The circular law under log-concavity

Radosław Adamczak

University of Warsaw

Asymptotic Geometric Analysis II
St. Petersburg 2013

Definition

For an $n \times n$ matrix A let μ_A denote its spectral measure, i.e.

$$\mu_A = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(A)},$$

where $\lambda_i(A)$ are the eigenvalues of A.

Theorem (Tao, Vu (2008))

Let $(X_{ij})_{i,j<\infty}$ be an infinite array of i.i.d. mean zero, variance one complex random variables. Let $A_n = (X_{ij})_{i,j\leq n}$. Then the spectral measure of $n^{-1/2}A_n$ converges almost surely as $n \to \infty$ to the uniform measure on the unit disc.

Previous contributions:

Ginibre, Mehta, Girko, Edelman, Bai, Götze-Tikhomirov, Pan-Zhou

Question:

- Can the independence assumption on the entries of A_n be relaxed?
- The first idea: independent entries –> independent rows with some geometric condition?
- The second idea: dependent rows, but an additional symmetry assumption?

Question:

- Can the independence assumption on the entries of A_n be relaxed?
- The first idea: independent entries –> independent rows with some geometric condition?
- The second idea: dependent rows, but an additional symmetry assumption?

Existing results:

- random Markov matrices - Bordenave, Caputo, Chafaï (2008)
- ± 1 matrices with a given row sum - Nguyen, Vu (2012)
- uniform doubly-stochastic matrices - Nguyen (2012)
- truncations of random unitary matrices – Dong, Jiang, Li (2012)
- matrices with independent log-concave isotropic rows – R.A. (2010-2013)
- matrices with log-concave isotropic unconditional distribution – Chafaï, R. A. (2013)

Isotropy, log-concavity

- A random vector X in $\mathbb{R}^n$ is **isotropic** if
$$\mathbb{E}X = 0$$
and
$$\mathbb{E}X \otimes X = \mathrm{Id}$$
or equivalently for all $y \in \mathbb{R}^n$,
$$\mathbb{E}\langle X, y\rangle^2 = |y|^2.$$
- A random vector X in $\mathbb{R}^n$ is log-concave if its law μ satisfies a Brunn-Minkowski type inequality
$$\mu(\theta A + (1-\theta)B) \geq \mu(A)^\theta \mu(B)^{1-\theta}.$$

Theorem (Borell)

A random vector not supported on any $(n-1)$ dimensional hyperplane is log-concave iff it has density of the form $\exp(-V(x))$, where $V\colon \mathbb{R}^n \to (-\infty, \infty]$ is convex.

Theorem (R.A. (2010–2013))

Let A_n be a sequence of $n \times n$ random matrices with independent rows $X_1^{(n)}, \ldots, X_n^{(n)}$ (defined on the same probability space). Assume that for each n and $i \leq n$, $X_i^{(n)}$ has a log-concave isotropic distribution. Then, with probability one, the spectral measure $\mu_{\frac{1}{\sqrt{n}}A_n}$ converges weakly to the uniform distribution on the unit disc.

Strategy of proof (Girko)

Definition

Let μ be a probability measure on $\mathbb{C}$ integrating $\log(|\cdot|)$ at infinity. The logarithmic potential of μ is defined as

$$U_\mu(z) = \int_{\mathbb{C}} \log(|x - z|) d\mu(x).$$

Fact

$$\mu = -\frac{1}{2\pi} \Delta U_\mu.$$

Strategy of proof (Girko)

Definition

Let μ be a probability measure on C integrating $\log(|\cdot|)$ at infinity. The logarithmic potential of μ is defined as

$$U_\mu(z) = \int_{\mathbb{C}} \log(|x - z|) d\mu(x).$$

For the empirical spectral measure of $n^{-1/2}A_n$,

$$U_{\mu_n}(z) = \frac{1}{n} \log|\det(n^{-1/2}A_n - z)| = \frac{1}{2n} \log|\det(A_n - z)|^2$$
$$= \frac{1}{2} \int \log x d\nu_{z,n}(x),$$

where $\nu_{z,n}$ is the empirical spectral measure of the (Hermitian) matrix $(n^{-1/2}A_n - z)(n^{-1/2}A_n - z)^*$.

Strategy of proof (Girko)

Definition

Let μ be a probability measure on C integrating $\log(|\cdot|)$ at infinity. The logarithmic potential of μ is defined as

$$U_\mu(z) = \int_{\mathbb{C}} \log(|x - z|) d\mu(x).$$

Strategy

- Prove that $(\mu_n)_n$ is tight and $\nu_{z,n}$ converge weakly. Use the log-potential to identify the limit.

Strategy of proof (Girko)

Definition

Let μ be a probability measure on C integrating $\log(|\cdot|)$ at infinity. The logarithmic potential of μ is defined as

$$U_\mu(z) = \int_{\mathbb{C}} \log(|x - z|) d\mu(x).$$

Strategy

- Prove that $(\mu_n)_n$ is tight and $\nu_{z,n}$ converge weakly. Use the log-potential to identify the limit.
- Problem: singularities of the logarithm

Strategy of proof (Girko)

Definition

Let μ be a probability measure on C integrating $\log(|\cdot|)$ at infinity. The logarithmic potential of μ is defined as

$$U_\mu(z) = \int_{\mathbb{C}} \log(|x - z|) d\mu(x).$$

Strategy

- Prove that $(\mu_n)_n$ is tight and $\nu_{z,n}$ converge weakly. Use the log-potential to identify the limit.
- Problem: singularities of the logarithm
- One needs to show that for all z, $\log(\cdot)$ is a.s. uniformly integrable with respect to the random measures $\nu_{z,n}$

Log-concave toolkit

Theorem (Prekopa-Leindler (1970's))

Marginals of log-concave isotropic random vectors are themselves isotropic and log-concave.

Theorem (Hensley (1980))

The density of a one-dimensional variance one log-concave variable is bounded by a universal constant.

Theorem (Klartag's thin shell concentration (2007))

Let X be an isotropic log-concave random vector in $\mathbb{R}^n$. There exist numerical positive constants C and c such that for all $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\left|\frac{|X|^2}{n} - 1\right| \geq \varepsilon\right) \leq C\exp(-c\varepsilon^C n^c).$$

Convergence of $\nu_{z,n}$

- We are interested in convergence of the empirical spectral measure of $(n^{-1/2}A_n - z\mathrm{Id})(n^{-1/2}A_n - z\mathrm{Id})^*$.

Convergence of $\nu_{z,n}$

- We are interested in convergence of the empirical spectral measure of $(n^{-1/2}A_n - z\mathrm{Id})(n^{-1/2}A_n - z\mathrm{Id})^*$.
- By general properties of random matrices with independent rows(exponential concentration for the Stieltjes transform), it is enough to prove the convergence of expected spectral measure.

Lemma (folklore(?)– Corollary to Azuma's inequality)

Let A be any $n \times N$ random matrix with independent rows and let $S\colon \mathbb{C}^+ \to \mathbb{C}$ be the Stieltjes transform of the spectral measure of $H = AA^*$. Then for any $\alpha = x + iy \in \mathbb{C}_+$ and any $\varepsilon > 0$,

$$\mathbb{P}(|S_n(\alpha) - \mathbb{E}S_n(\alpha)| \geq \varepsilon) \leq C\exp(-cn\varepsilon^2 y^2).$$

Theorem (R.A. (2011), following Dozier-Silverstein)

Let $N = N_n$ and assume that $n/N \to c > 0$. Let R_n be a deterministic $n \times N$ matrix such that the spectral measure of $\frac{1}{N} R_n R_n^*$ converges to some probability measure H. Let A_n be an $n \times N$ random matrix with independent rows $X_i = X_i^{(n)}$ such that

$$\frac{1}{n} \sum_{i=1}^{n} \sup_{\|C\| \leq 1} \frac{1}{N} \mathbb{E} |\langle C X_i, X_i \rangle - \operatorname{tr} C| = o(1).$$

Then the spectral measure of the matrix $M_n = \frac{1}{N}(R_n + A_n)(R_n + A_n)^*$ converges a.s. to a deterministic probability measure μ, whose Stieltjes transform $S(z) = \int_0^\infty \frac{1}{x-z} \mu(dx)$ is characterized by

$$S(z) = \int_0^\infty \frac{1}{\frac{t}{1+cS(z)} - (1 + S(z))z + 1 - c} H(dt).$$

Remarks

- The Marchenko-Pastur theorem for random matrices with independent rows distributed uniformly on ℓ_p^n balls was proved by Aubrun (2006) by reduction to the independent case

Remarks

- The Marchenko-Pastur theorem for random matrices with independent rows distributed uniformly on ℓ_p^n balls was proved by Aubrun (2006) by reduction to the independent case
- The Marchenko-Pastur theorem for general matrices with independent rows such that

$$\max_{\|C\|\leq 1} \frac{1}{N^2}\mathbb{E}|\langle CX, X\rangle - \mathrm{tr}\, C|^2 = o(1)$$

  was obtained by Pajor and Pastur (2007). They also verified that log-concave isotropic vectors satisfy this assumption.

Remarks

- The Marchenko-Pastur theorem for random matrices with independent rows distributed uniformly on ℓ_p^n balls was proved by Aubrun (2006) by reduction to the independent case
- The Marchenko-Pastur theorem for general matrices with independent rows such that

$$\max_{\|C\|\leq 1} \frac{1}{N^2}\mathbb{E}|\langle CX, X\rangle - \operatorname{tr} C|^2 = o(1)$$

  was obtained by Pajor and Pastur (2007). They also verified that log-concave isotropic vectors satisfy this assumption.
- One can state a more general condition, which works when the rows are dependent. Examples: 1) a generalization of a recent result by O'Rourke for random matrices with decaying correlations, 2) random matrices with exchangeable entries.

Uniform integrability of the $\log(\cdot)$

Following Tao and Vu, one needs three ingredients

- a polynomial bound on the largest singular value
- a polynomial bound on the smallest singular value
- a bound on the distance of a single row of the matrix from the span of some other k-rows. It should be with high probability of the order $\sqrt{n-k}$.

- largest singular value

Theorem (Litvak, Pajor,Tomczak-Jaegermann, R.A. (2010))

With high probability we have

$$\|A\| \leq C\sqrt{n}.$$

In fact for the circular law a weaker bound suffices, so one can simply use Klartag' thin shell or Paouris large deviation inequality for the Hilbert-Schmidt (Euclidean) norm of the matrix.

- smallest singular value

Proposition

Let A_n be an $n \times n$ matrix with independent log-concave isotropic rows and let M_n be any deterministic matrix. Let σ_n be the smallest singular value of $A_n + M_n$. Then with probability at least $1 - n^{-2}$,

$$\sigma_n \geq cn^{-4}.$$

- largest singular value

Theorem (Litvak, Pajor, Tomczak-Jaegermann, R.A. (2010))

With high probability we have

$$\|A\| \leq C\sqrt{n}.$$

In fact for the circular law a weaker bound suffices, so one can simply use Klartag' thin shell or Paouris large deviation inequality for the Hilbert-Schmidt (Euclidean) norm of the matrix.

- smallest singular value

Proof (now standard, following Rudelson-Vershynin).

Let X_i be the rows of $A_n + M_n$. We have

$$\sigma_n \geq \frac{1}{\sqrt{n}} \min_{i \leq n} (\mathrm{dist}(X_i, span\{X_j\}_{j \neq i}).$$

LHS easily bounded by independence of rows and bounded density of marginals. □

Digression: $M_n = 0$

Theorem (Guédon, Litvak, Pajor, Tomczak-Jaegermann, R.A. (2010))

Let A_n be an $n \times n$ matrix with independent log-concave isotropic rows and let σ_n be the smallest singular value of A_n. Then for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\sigma_n \leq c\varepsilon n^{-1/2}\right) \leq C\varepsilon \log^2\left(\frac{2}{\varepsilon}\right).$$

Digression: $M_n = 0$

Theorem (Guédon, Litvak, Pajor, Tomczak-Jaegermann, R.A. (2010))

Let A_n be an $n \times n$ matrix with independent log-concave isotropic rows and let σ_n be the smallest singular value of A_n. Then for every $\varepsilon \in (0, 1)$,

$$\mathbb{P}\left(\sigma_n \le c\varepsilon n^{-1/2}\right) \le C\varepsilon \log^2\left(\frac{2}{\varepsilon}\right).$$

This may be considered a counterpart of the Edelman-Szarek result for the Gaussian case. Remark: everything is smooth so no discrete problems present e.g. for sign matrices.

Digression: $M_n = 0$

Theorem (Guédon, Litvak, Pajor, Tomczak-Jaegermann, R.A. (2010))

Let A_n be an $n \times n$ matrix with independent log-concave isotropic rows and let σ_n be the smallest singular value of A_n. Then for every $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\sigma_n \le c\varepsilon n^{-1/2}\right) \le C\varepsilon \log^2\left(\frac{2}{\varepsilon}\right).$$

This may be considered a counterpart of the Edelman-Szarek result for the Gaussian case. Remark: everything is smooth so no discrete problems present e.g. for sign matrices.

Problems:

- get rid of the log,
- extend to nonzero M_n (for Gaussian matrix - Sankar, Spielman, Teng (2003)).

- distance from the subspace
  We need a good lower estimate on $\mathrm{dist}(X, E)$, where E is a deterministic subspace of $\mathbb{C}^n$ of dimension k.
  For $\mathbb{R}^n$ it follows directly from Klartag's result, since $P_{E^c}X$ is an isotropic log-concave random vector on E^c (by Prekopa-Leindler) and thus
  $$\mathbb{P}\Big(\big||P_{E^c}X|^2 - (n-k)\big| \geq \varepsilon(n-k)\Big) \leq C\exp(-c\varepsilon^C(n-k)^c).$$
  For $\mathbb{C}^n$ one needs some simple adjustments.

- distance from the subspace
  We need a good lower estimate on $\mathrm{dist}(\mathrm{X},\mathrm{E})$, where E is a deterministic subspace of $\mathbb{C}^n$ of dimension k.
  For $\mathbb{R}^n$ it follows directly from Klartag's result, since $P_{E^c}X$ is an isotropic log-concave random vector on E^c (by Prekopa-Leindler) and thus

$$\mathbb{P}\Big(\big||P_{E^c}X|^2-(n-k)\big|\geq\varepsilon(n-k)\Big)\leq C\exp(-c\varepsilon^C(n-k)^c).$$

  For $\mathbb{C}^n$ one needs some simple adjustments.
- Instead of Klartag's result one may also use the following

Theorem (Paouris (2009))

Let X be an isotropic log-concave random vector in $\mathbb{R}^n$ and let A be an $n\times n$ real nonzero matrix. Then for $y\in\mathbb{R}^n$ and $\varepsilon\in(0,c_1)$,

$$\mathbb{P}(|AX-y|\leq\varepsilon\|A\|_{HS})\leq\varepsilon^{c_1(\|A\|_{HS}/\|A\|)},$$

where $c_1>0$ is a universal constant.

Here (after passing to real matrices) $\|A\|_{HS}=\sqrt{n-k}$, $\|A\|\leq 1$.

Beyond independent rows (joint work with D. Chafaï)

Definition

A random vector $X = (X_1, \ldots, X_n)$ *is called* **unconditional** *if its distribution is equal to the distribution of* $(\varepsilon_1 X_1, \ldots, \varepsilon_n X_n)$ *for any choice of* $\varepsilon_i \in \{-1, +1\}$.

Beyond independent rows (joint work with D. Chafaï)

Definition

A random vector $X = (X_1, \dots, X_n)$ is called **unconditional** *if its distribution is equal to the distribution of $(\varepsilon_1 X_1, \dots, \varepsilon_n X_n)$ for any choice of $\varepsilon_i \in \{-1, +1\}$.*

Theorem (Chafaï, R.A. (2013))

Let us identify the space of $n \times n$ real matrices with $\mathbb{R}^{n^2}$ in a natural way. Assume that for each n, A_n is a random matrix with log-concave isotropic unconditional distribution. Then, with probability one, the empirical spectral measure of $\frac{1}{\sqrt{n}} A_n$ converges to the uniform measure on the unit disc.

Remark:

There are models with log-concave isotropic distribution for which the limiting spectral measure is not the circular law (Feinberg-Zee,Guionnet-Krishnapur-Zeitouni).

Convergence of $\nu_{z,n}$

Theorem (R.A. (2010))

Let $A_n = [X_{ij}^{(n)}]_{1\le i\le n, 1\le j\le n}$. *Let us assume that the following assumptions are satisfied*

(A1) *for every* $k \in N$, $\sup_n \max_{i\le n, j\le n} \mathbb{E}|X_{ij}^{(n)}|^k < \infty$,

(A2) *for every* n, i, j, $\mathbb{E}(X_{ij}^{(n)}|\mathcal{F}_{ij}) = 0$, *where* $\mathcal{F}_{ij}$ *is the* σ*-field generated by* $\{X_{kl}^{(n)} : (k,l) \neq (i,j)\}$,

(A3) $|R_n|/\sqrt{n}, |C_n|/\sqrt{n} \to 1$ *in probability, where* R_n *and* C_n *are resp. random row and column of* A_n.

Then the expected spectral measure of

$$(n^{-1/2}A_n - z\mathrm{Id})(n^{-1/2}A_n - z\mathrm{Id})^*$$

converges to a measure which does not depend on the distribution of A_n.

Convergence of $\nu_{z,n}$

- In the log-concave unconditional case the assumptions are satisfied thanks to Klartag's thin shell inequality
- This gives convergence of $\mathbb{E}v_{z,n}$ to the same measure as in the Gaussian case.

Convergence of $\nu_{z,n}$

- In the log-concave unconditional case the assumptions are satisfied thanks to Klartag's thin shell inequality
- This gives convergence of $\mathbb{E}\nu_{z,n}$ to the same measure as in the Gaussian case.
- The a.s. convergence follows by concentration implied by the Poincaré inequality for unconditional log-concave measures, applied to the Stieltjes transform of $\nu_{z,n}$

Theorem (Klartag)

If X is an isotropic unconditional log-concave random vector in $\mathbb{R}^n$, then for every smooth $f\colon \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var} f(X) \le C \log^2(n+1)\mathbb{E}|\nabla f(X)|^2.$$

The largest singular value

Theorem (Latała)

Let X be an unconditional isotropic log-concave random vector in $\mathbb{R}^n$ and let Y be a random vector in $\mathbb{R}^n$ whose components are i.i.d. standard symmetric exponential variables. Then for every norm $\|\cdot\|$ on $\mathbb{R}^n$ and every $t > 0$,

$$\mathbb{P}(\|X\| \geq t) \leq C\mathbb{P}(\|Y\| > t/C).$$

Combining this with known bounds on the operator norm of random matrices with i.i.d. entries and the Poincaré inequality for teh exponential distribution, we get

Lemma

Let M_n be a deterministic $n \times n$ matrix with $\|M_n\| \leq R\sqrt{n}$ for some $R > 0$. Then for all $t \geq 1$,

$$\mathbb{P}(\|A_n + M_n\| \geq (R + C)\sqrt{n} + t) \leq 2\exp(-ct).$$

The smallest singular value

Lemma (Chafaï, R.A.)

Let A_n be an $n \times n$ random matrix with log-concave isotropic unconditional distribution and let M_n be a deterministic $n \times n$ matrix. Then

$$\mathbb{P}(s_n(A_n + M_n) \le n^{-6.5}) \le Cn^{-3/2}.$$

- A better but still suboptimal result can be obtained
- Can one get $\mathbb{P}(s_n(A_n + M_n) \le \varepsilon n^{-1/2}) \le C\varepsilon \log^C(1/\varepsilon)$ at least for $M_n = 0$?
- The proof uses the fact that conditional distribution of a single column given all the remaining columns is log-concave and unconditional. It is not isotropic, but by isotropy, Hensley and Markov's ineq. one can get a lower bound on conditional variances. Then we again use the boundedness of density for one-dimensional log-concave vectors.

Distance from a subspace

Since the rows of A_n are not independent we cannot reduce to a deterministic subspace.

Distance from a subspace

Since the rows of A_n are not independent we cannot reduce to a deterministic subspace.

Lemma (Chafaï, R.A.)

Let M_n a deterministic $n \times n$ matrix with $\|M_n\| \leq R\sqrt{n}$. Denote the rows of $A_n + M_n$ by $Z_1, \ldots, Z_n$ and let H be the space spanned by $Z_1, \ldots, Z_k$ ($k < n$). Then with probability at least $1 - 2n\exp(-c_R(n-k)^{1/3})$,

$$\mathrm{dist}(Z_{k+1}, H) \geq c_R\sqrt{n-k}.$$

Distance from a subspace

Since the rows of A_n are not independent we cannot reduce to a deterministic subspace.

Lemma (Chafaï, R.A.)

Let M_n a deterministic $n \times n$ matrix with $\|M_n\| \le R\sqrt{n}$. Denote the rows of $A_n + M_n$ by $Z_1, \ldots, Z_n$ and let H be the space spanned by $Z_1, \ldots, Z_k$ ($k < n$). Then with probability at least $1 - 2n\exp(-c_R(n-k)^{1/3})$,

$$\mathrm{dist}(Z_{k+1}, H) \ge c_R\sqrt{n-k}.$$

Some (not too well motivated) questions:

- What is the right probability bound?
- Does a similar bound hold without unconditionality (i.e. for log-concave isotropic matrices)?
- Is the dependence on R necessary?

Outline of the proof of the lemma

- For simplicity $M_n = 0$.

Outline of the proof of the lemma

- For simplicity $M_n = 0$.
- Let $e_1, \dots, e_{n-k}$ be an orthonormal basis in $H^\perp$ and X be the $(k+1)th$ row of A_n.

Outline of the proof of the lemma

- For simplicity $M_n = 0$.
- Let $e_1, \dots, e_{n-k}$ be an orthonormal basis in $H^{\perp}$ and X be the $(k+1)th$ row of A_n.
- $X = (X_1\varepsilon_1, \dots, X_n\varepsilon_n)$, where ε's – i.i.d. Rademachers

Outline of the proof of the lemma

- For simplicity $M_n = 0$.
- Let $e_1, \dots, e_{n-k}$ be an orthonormal basis in $H^\perp$ and X be the $(k+1)th$ row of A_n.
- $X = (X_1\varepsilon_1, \dots, X_n\varepsilon_n)$, where ε's – i.i.d. Rademachers
- By applying Talagrand's concentration inequality to ε's we get that

$$\mathbb{P}_\varepsilon\Big(\mathrm{dist}(X,H)^2 \le c\sum_{i=1}^{n-k}\sum_{j=1}^{n} X_j^2|e_{ij}|^2\Big) \le 2\exp\Big(-c\frac{\sum_{i=1}^{n-k}\sum_{j=1}^{n} X_j^2|e_{ij}|^2}{\max_i X_i^2}\Big).$$

Outline of the proof of the lemma

- For simplicity $M_n = 0$.
- Let $e_1, \dots, e_{n-k}$ be an orthonormal basis in $H^\perp$ and X be the $(k+1)th$ row of A_n.
- $X = (X_1\varepsilon_1, \dots, X_n\varepsilon_n)$, where ε's – i.i.d. Rademachers
- By applying Talagrand's concentration inequality to ε's we get that

$$\mathbb{P}_\varepsilon\Big(\mathrm{dist}(X,H)^2 \le c\sum_{i=1}^{n-k}\sum_{j=1}^{n} X_j^2|e_{ij}|^2\Big) \le 2\exp\Big(-c\frac{\sum_{i=1}^{n-k}\sum_{j=1}^{n} X_j^2|e_{ij}|^2}{\max_i X_i^2}\Big).$$

- $\max_i |X_i|$ can be controlled easily by log-concavity, so it remains to prove that with high probability

$$\sum_{i=1}^{n-k}\sum_{j=1}^{n} X_j^2|e_{ij}|^2 \ge c(n-k)$$

- $\max_i |X_i|$ can be controlled easily by log-concavity, so it remains to prove that with high probability

$$\sum_{i=1}^{n-k} \sum_{j=1}^{n} X_j^2 |e_{ij}|^2 \geq c(n-k)$$

- $\max_i |X_i|$ can be controlled easily by log-concavity, so it remains to prove that with high probability

$$\sum_{i=1}^{n-k} \sum_{j=1}^{n} X_j^2 |e_{ij}|^2 \geq c(n-k)$$

- for this we prove that with high probability
  1) e_i are incompressible, so a linear proportion of their coordinates (say at least αn) is greater then $\frac{\theta}{\sqrt{n}}$
  2) at least $(1-\alpha/2)n$ coordinates of X is larger then θ

- $\max_i |X_i|$ can be controlled easily by log-concavity, so it remains to prove that with high probability

$$\sum_{i=1}^{n-k} \sum_{j=1}^{n} X_j^2 |e_{ij}|^2 \geq c(n-k)$$

- for this we prove that with high probability
  1) e_i are incompressible, so a linear proportion of their coordinates (say at least αn) is greater then $\frac{\theta}{\sqrt{n}}$
  2) at least $(1-\alpha/2)n$ coordinates of X is larger then θ

  Thus for each i there are at least $\alpha n/2$ $j's$ s.t. $|X_j| \geq \theta$ and $|e_{ij}| \geq \theta/\sqrt{n}$, which gives

$$\sum_{i=1}^{n-k} \sum_{j=1}^{n} X_j^2 |e_{ij}|^2 \geq \frac{\theta^4 \alpha}{2}(n-k)$$

- $\max_i |X_i|$ can be controlled easily by log-concavity, so it remains to prove that with high probability

$$\sum_{i=1}^{n-k}\sum_{j=1}^{n} X_j^2 |e_{ij}|^2 \geq c(n-k)$$

- for this we prove that with high probability
  1) e_i are incompressible, so a linear proportion of their coordinates (say at least αn) is greater then $\frac{\theta}{\sqrt{n}}$
  2) at least $(1-\alpha/2)n$ coordinates of X is larger then θ

  Thus for each i there are at least $\alpha n/2$ $j's$ s.t. $|X_j| \geq \theta$ and $|e_{ij}| \geq \theta/\sqrt{n}$, which gives

$$\sum_{i=1}^{n-k}\sum_{j=1}^{n} X_j^2 |e_{ij}|^2 \geq \frac{\theta^4\alpha}{2}(n-k)$$

- To prove 1) and 2) one uses the fact that unconditional log-concave measures satisfy the hyperplane conjecture so, their densities in dimension m are bounded by C^m.

Thank you