# Permanent estimators via random matrices

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joint work with Ofer Zeitouni

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# Permanent of a matrix

Let $A$ be an $n \times n$ matrix with $a_{i,j} \geq 0$.

**Permanent of $A$:**

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Evaluation of permanents is $\#P$-complete (Valiant 1979)
if there exists a polynomial-time algorithm for permanent evaluation, then any $\#P$ problem can be solved in polynomial time.
Fast computation $\Rightarrow$ P=NP.

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# Applications of permanents

### Wick's formula

Let $f_1, \ldots, f_n, g_1, \ldots, g_n$ be complex centered normal random variables. Then

$$\mathbb{E} \prod_{j=1}^{n} f_j \bar{g}_j = \operatorname{perm}(A),$$

where $A$ is the correlation matrix: $a_{i,j} = \mathbb{E} f_i \bar{g}_j$.

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## Perfect matchings

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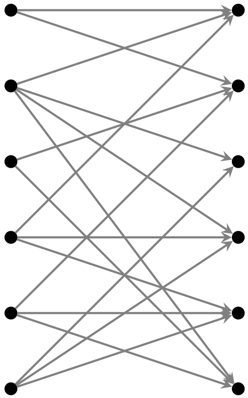

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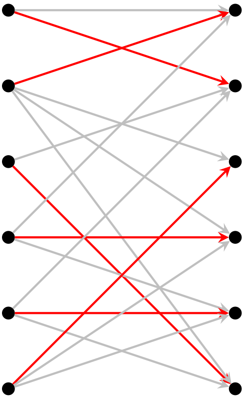

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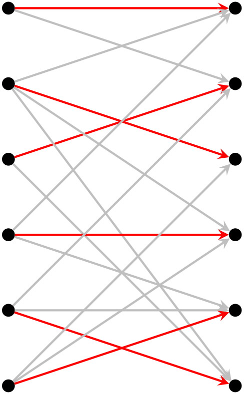

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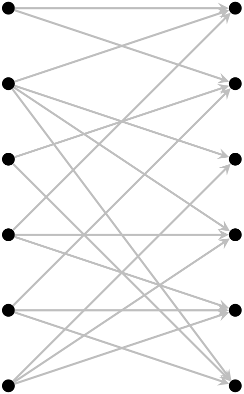
$$\#(\text{perfect matchings}) = \text{perm}(A),$$

where $A$ is the adjacency matrix of the graph:

$$a_{i,j} = 1 \quad \text{if } i \to j.$$



# Deterministic bounds

- Linial–Samorodnitsky–Wigderson algoritm: if $\operatorname{perm}(A) > 0$, then one can find in polynomial time diagonal matrices $D, D'$ such that the renormalized matrix $A' = D'AD$ is almost doubly stochastic:

$$1 - \varepsilon < \sum_{i=1}^{n} a'_{i,j} < 1 + \varepsilon, \qquad \text{for all } j = 1, \dots, n$$

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- $\operatorname{perm}(A) = \prod_{i=1}^{n} d_i \cdot \prod_{j=1}^{n} d'_j \cdot \operatorname{perm}(A')$

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- Linial–Samorodnitsky–Wigderson algoritm: reduces permanent estimates to almost doubly stochastic matrices
- Van der Waerden conjecture, proved by Falikman and Egorychev: if $A$ is doubly stochastic, then

$$1 \geq \text{perm}(A) \geq \frac{n!}{n^n} \approx e^{-n}$$

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- Bregman's theorem (1973) implies that if $A$ is doubly stochastic, and $\max a_{i,j} \leq t \cdot \min a_{i,j}$, then

$$\mathrm{perm}(A) \leq e^{-n} \cdot n^{O(t^2)}$$

- Conclusion: if $\max a_{i,j} \leq t \cdot \min a_{i,j}$, then Linial–Samorodnitsky–Wigderson algoritm with multiplicative error $n^{O(t^2)}$
- Doesn't cover matrices with zeros.

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  Let $R$ be an $n \times n$ random matrix with i.i.d. $\pm 1$ entries.
  Form the Hadamard product $R \odot A_{1/2}$: $\quad w_{i,j} = \sqrt{a_{i,j}} \cdot r_{i,j}$.
  Then
  $$\mathrm{perm}(A) = \mathbb{E} \det^2(R \odot A_{1/2}).$$
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- Advantage: Godsil–Gutman estimator is faster than any other algorithm.
- Deficiency: Godsil–Gutman estimator performs well for "generic" matrices, but fails for large classes of $\{0, 1\}$ matrices, because of arithmetic issues.

# Barvinok's estimator

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**Theorem (Barvinok)**

*Let $A$ be any $n \times n$ matrix. Then, with high probability,*

$$((1-\varepsilon)\cdot\theta)^n \,\mathrm{perm}(A) \le \det^2(G \odot A_{1/2}) \le C\,\mathrm{perm}(A),$$

*where $C$ is an absolute constant and*

- $\theta = 0.28$ *for real Gaussian matrices;*
- $\theta = 0.56$ *for complex Gaussian matrices;*

# Subexponential bounds for Barvinok's estimator

- Identity matrix: multiplicative error at least $\exp(cn)$ w.h.p.
- Matrix of all ones: multiplicative error at most $\exp(C\sqrt{\log n})$ (Goodman, 1963).
- What happens for other matrices?

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- Balanced entries (Friedland, Rider, Zeitouni, 2004):
  if $\max a_{i,j} \le t \cdot \min a_{i,j}$, then

$$e^{-o(n)} \le \frac{\det^2(G \odot A_{1/2})}{\operatorname{perm}(A)} \le e^{o(n)}$$

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- The bound is asymptotic.
- Not applicable for matrices with zeros.
- Linial–Samorodnitsky–Wigderson algorithm estimates the permanent with polynomial error for balanced matrices.

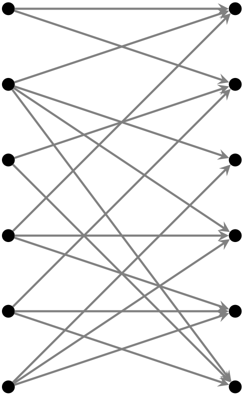
# Subexponential bounds for Barvinok's estimator

Question:

for which graphs would Barvinok's estimator yield a small error?

# Strongly connected bipartite graphs

Let $\Gamma = (L, R, V)$ be an $n \times n$ bipartite graph.



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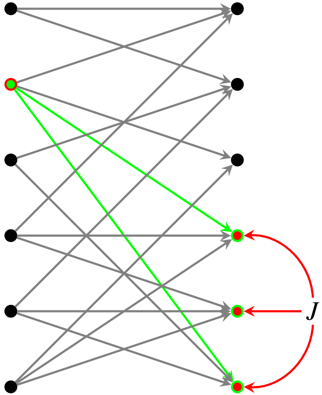
A vertex $i \in L$ is $\delta$-strongly connected to a set $J \subset R$ if it is connected to at least $(\delta/2) \cdot |J|$ vertices of $J$.



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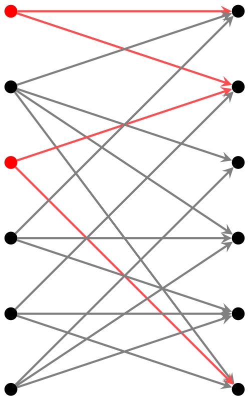

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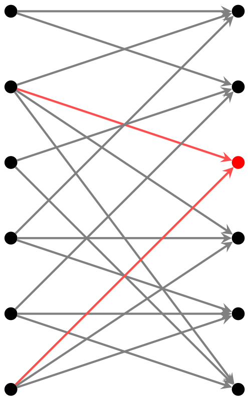

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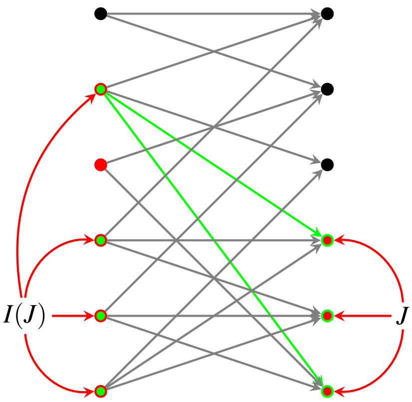
3. Strong expansion condition: for any set $J \subset [m]$ the set of its $\delta$-strongly connected neighbors has the cardinality $|I(J)| \geq \min\left((1+\kappa)|J|, n\right)$.



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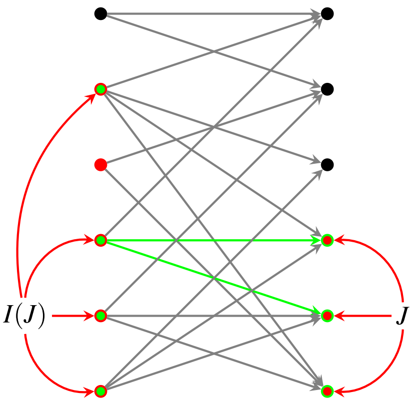


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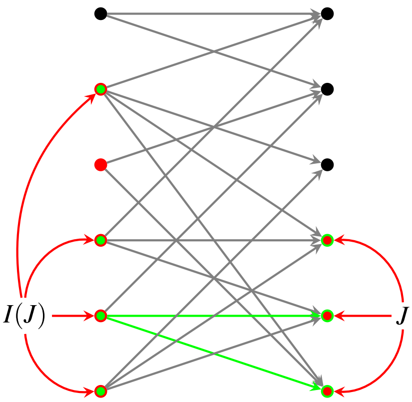


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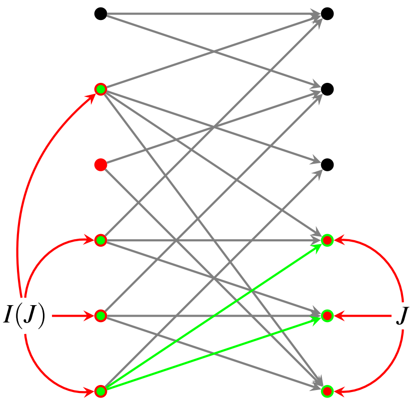

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Theorem (R'–Zeitouni, 2013)

*Let A be the adjacency matrix A of an $n \times n$ bipartite graph, which has*

- *the minimal degree at least $\delta n$ with some $\delta > 0$;*
- *expander-type property*

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# Results for matrices

### Large entries graph

Let $s > 0$ and let $B$ be an $n \times n$ matrix $B$ with non-negative entries.
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$$B = \begin{pmatrix} 0.7 & 0 & 0.1 & 0.5 \\ 0.1 & 0.6 & 0.8 & 0.2 \\ 0.6 & 0.6 & 0.3 & 0.5 \\ 0.2 & 0.8 & 0.7 & 0.3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad (s = 0.5)$$

Consider matrices with strongly connected large entries graphs.

# Results for matrices

### Theorem

*Let $B$ be an $n \times n$ matrix such that*

$$\sum_{i=1}^{n} b_{i,j} \le 1 \quad \textit{for all } j \in [n]; \qquad \textit{and} \qquad \sum_{j=1}^{n} b_{i,j} \le 1 \quad \textit{for all } i \in [n],$$

*and $0 \le b_{i,j} \le b_n/n$, where $0 < b_n \le n$.*
*Assume that the large entries graph $\Gamma_B(1/n)$ is $(\delta, \kappa)$-strongly connected.*

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- Large maximal entry: $\max b_{i,j} = \Omega(1)$ or $b_n = \Omega(n)$:
  - Barvinok's estimator is well-concentrated: $(\tau b_n n)^{1/3} = O(n^{2/3})$;
  - It may be concentrated exponentially far from the permanent: $\sqrt{b_n n} = \Omega(n)$.
  - Consistent failure is possible.

# Example of a consistent failure

Let $B$ be the $n \times n$ matrix with entries

$$b_{i,j} = \begin{cases} \theta & \text{if } i = j \\ \frac{1-\theta}{n-1} & \text{if } i \neq j \end{cases}.$$

- The matrix $B$ is doubly stochastic for $\theta \in (0, 1)$.
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**Theorem**

*There exists $\theta_0 < 1$ such that for any $\theta \in (\theta_0, 1)$*

$$\det^2(B_{1/2} \odot G) < e^{-cn} \operatorname{perm}(B)$$

*with high probability.*

# Approach to concentration

- Aim: $X(G) := \det^2(A_{1/2} \odot G)$ is concentrated.
- $\det^2(A_{1/2} \odot G)$ is highly non-linear $\Rightarrow$ $\log(\det^2(A_{1/2} \odot G))$ is easier to control.
- Modified aim : $Y(G) = \log \det^2(A_{1/2} \odot G)$ is concentrated around its expectation.
  We will have to compare the concentration for $X(G)$ and $Y(G)$ at the end.
- There exists a subgaussian concentration inequality for Lipschitz functions on $\mathbb{R}^{n \times n}$ with respect to the gaussian measure.
- $\log \det^2(A_{1/2} \odot G)$ is not Lipschitz.
- Main challenge: using the Lipschitz concentration for a non–Lipschitz function.

# Concentration for Gaussian measure

Aim: $Y(G) = \log \det^2(A_{1/2} \odot G)$ is concentrated around its expectation. There exists a subgaussian concentration inequality for Lipschitz functions on $\mathbb{R}^{n \times n}$ with respect to the gaussian measure:

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- $\log\det^2(A_{1/2} \odot G) = 2\sum_{j=1}^{n} \log s_j(A_{1/2} \odot G)$.
- The maping $G \to A_{1/2} \odot G$ is Lipschitz.
- The mapping $M \to \big(s_1(M), \ldots, s_n(M)\big)$ is Lipschitz.
- Truncated logarithm $\log_\varepsilon x = \max(\log x, \varepsilon)$ is a Lipschitz function.

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   $\Rightarrow$ contribution of the last singular values is limited.
5. How to choose the threshold $k$?

- Smaller $k$ $\Rightarrow$ smaller error.
- Larger $k$ $\Rightarrow$ stronger concentration.

# Choosing the right threshold

$$\log \det{}^2 (A_{1/2} \odot G) = \sum_{j=1}^{n-k} \log_{\varepsilon_j} s_j(A_{1/2} \odot G) + \sum_{j=n-k+1}^{n} \text{error terms}$$

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$$\mathbb{E}\log\det^2(A_{1/2}\odot G) \approx \log\mathbb{E}\det^2(A_{1/2}\odot G) = \log\mathrm{perm}(A)$$

  up to the error terms.
- We had to use a random variable to connect two deterministic quantities.