q_\varphi^{(s)}) = \frac{D_f(P_{K_s(\varphi)}, Q_{K_s(\varphi)})}{s^{\frac{n}{2}} \left|S^{\frac{1}{s}-1}\right|}$$

Thank you!

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