Random almost spherical sections of centrally-symmetric convex sets

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Introduction

Dependence on n and ϵ in Dvoretzky's theorem

Given $n \in \mathbb{N}$ and $\epsilon > 0$, what is the largest m such that any n-dimensional normed space contains a $(1+\epsilon)$-Euclidean subspace?

The right dependence of m on the dimension n was found by V. Milman. The best known (so far) lower bound for m is due to G. Schechtman:

Theorem [G. Schechtman]

$m \geq c \frac{\epsilon}{\ln^2(1/\epsilon)} \ln n$, for an absolute constant c.

Outline of the proof of the theorem of G. Schechtman

Assume that $X = (\mathbb{R}^n, \|\cdot\|)$ is a normed space with the unit ball $B_{\|\cdot\|}$ in John's position. Let $g = (g_1, g_2, \ldots, g_n)$ be the standard Gaussian vector. There are two possibilities:

X **is "far from** l_∞^n**".** Precisely, $\mathbb{E}\|g\| \geq C\sqrt{\frac{\ln n}{\epsilon}}$. Then from the classical argument, it follows that X contains a $(1+\epsilon)$-Euclidean subspace of dimension $c\epsilon \ln n / \ln \frac{1}{\epsilon}$.

Otherwise, by applying certain argument (which involves James' blocking), one can find a subspace $E \subset X$ with $\dim E \geq n^{c\epsilon/\ln\frac{1}{\epsilon}}$, which is $(1+\epsilon)$-isometric to $l_\infty^{\dim E}$. It is well known that l_∞^k contains a $(1+\epsilon)$-Euclidean subspace of dimension $c\frac{\ln k}{\ln(1/\epsilon)}$. Thus, X contains a $(1+C\epsilon)$-Euclidean subspace of dimension

$$c\frac{\ln(n^{\epsilon/\ln(1/\epsilon)})}{\ln(1/\epsilon)} = c\frac{\epsilon}{\ln^2(1/\epsilon)}\ln n.$$

What is done

In this talk, a modified proof of

the theorem of G. Schechtman is considered,

without the construction of l_∞-subspaces

and not using James' blocking.

Motivation

The space l_∞^n contains $(1+\epsilon)$-Euclidean subspaces of dimension $c\frac{\ln n}{\ln(1/\epsilon)}$. But how *many* almost Euclidean subspaces are there?

[G. Schechtman]

For any natural n and m, if the Haar measure of "$(1+\epsilon)$-spherical" m-dimensional sections of l_∞^n is greater than $1-n^{-C\epsilon}$ then with necessity $m \leq c\epsilon \ln n$.

In other words, in case of l_∞^n, if we put an additional restriction on the Haar measure of almost Euclidean subspaces then the dependence on ϵ becomes much worse.

Motivation

In case of an arbitrary normed space $(\mathbb{R}^n, \|\cdot\|)$, the theorem of G. Schechtman tells us that there always exists a $(1+\epsilon)$-Euclidean subspace of dimension $m = c\frac{\epsilon}{\ln^2(1/\epsilon)} \ln n$.

But, as in the case of l_∞^n, we can also ask how many $(1+\epsilon)$-spherical subspaces of dimension m the space contains.

?

For a normed space $(\mathbb{R}^n, \|\cdot\|)$, can we always find a bijective linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ such that for the norm $\|T\cdot\|$ defined as

$$\|T\cdot\| : x \in \mathbb{R}^n \to \|Tx\|,$$

the vast majority (with respect to the Haar measure) of subspaces of $(\mathbb{R}^n, \|T\cdot\|)$ of dimension $c\frac{\epsilon}{\ln^2(1/\epsilon)} \ln n$ are $(1+\epsilon)$-Euclidean (and even "$(1+\epsilon)$-spherical")?

The Result

A positive answer to the last question can be obtained **if in the original proof of G. Schechtman we replace the construction of $(1+\epsilon)$-isometric l_∞-sections by a different procedure.**

The Result

Given any normed space $(\mathbb{R}^n, \|\cdot\|)$ and any $\epsilon > 0$, there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ (depending on $\|\cdot\|$ and ϵ) such that the Haar measure of $(1+\epsilon)$-spherical sections of $(\mathbb{R}^n, \|T\cdot\|)$ of dimension $c\frac{\epsilon}{\ln^2(1/\epsilon)}\ln n$ is greater than $1 - n^{-c\epsilon/\ln\frac{1}{\epsilon}}$.

The Proposition

Recall that $g = (g_1, g_2, \dots, g_n)$ is the standard Gaussian vector in $\mathbb{R}^n$. For any subspace $E \subset \mathbb{R}^n$, let Proj_E be the orthogonal projection onto E. The Result follows from

Proposition

Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$ with the unit ball in John's position. Then for any $\epsilon \in (0, 1/2]$ there is a subspace $E = E(\epsilon, \|\cdot\|) \subset \mathbb{R}^n$ of dimension at least $\sqrt{n}$ such that

$$\mathbb{P}\left\{\left|\|\mathrm{Proj}_E g\| - \mathrm{Med}\,\|\mathrm{Proj}_E g\|\right| > \epsilon \mathrm{Med}\,\|\mathrm{Proj}_E g\|\right\} \le 2\exp\left(-c\epsilon \ln n / \ln \frac{1}{\epsilon}\right).$$

Proof of the Proposition

So, we assume that $\|\cdot\|$ is a norm in $\mathbb{R}^n$ and the unit ball $B_{\|\cdot\|}$ is in John's position. As in the original proof of G. Schechtman, we consider two possibilities:

* The space $(\mathbb{R}^n, \|\cdot\|)$ is "far" from l_∞^n, precisely, $\mathrm{Med}\|g\| \geq C\sqrt{\frac{\ln n}{\epsilon}}$. Then the standard concentration inequality for Gaussians gives

$$\mathbb{P}\left\{|\|g\| - \mathrm{Med}\,\|g\|| > \epsilon \mathrm{Med}\,\|g\|\right\} \leq 2\exp\bigl(-c\epsilon \ln n\bigr),$$

  so in this case the Proposition holds with $E = \mathbb{R}^n$.

Proof of the Proposition

* Otherwise, $\mathrm{Med}\|g\| \leq C\sqrt{\frac{\ln n}{\epsilon}}$. Without loss of generality (rotation + Dvoretzky–Rogers), we can assume that the norm of the first $n/2$ standard unit vectors

$$\|e_1\|, \|e_2\|, \dots, \|e_{n/2}\| \geq 1/4.$$

For any $\delta > 0$ and any subset $J \subset \{1, 2, \dots, n\}$, let $M(J, \delta)$ be the number such that

$$\mathbb{P}\{\|g\chi_J\| > M(J, \delta)\} = \delta.$$

Let $k = \exp\left(c\epsilon \ln n / \ln \frac{1}{\epsilon}\right)$. Next, we apply some procedure to generate a subset $A \subset \{1, 2, \dots, n/2\}$ of cardinality at least $\sqrt{n}$ such that there is a partition of A: $\{A_1, A_2, \dots, A_k\}$ such that

$$M(A_i, k^{-1/2}) \geq (1 - \epsilon) M(A, k^{-1/2}) \text{ for all } i = 1, 2, \dots, k.$$

Proof of the Proposition

So, A is of cardinality at least $\sqrt{n}$ and there is a partition $\{A_1, A_2, \ldots, A_k\}$ of A such that

$$M(A_i, k^{-1/2}) \geq (1-\epsilon)M(A, k^{-1/2}) \text{ for all } i = 1, 2, \ldots, k,$$

where for any subset J: $\mathbb{P}\{\|g\chi_J\| > M(J, k^{-1/2})\} = k^{-1/2}$.

Claim

$$\mathbb{P}\left\{\|g\chi_A\| < (1-2\epsilon)M(A, k^{-1/2})\right\} \leq \exp(-\sqrt{k}) + k^{-1/2}.$$

Proof of the claim is elementary when the norm $\|\cdot\|$ is unconditional: Since $A_1, A_2, \ldots, A_k$ are pairwise disjoint, the random variables $\|g\chi_{A_1}\|, \|g\chi_{A_2}\|, \ldots, \|g\chi_{A_k}\|$ are independent, so

$$\mathbb{P}\left\{\|g\chi_A\| < (1-\epsilon)M(A, k^{-1/2})\right\} \leq \mathbb{P}\left\{\max_i \|g\chi_{A_i}\| < \ldots\right\}$$

$$\leq \prod_{i=1}^{k} \mathbb{P}\left\{\|g\chi_{A_i}\| < M(A_i, k^{-1/2})\right\} = \left(1 - k^{-1/2}\right)^k \leq \exp(-\sqrt{k}).$$

In fact, the unconditionality is unnecessary.

Proof of the Proposition

Thus, A is a set of cardinality at least $\sqrt{n}$, and, by the claim,

$$\mathbb{P}\left\{\|g\chi_A\| < (1-2\epsilon)M(A,k^{-1/2})\right\} \le 2k^{-1/2}.$$

On the other hand, by the definition of $M(A,k^{-1/2})$,

$$\mathbb{P}\left\{\|g\chi_A\| > M(A,k^{-1/2})\right\} = k^{-1/2}.$$

Recall that we defined k as $k = \exp\left(c\epsilon \ln n / \ln \frac{1}{\epsilon}\right)$. Then from the above estimates we get for $L = (1-\epsilon)M(A,k^{-1/2})$

$$\mathbb{P}\left\{|\|g\chi_A\| - L| > \epsilon M(A,k^{-1/2})\right\} \le 3\exp\left(-c\epsilon \ln n / \ln \frac{1}{\epsilon}\right).$$

It is easy to derive from the last inequality

$$\mathbb{P}\left\{|\|g\chi_A\| - \mathrm{Med}\|g\chi_A\|| > C\epsilon \mathrm{Med}\|g\chi_A\|\right\} \le 2\exp\left(-\tilde{c}\epsilon \ln n / \ln \frac{1}{\epsilon}\right),$$

So the Proposition holds with $E = \mathrm{span}\{e_i : \ i \in A\}$.

The Result

We've just proved

Proposition

Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$ with the unit ball in John's position. Then for any $\epsilon \in (0, 1/2]$ there is a subspace $E = E(\epsilon, \|\cdot\|) \subset \mathbb{R}^n$ of dimension at least $\sqrt{n}$ such that

$$\mathbb{P}\left\{\left|\|\mathrm{Proj}_E g\| - \mathrm{Med}\,\|\mathrm{Proj}_E g\|\right| > \epsilon \mathrm{Med}\,\|\mathrm{Proj}_E g\|\right\} \le 2\exp\left(-c\epsilon \ln n / \ln \frac{1}{\epsilon}\right).$$

Clearly, we can find a bijective linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|Tx\| \approx \|\mathrm{Proj}_E x\|, \quad x \in S^{n-1}.$$

Then, in particular, we obtain

$$\mathbb{P}\left\{\left|\|Tg\| - \mathrm{Med}\,\|Tg\|\right| > \epsilon \mathrm{Med}\,\|Tg\|\right\} \le 2\exp\left(-c\epsilon \ln n / \ln \frac{1}{\epsilon}\right).$$

The Result

The last identity implies The Result:

The Result

Given any normed space $(\mathbb{R}^n, \|\cdot\|)$ and any $\epsilon > 0$, there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ (depending on $\|\cdot\|$ and ϵ) such that the Haar measure of $(1+\epsilon)$-spherical sections of $(\mathbb{R}^n, \|T\cdot\|)$ of dimension $c\frac{\epsilon}{\ln^2(1/\epsilon)} \ln n$ is greater than $1 - n^{-c\epsilon/\ln\frac{1}{\epsilon}}$.

The unit ball of the norm $\|T\cdot\|$ —

$$\{x \in \mathbb{R}^n : \; \|Tx\| \leq 1\}$$

— looks like this: