Thin shell concentration for convex measures.

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Asymptotic Geometric Analysis.
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Kannan Lovász Simonovits Conjecture.

A random vector X is said to be log-concave if it satisfies

$$\mathbb{P}(X \in (1-\theta)A + \theta B) \geq \mathbb{P}(X \in A)^{1-\theta}\mathbb{P}(X \in B)^{\theta}$$

Assume that X is a symmetric log-concave isotropic that is $\mathbb{E}\langle X, \theta \rangle^2 = |\theta|_2^2$.

The KLS conjecture. (Poincaré inequality). There exist a universal constant h such that for every function F,

$$h^2 \operatorname{Var} F(X) \leq \mathbb{E}|\nabla F(X)|_2^2.$$

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Variance conjecture.

$$h^2 \operatorname{Var} |X|_2^2 \leq \mathbb{E}|X|_2^2 = n.$$

Kannan Lovász Simonovits Conjecture.

This conjecture implies :

Strong concentration of the measure in a thin Euclidean shell

$$\mathbb{P}\left(\left||X|_2-\sqrt{n}\right|\geq t\sqrt{n}\right)\leq C\,\exp(-c\,t\,\sqrt{n})$$

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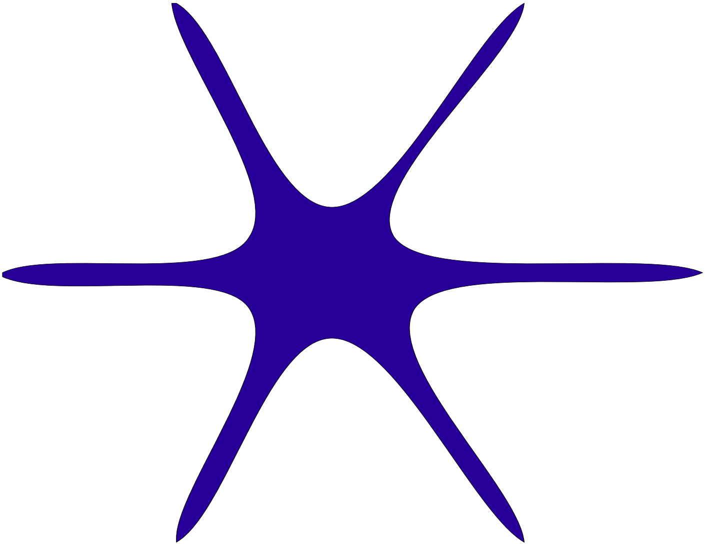
Theorem [Guédon-Milman '11]

$$\forall t \geq 0, \quad \mathbb{P}\left(\left||X|_2 - \sqrt{n}\right| \geq t\sqrt{n}\right) \leq C\exp(-c\sqrt{n}\,\min(t^3, t))$$

$$\mathrm{Var}\,|X|_2^2 \leq C\,n^{5/3} \quad \textit{and} \quad h \geq c\,n^{-5/12}$$

Improvement of results due to Paouris ('06), Klartag ('07), Fleury Guédon Paouris ('07), Fleury ('09).

Pictures - Intuition in high dimension.



convex body in ”isotropic position”.

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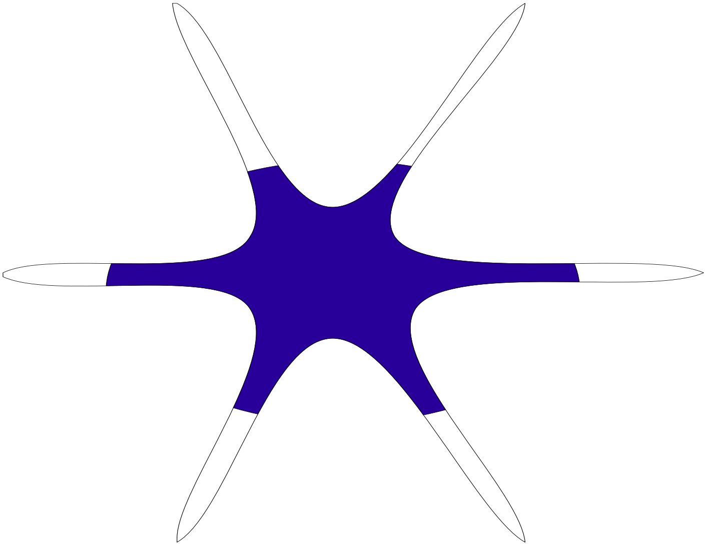

intersection with a ball of radius $\sqrt{n}$.

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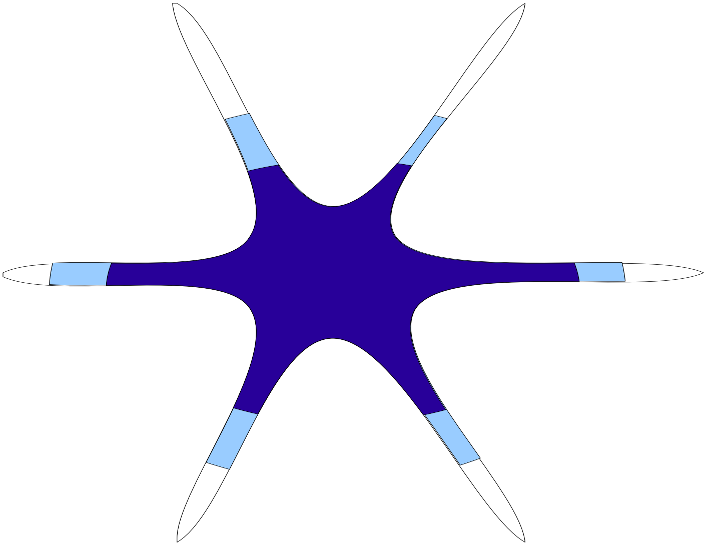
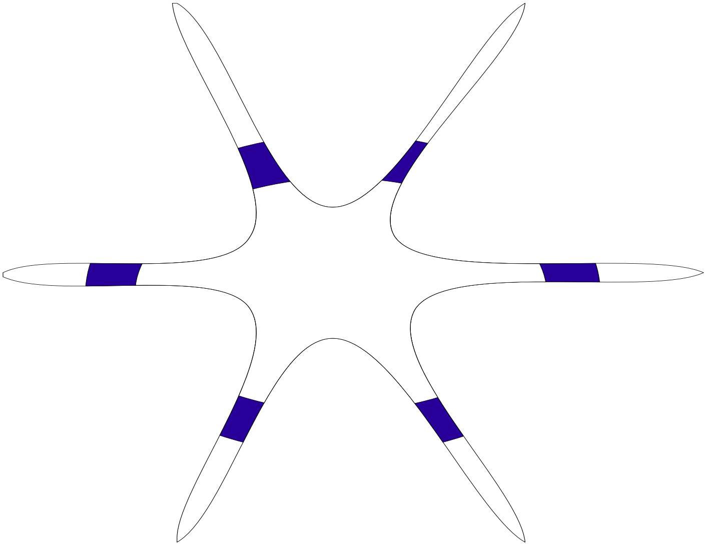


volume inside a ball of radius $100\sqrt{n}$

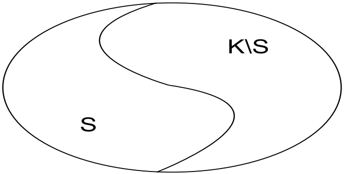
Pictures - Intuition in high dimension.



volume inside a shell of width $\sqrt{n}/n^{1/6}$

Isoperimetric problem.

Let μ be the log-concave probability associated to the log-concave r.v. X.



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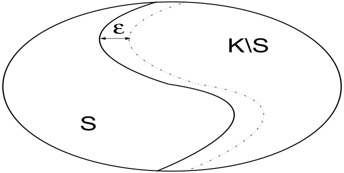
Let μ be the log-concave probability associated to the log-concave r.v. X.

Define

$$\mu^+(S) = \liminf_{\varepsilon \to 0} \frac{\mu(S + \varepsilon B_2^n) - \mu(S)}{\varepsilon}$$



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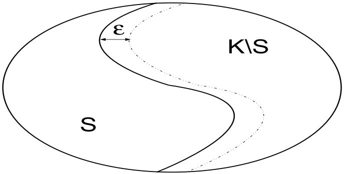

Question. Find the largest h such that

$$\forall\, S \subset K,\ \mu^+(S) \geq \ h\ \mu(S)(1 - \mu(S)) \quad ?$$

μ is log-concave with log concave density.

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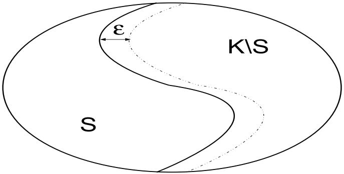

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- E. Milman ('09), Gozlan Roberto Samson ('12)
- Eldan-Klartag ['11], Eldan ['12] : Variance conjecture implies hyperplane conjecture, KLS

Central limit theorem and thin shell

CLT : classical case. $x_1, \dots, x_n$, n i.i.d random variables,

$$\mathbb{E}x_i^2 = 1, \mathbb{E}x_i = 0, \mathbb{E}x_i^3 = \tau$$

then $\forall \theta \in S^{n-1}$

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}\left(\sum_{i=1}^{n} \theta_i x_i \leq t \right) - \int_{-\infty}^{t} e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right| \leq \tau |\theta|_4^2 = \frac{\tau}{\sqrt{n}}.$$

Central limit theorem and thin shell

Question. [Ball '97], [Brehm-Voigt '98] Let K be an isotropic convex body, find a direction $\theta \in S^{n-1}$ such that

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The approach. [Anttila-Ball-Perissinaki '03]

Thin shell estimate implies CLT : $\forall n, \exists \varepsilon_n$ such that for every random vector uniformly distributed in an isotropic convex body

$$\mathbb{P}\left(\left| \frac{|X|_2}{\sqrt{n}} - 1 \right| \ge \varepsilon_n \right) \le \varepsilon_n$$

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Theorem [Bobkov '03] *General isotropic random vector.*

Concentration of the mass in a Euclidean ball or shell

$\Leftrightarrow$

Behavior of $\left(\mathbb{E}|X|_2^p\right)^{1/p}$ for some values of p.

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Behavior of $\left(\mathbb{E}|X|_2^p\right)^{1/p}$ for some values of p.

• X log-concave random vector. *Paouris Theorem (large deviation) may be written as (ALLOPT '12)*

$$\forall p \geq 1, \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq C\, \mathbb{E}|X|_2 + c\, \sigma_p(X) \qquad (\star)$$

where $\sigma_p(X) = \sup_{|z|_2 \leq 1} \left(\mathbb{E}\langle z, X\rangle^p\right)^{1/p}$.

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In isotropic position, $\mathbb{E}|X|_2 \leq \left(\mathbb{E}|X|_2^2\right)^{1/2} = \sqrt{n}$.
By Borell's inequality (Khintchine type inequality)

$$\forall p \geq 1, \quad \left(\mathbb{E}\langle z, X\rangle^p\right)^{1/p} \leq C\,p\,\left(\mathbb{E}\langle z, X\rangle^2\right)^{1/2} = C\,p\,|z|_2$$

Hence $\qquad \forall p \geq 1, \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq C\sqrt{n} + cp$

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Take $p = t\sqrt{n}$, Markov gives

$$\forall t \geq 1, \quad \mathbb{P}\left(|X|_2 \geq t\,\sqrt{n}\right) \leq e^{-c\,t\,\sqrt{n}}.$$

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- Small Ball Estimates of Paouris - Negative moments.

- Variance conjecture - slightly more, cf KLS. In isotropic position,

$$\forall p \in [2, c\sqrt{n}], \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq \sqrt{n} + c\,\frac{p}{\sqrt{n}} = \left(\mathbb{E}|X|_2^2\right)^{1/2}\left(1 + \frac{c\,p}{n}\right).$$

- In view of $(\star)$, more tractable conjecture :

$$\forall p \geq 1, \quad \left(\mathbb{E}|X|_2^p\right)^{1/p} \leq \mathbb{E}|X|_2 + c\,\sigma_p(X)$$

Other probabilistic questions.

For which random vector do we have that for any norm,

$$(\mathbb{E}\|X\|^p)^{1/p} \le C\,\mathbb{E}\|X\| + c \sup_{\|z\|_\star \le 1} (\mathbb{E}\langle z, X\rangle^p)^{1/p}.$$

Examples : Gaussian and Rademacher vectors, for all $p \ge 1$. Other example of the form $X = \sum \xi_i v_i$ with ξ_i independant, symmetric random variables with logarithmicaly concave tails (see the work of Gluskin, Kwapien, Latała).

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Paouris Theorem tells that it is *true for log-concave and the Euclidean norm*.

Latała *True for any ℓ_p-norm for $p \geq 1$.*

New class of random vectors

The hypothesis $H(p, \lambda)$:
Let $p > 0$, $m = \lceil p \rceil$, and $\lambda \geq 1$. A random vector X in E satisfies the assumption $H(p, \lambda)$ if for every linear mapping $A : E \to \mathbb{R}^m$ s. t. $Y = AX$ is non-degenerate there exists a gauge $\| \cdot \|$ on $\mathbb{R}^m$ s. t. $\mathbb{E}\|Y\| < \infty$ and

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• Any m-dimensional norm can be approx. by e^m numbers of linear forms

$$(\mathbb{E}\|Y\|^p)^{1/p} \leq C \left(\mathbb{E} \sup_{i=1,\dots,e^m} |\varphi_i(Y)|^p \right)^{1/p}$$

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$\to$ Rademacher, Gaussian, ψ_2 vectors satisfy $H(p,C\psi^2)$ for every $p\leq n$. Wlog, assume isotropicity of the vector AX

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Results. (AGLLOPT* '12)

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Theorem 1 *Let $p>0$ and $\lambda\geq 1$. If a random vector X satisfies $H(p,\lambda)$ then*

$$(\mathbb{E}|X|_2^p)^{1/p}\leq c\,(\lambda\mathbb{E}|X|_2+\sigma_p(X))$$

where c is a universal constant.

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s-concave random vectors, $s < 0$

Convex measures : definition

Let $s < 1/n$. A probability Borel measure μ on $\mathbb{R}^n$ is called s-concave if $\forall A, B \subset \mathbb{R}^n, \forall \theta \in [0,1]$,

$$\mu((1-\theta)A + \theta B) \geq ((1-\theta)\mu(A)^s + \theta\mu(B)^s)^{1/s}$$

whenever $\mu(A)\mu(B) > 0$.

For $s = 0$, this corresponds to log-concave measures.

The class of s-concave measures was introduced and studied by Borell in the 70's. A s-concave probability ($s \leq 0$) is supported on some convex subset of an affine subspace where it has a density.

s-concave random vectors, $s < 0$

Convex measures : properties

Let $s = -1/r$.

When the support generates the whole space, a convex measure has a density g which has the form

$$g = f^{-\alpha} \quad \text{with} \quad \alpha = n + r$$

and f is a positive convex function on $\mathbb{R}^n$. (Borell).

Example :

$$g(x) = c(1 + \|x\|)^{-n-r}, r > 0.$$

• A log-concave prob is $(-1/r)$-concave for any $r > 0$

• The linear image of a $(-1/r)$-concave vector is also $(-1/r)$-concave.

• The Euclidean norm of a $(-1/r)$-concave random vector has moments of order $0 < p < r$.

Convex measures and $H(p, \lambda)$

Theorem 2. *Let $r \geq 2$ and X be a $(-1/r)$-concave random vector. Then for every $0 < p < r/2$, X satisfies the assumption $H(p, C)$, C being a universal constant.*

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Recall $H(p, \lambda)$: for every linear mapping $A : E \to \mathbb{R}^m$ s. t. $Y = AX$ is non-degenerate there exists a gauge $\| \cdot \|$ on $\mathbb{R}^m$ s. t. $\mathbb{E}\|Y\| < \infty$ and

$$(\mathbb{E}\|Y\|^p)^{1/p} \leq \lambda\, \mathbb{E}\|Y\|.$$

For $Y = AX$ symmetric, the norm is defined by a level set of the density of g_Y. Its unit ball is

$$K_\alpha = \{t \in \mathbb{R}^m : g_y(t) \leq \alpha^m \|g_Y\|_\infty\}$$

Convex measures and $H(p, \lambda)$

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Theorem 3. *Let $r \geq 2$ and X be a $(-1/r)$-concave random vector. Then for every $0 < p < r/2$,*

$$(\mathbb{E}|X|_2^p)^{1/p} \leq C(\mathbb{E}|X|_2 + \sigma_p(X)).$$

Convex measures. Concentration of $|X|_2$. Large deviation

Corollary. *Let* $r \geq 2$ *and* X *be a* $(-1/r)$*-concave random vector. Then for every* $t > 0$*,*

$$\mathbb{P}\big(|X|_2 > t\sqrt{n}\big) \leq \left(\frac{c\max(1, r/\sqrt{n})}{t}\right)^{r/2}$$

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Corollary. *Let $r \geq \log n$ and X be a $(-1/r)$-concave isotropic random vector. Let $X_1, \ldots, X_N$ be independent copies of X. Then for every $\varepsilon \in (0,1)$ and every $N \geq C(\varepsilon)n$, one has*

$$\mathbb{E}\left\|\frac{1}{N}\sum_{i=1}^{N} X_i X_i - I\right\| \leq \varepsilon.$$

Convex measures. Concentration of $|X|_2$. Thin shell. (FGP* '13)

Theorem 4. *Let X be a $(-1/r)$-concave isotropic random vector on $\mathbb{R}^n$. Then for any $p \in \mathbb{R}$ such that $0 < |p| \leq \frac{1}{2}\min(r, n^{1/3})$, we have*

$$\left| \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right| \leq \frac{C\,|p-2|}{r} + \left(\frac{C\,|p-2|}{n^{1/3}} \right)^{3/5}$$

where C is a universal constant.

$\star$ Fradelizi, Guédon, Pajor

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where C is a universal constant.

Optimality. There are $(-1/r)$-concave isotropic random vector on $\mathbb{R}^n$ for which

$$\left| \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right| \geq \frac{C\,|p-2|}{r}.$$

So r must go to infinity with the dimension n.

$\star$ Fradelizi, Guédon, Pajor

Convex measures. Concentration of $|X|_2$. Thin shell. (FGP* '13)

Theorem 4. *Let X be a $(-1/r)$-concave isotropic random vector on $\mathbb{R}^n$. Then for any $p \in \mathbb{R}$ such that $0 < |p| \leq \frac{1}{2}\min(r, n^{1/3})$, we have*

$$\left| \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \right| \leq \frac{C\,|p-2|}{r} + \left(\frac{C\,|p-2|}{n^{1/3}} \right)^{3/5}$$

where C is a universal constant.

Central Limit Theorem.

$$\sigma_{n-1}\left(\theta \in S^{n-1} \;:\; \sup_{t \in \mathbb{R}} |\mathbb{P}(\langle X, \theta \rangle \leq t) - \Phi(t)| \geq 4\varepsilon(X) + \delta \right) \leq 4n^{3/8} e^{-cn\delta^4}$$

$\star$ Fradelizi, Guédon, Pajor

Adaptation of several tools valid for log-concave measures to the class of $-1/r$-concave measures.

Adaptation of several tools valid for log-concave measures to the class of $-1/r$-concave measures.

Theorem 5. *Let* $\alpha > 0$ *and* $f : [0, \infty) \to [0, \infty)$ *be* $(-1/\alpha)$*-concave and integrable. Define* $H_f : [0, \alpha) \to \mathbb{R}_+$ *by*

$$H_f(p) = \begin{cases} \dfrac{1}{B(p, \alpha - p)} \displaystyle\int_0^{+\infty} t^{p-1} f(t) dt, & 0 < p < \alpha \\ f(0) & p = 0 \end{cases}$$

where

$$B(p, \alpha - p) = \int_0^1 u^{p-1}(1-u)^{\alpha-p-1} du = \int_0^{+\infty} t^{p-1}(t+1)^{-\alpha} dt.$$

Then H_f *is log-concave on* $[0, \alpha)$*.*

Adaptation of several tools valid for log-concave measures to the class of $-1/r$-concave measures.

Berwald type inequality with sharp constants.

Corollary. *Let $r > 0$ and μ be a $(-1/r)$-concave measure on $\mathbb{R}^n$. Let $\phi : \mathbb{R}^n \to \mathbb{R}_+$ be concave on its support. Then for any $0 < p \le q < r$,*

$$\left(\int_{\mathbb{R}^n} \phi(x)^q d\mu(x)\right)^{1/q} \le \frac{(qB(q, r-q))^{1/q}}{(pB(p, r-p))^{1/p}} \left(\int_{\mathbb{R}^n} \phi(x)^p d\mu(x)\right)^{1/p}$$

and for every $\theta \in S^{n-1}$,

$$\begin{aligned}\left(\int_{\mathbb{R}^n} \langle x, \theta\rangle_+^q \, d\mu(x)\right)^{1/q} &\le \frac{(qB(q, r-q))^{1/q}}{(pB(p, r-p))^{1/p}} \left(\int_{\mathbb{R}^n} \langle x, \theta\rangle_+^p \, d\mu(x)\right)^{1/p} \\ &\le \frac{C\,q}{p} \left(\int_{\mathbb{R}^n} \langle x, \theta\rangle_+^p \, d\mu(x)\right)^{1/p}\end{aligned}$$

Strategy of the proof

Let $1 \leq k < n$.

$$\frac{\mathbb{E}|X|_2^p}{\mathbb{E}|G_n|_2^p} = \frac{\mathbb{E}_{F,X}|P_F X|_2^p}{\mathbb{E}_{F,G}|P_F G_n|_2^p} = \frac{\mathbb{E}_{F,X}|P_F X|_2^p}{\mathbb{E}_G|G_k|_2^p}.$$

We get

$$\mathbb{E}|X|_2^p = \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\mathbb{E}_{F,X}|P_F X|_2^p$$

Rewriting using the invariance of the Haar measure and polar coordinates :

$$\mathbb{E}_{F,X}|P_F X|_2^p = \mathbb{E}_U h_{k,p}(U)$$

where U is uniformly distributed over $SO(n)$, and $h_{k,p} : SO(n) \to \mathbb{R}_+$ is defined as :

$$h_{k,p}(U) := \mathrm{vol}(S^{k-1}) \int_0^\infty t^{p+k-1} \pi_{U(E_0)} w(tU(\theta_0))dt,$$

$E_0 \in G_{n,k}$, $\theta_0 \in S(E_0)$, w denotes the density of X in $\mathbb{R}^n$.

Strategy of the proof

Properties of $h_{k,p}$?

$$h_{k,p}(U) := \mathrm{vol}(S^{k-1}) \int_0^\infty t^{p+k-1} \pi_{U(E_0)} w(tU(\theta_0))dt \quad , \quad U \in SO(n).$$

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Some log-concavity

Corollary *Let* $r > 0$*. For any* $-1/(r+n)$*-concave function* $w : \mathbb{R}^n \to \mathbb{R}_+$*, and any subspace* F *of dimension* $k \leq n$*,*

$$p \mapsto \begin{cases} \dfrac{h_{k,p}(U)}{B(p+k, r-p)}, & p > -k+1, \\ \mathrm{vol}(S^{k-1})\pi_{U(E_0)}w(0), & p = -k+1 \end{cases}$$

is log-concave on $[-k+1, r)$*.*

Strategy of the proof

An estimate from above of its log-Lipschitz constant.
The Z_p^+ bodies associated to a random vector X

$$h_{Z_p^+(X)}(\theta) = \left(\mathbb{E}\langle X, \theta\rangle_+^p\right)^{1/p}$$

Proposition *Let $n \geq 1$, $r > 2$ and w be a $-1/(r+n)$-concave density of a probability measure on $\mathbb{R}^n$. Let $1 \leq k \leq \min(\frac{n-1}{2}, \frac{r}{2} - 1)$ and $-\frac{k}{2} + 1 \leq p \leq r - 1$. Denote by $L_{k,p}$ the log-Lipschitz constant of $U \mapsto h_{k,p}(U)$. Then*

$$L_{k,p} \leq C \max(k,p) d(Z^+_{\max(k,p)}(w), B_2^n)$$

where C is a universal constant.

Strategy of the proof

An estimate from above of its log-Lipschitz constant.

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Proposition *Let* $n \geq 1$*,* $r > 2$ *and* w *be a* $-1/(r+n)$*-concave density of a probability measure on* $\mathbb{R}^n$*. Let* $1 \leq k \leq \min(\frac{n-1}{2}, \frac{r}{2} - 1)$ *and* $-\frac{k}{2} + 1 \leq p \leq r - 1$*. Denote by* $L_{k,p}$ *the log-Lipschitz constant of* $U \mapsto h_{k,p}(U)$*. Then*

$$L_{k,p} \leq C \max(k,p) d(Z^+_{\max(k,p)}(w), B_2^n)$$

where C *is a universal constant.*

End of the proof.

Log-Sobolev inequality on $SO(n)$.

THANK YOU