L_p-affine surface areas for log concave functions

joint with

U. Caglar, M. Fradelizi, O. Guedon, J. Lehec, C. Schütt

γ_n standard Gaussian measure on $\mathbb{R}^n$ with density $\frac{e^{-\frac{\|x\|^2}{2}}}{(2\pi)^{\frac{n}{2}}}$

μ probability measure on $\mathbb{R}^n$

γ_n standard Gaussian measure on $\mathbb{R}^n$ with density $\frac{e^{-\frac{\|x\|^2}{2}}}{(2\pi)^{\frac{n}{2}}}$

μ probability measure on $\mathbb{R}^n$

Log Sobolev inequality (Stam, Federbush, Gross)

$$H(\mu \mid \gamma_n) \leq \frac{1}{2} \;\; I(\mu \mid \gamma_n)$$

relative entropy $H(\mu \mid \gamma_n) = \int_{\mathbb{R}^n} \log\left(\frac{d\mu}{d\gamma_n}\right) d\mu$

Fisher information $I(\mu \mid \gamma) = \int_{\mathbb{R}^n} \left\| \nabla \log\left(\frac{d\mu}{d\gamma_n}\right) \right\|^2 d\mu$

$$H(\mu \mid \gamma_n) \leq \frac{1}{2} \; I(\mu \mid \gamma_n)$$

$$H(\mu \mid \gamma_n) \leq \frac{1}{2} \; I(\mu \mid \gamma_n)$$

Slight improvement (e.g. Bakry+Ledoux)

$$H(\mu \mid \gamma_n) \leq \frac{C(\mu)}{2} + \frac{n}{2} \; \log\left(1 + \frac{I(\mu \mid \gamma_n) - C(\mu)}{n}\right)$$

$$C(\mu) = \int_{\mathbb{R}^n} \|x\|^2 \, d\mu - n$$

$$H(\mu \mid \gamma_n) \leq \frac{1}{2} \; I(\mu \mid \gamma_n)$$

Slight improvement (e.g. Bakry+Ledoux)

$$H(\mu \mid \gamma_n) \leq \frac{C(\mu)}{2} + \frac{n}{2} \; \log\left(1 + \frac{I(\mu \mid \gamma_n) - C(\mu)}{n}\right)$$

$$C(\mu) = \int_{\mathbb{R}^n} \|x\|^2 \, d\mu - n$$

Let $\psi = -\log(d\mu/dx)$. The **Shannon entropy** of μ

$$S(\mu) = \int_{\mathbb{R}^n} \psi \, \mu(dx) = -H(\mu \mid dx)$$

The inequality

$$H(\mu \mid \gamma_n) \le \frac{C(\mu)}{2} + \frac{n}{2} \log\left(1 + \frac{I(\mu \mid \gamma_n) - C(\mu)}{n}\right)$$

is equivalent to

$$2\left(-S(\mu) + S(\gamma_n)\right) \le n \log\left(\frac{\int_{\mathbb{R}^n} \Delta\psi \, d\mu}{n}\right)$$

$\Delta\psi$ is the Laplace operator of ψ

If the measure μ is log-concave, then a reverse log Sobolev inequality holds

Theorem 1 (Artstein, Klartag, Schütt, W)

If $\psi = -\log(d\mu/dx)$ *is a convex function that is* $\mathcal{C}^2$*-smooth and strictly convex on its domain, then*

$$\int_{\mathbb{R}^n} \log\big(\det(Hess\psi)\big)\, d\mu \leq 2\big(-S(\mu) + S(\gamma_n)\big).$$

If the measure μ is log-concave, then a reverse log Sobolev inequality holds

Theorem 1 (Artstein, Klartag, Schütt, W)

If $\psi = -\log(d\mu/dx)$ is a convex function that is $\mathcal{C}^2$-smooth and strictly convex on its domain, then

$$\int_{\mathbb{R}^n} \log\big(\det(Hess\psi)\big)\, d\mu \le 2\big(-S(\mu) + S(\gamma_n)\big).$$

Equality (Caglar, Fradelizi, Guedon, Lehec, Schütt, W)

Equality holds if and only if μ is a Gaussian (with any mean and any positive definite covariant matrix).

If the measure μ is log-concave, then a reverse log Sobolev inequality holds

Theorem 1 (Artstein, Klartag, Schütt, W)

If $\psi = -\log(d\mu/dx)$ *is a convex function that is* $\mathcal{C}^2$*-smooth and strictly convex on its domain, then*

$$\int_{\mathbb{R}^n} \log\big(\det(Hess\psi)\big)\, d\mu \leq 2\big(-S(\mu) + S(\gamma_n)\big).$$

Equality (Caglar, Fradelizi, Guedon, Lehec, Schütt, W)

Equality holds if and only if μ *is a Gaussian (with any mean and any positive definite covariant matrix).*

The left hand side and the right hand side of the inequality are invariant under affine transformations

Functional Blaschke Santaló inequality

(Artstein+Klartag+Milman, Ball, Fradelizi+Meyer, Lehec)

f and g non-negative functions on $\mathbb{R}^n$ such that for all x, y

$$f(x)\ g(y) \leq e^{-\langle x,y \rangle}$$

If $\int xf(x)dx = 0$, then

$$\left(\int f dx\right)\left(\int g dx\right) \leq (2\pi)^n$$

with equality off there is a positive definite matrix A and $c > 0$ such that

$$f(x) = c\ e^{-\frac{\langle Ax,x\rangle}{2}} \qquad g(y) = \frac{1}{c}\ e^{-\frac{\langle A^{-1}y,y\rangle}{2}}$$

Proof of Theorem 1

$$\psi = -\log(d\mu/dx) \quad \text{or} \quad d\mu = e^{-\psi}dx$$

Put

$$f(x) = e^{-\psi(x)} \qquad g(y) = e^{-\psi^*(y)}$$

Legendre transform $\quad \psi^*(y) = \sup_x (\langle x, y\rangle - \psi(x))$

Proof of Theorem 1

$$\psi = -\log(d\mu/dx) \quad \text{or} \quad d\mu = e^{-\psi}dx$$

Put

$$f(x) = e^{-\psi(x)} \qquad g(y) = e^{-\psi^*(y)}$$

Legendre transform $\quad \psi^*(y) = \sup_x\left(\langle x, y\rangle - \psi(x)\right)$

Then

$$f(x)\, g(y) \le e^{-\langle x,y\rangle}$$

Proof of Theorem 1

$$\psi = -\log(d\mu/dx) \quad \text{or} \quad d\mu = e^{-\psi}dx$$

Put

$$f(x) = e^{-\psi(x)} \qquad g(y) = e^{-\psi^*(y)}$$

Legendre transform $\quad \psi^*(y) = \sup_x (\langle x, y\rangle - \psi(x))$

Then

$$f(x)\ g(y) \leq e^{-\langle x,y\rangle}$$

Invariance of μ under translations $\implies$ we can assume that

$$\int x d\mu = \int x e^{-\psi} dx = \int x f dx = 0$$

Blaschke Santaló inequality $\implies$

$$(2\pi)^n \geq \left(\int e^{-\psi(x)} dx\right) \left(\int e^{-\psi^*(y)} dy\right) = \int e^{-\psi^*(y)} dy$$

Blaschke Santaló inequality $\Longrightarrow$

$$(2\pi)^n \geq \left(\int e^{-\psi(x)} dx\right) \left(\int e^{-\psi^*(y)} dy\right) = \int e^{-\psi^*(y)} dy$$

Definition of Legendre transform $\Longrightarrow$ $\psi(x) + \psi^*(y) \geq \langle x, y \rangle$

with equality iff $y = \nabla \psi(x)$

Blaschke Santaló inequality $\implies$

$$(2\pi)^n \geq \left(\int e^{-\psi(x)} dx\right) \left(\int e^{-\psi^*(y)} dy\right) = \int e^{-\psi^*(y)} dy$$

Definition of Legendre transform $\implies$ $\psi(x) + \psi^*(y) \geq \langle x, y \rangle$

with equality iff $y = \nabla\psi(x)$

Change of variable $\quad y = \nabla\psi(x)$

Blaschke Santaló inequality $\implies$

$$(2\pi)^n \geq \left(\int e^{-\psi(x)}dx\right)\left(\int e^{-\psi^*(y)}dy\right) = \int e^{-\psi^*(y)}dy$$

Definition of Legendre transform $\implies$ $\psi(x) + \psi^*(y) \geq \langle x, y\rangle$

with equality iff $y = \nabla\psi(x)$

Change of variable $\quad y = \nabla\psi(x)$

$$(2\pi)^n \geq \int e^{-\psi^*(y)}dy = \int e^{-\psi^*(\nabla\psi(x))}\ \det(\text{Hess}\psi)dx$$

Blaschke Santaló inequality $\implies$

$$(2\pi)^n \geq \left(\int e^{-\psi(x)}dx\right)\left(\int e^{-\psi^*(y)}dy\right) = \int e^{-\psi^*(y)}dy$$

Definition of Legendre transform $\implies$ $\psi(x) + \psi^*(y) \geq \langle x, y\rangle$

with equality iff $y = \nabla\psi(x)$

Change of variable $\quad y = \nabla\psi(x)$

$$(2\pi)^n \geq \int e^{-\psi^*(y)}dy = \int e^{-\psi^*(\nabla\psi(x))} \det(\mathrm{Hess}\psi)dx$$

$$\psi^*(\nabla\psi(x)) = \langle x, \nabla\psi(x)\rangle - \psi(x)$$

Blaschke Santaló inequality $\implies$

$$(2\pi)^n \geq \left(\int e^{-\psi(x)} dx\right) \left(\int e^{-\psi^*(y)} dy\right) = \int e^{-\psi^*(y)} dy$$

Definition of Legendre transform $\implies$ $\psi(x) + \psi^*(y) \geq \langle x, y \rangle$

with equality iff $y = \nabla\psi(x)$

Change of variable $\quad y = \nabla\psi(x)$

$$(2\pi)^n \geq \int e^{-\psi^*(y)} dy = \int e^{-\psi^*(\nabla\psi(x))} \ \det(\text{Hess}\psi) dx$$

$$\psi^*(\nabla\psi(x)) = \langle x, \nabla\psi(x) \rangle - \psi(x)$$

$$= \int e^{\psi(x) - \langle x, \nabla\psi(x) \rangle} \ \det(\text{Hess}\psi) dx$$

$$(2\pi)^n \geq \int e^{\psi(x) - \langle x, \nabla\psi(x) \rangle} \det(\text{Hess}\psi) dx$$

$$d\mu = e^{-\psi} dx$$

$$(2\pi)^n \geq \int e^{\psi(x)-\langle x,\nabla\psi(x)\rangle} \ \det(\mathrm{Hess}\psi)dx$$

$$d\mu = e^{-\psi}dx$$

$$= \int e^{2\psi(x)-\langle x,\nabla\psi(x)\rangle} \ \det(\mathrm{Hess}\psi)d\mu(x)$$

$$(2\pi)^n \geq \int e^{\psi(x) - \langle x, \nabla\psi(x)\rangle} \ \det(\mathrm{Hess}\psi) dx$$

$$d\mu = e^{-\psi} dx$$

$$= \int e^{2\psi(x) - \langle x, \nabla\psi(x)\rangle} \ \det(\mathrm{Hess}\psi) d\mu(x)$$

$$\log \downarrow$$

$$\log(2\pi)^n \geq \log\left(\int e^{2\psi(x) - \langle x, \nabla\psi(x)\rangle} \ \det(\mathrm{Hess}\psi) d\mu(x)\right)$$

$$(2\pi)^n \geq \int e^{\psi(x) - \langle x, \nabla\psi(x)\rangle} \det(\mathrm{Hess}\psi) dx$$

$d\mu = e^{-\psi} dx$

$$= \int e^{2\psi(x) - \langle x, \nabla\psi(x)\rangle} \det(\mathrm{Hess}\psi) d\mu(x)$$

$\log \downarrow$

$$\log(2\pi)^n \geq \log\left(\int e^{2\psi(x) - \langle x, \nabla\psi(x)\rangle} \det(\mathrm{Hess}\psi) d\mu(x)\right)$$

Jensen

$$\geq 2\int \psi(x) d\mu(x) - \int \langle x, \nabla\psi(x)\rangle d\mu(x) +$$

$$\int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x)$$

$$\log(2\pi)^n \geq \log\left(\int e^{2\psi(x)-\langle x,\nabla\psi(x)\rangle} \det(\mathrm{Hess}\psi) d\mu(x)\right)$$

Jensen

$$\geq 2\int \psi(x) d\mu(x) - \int \langle x, \nabla\psi(x)\rangle d\mu(x) +$$

$$\int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x)$$

$$= 2\ S(\mu) - n + \int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x)$$

$$\log(2\pi)^n \geq \log\left(\int e^{2\psi(x)-\langle x,\nabla\psi(x)\rangle}\ \det(\mathrm{Hess}\psi)d\mu(x)\right)$$

Jensen

$$\geq 2\int \psi(x)d\mu(x) - \int \langle x,\nabla\psi(x)\rangle d\mu(x) +$$

$$\int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x)$$

$$= 2\ S(\mu) - n + \int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x)$$

$$\int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x) \leq \log(2\pi)^n + n - 2\ S(\mu) =$$

$$\log(2\pi)^n \geq \log\left(\int e^{2\psi(x)-\langle x,\nabla\psi(x)\rangle} \det(\mathrm{Hess}\psi) d\mu(x)\right)$$

Jensen

$$\geq 2\int \psi(x) d\mu(x) - \int \langle x, \nabla\psi(x)\rangle d\mu(x) +$$

$$\int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x)$$

$$= 2\ S(\mu) - n + \int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x)$$

$$\int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x) \leq \log(2\pi)^n + n - 2\ S(\mu) =$$

$$2\left(\log(2\pi e)^{\frac{n}{2}} - S(\mu)\right) =$$

$$\log(2\pi)^n \geq \log\left(\int e^{2\psi(x)-\langle x,\nabla\psi(x)\rangle} \ \det(\mathrm{Hess}\psi) d\mu(x)\right)$$

Jensen

$$\geq 2\int \psi(x) d\mu(x) - \int \langle x, \nabla\psi(x)\rangle d\mu(x) + \int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x)$$

$$= 2\ S(\mu) - n + \int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x)$$

$$\int \log\left(\det(\mathrm{Hess}\psi)\right) d\mu(x) \leq \log(2\pi)^n + n - 2\ S(\mu) = 2\left(\log(2\pi e)^{\frac{n}{2}} - S(\mu)\right) = 2\left(S(\gamma_n) - S(\mu)\right)$$

Generalizations: Starting point is transformation formula

$$\int e^{-\psi^*(y)} dy = \int e^{\psi(x) - \langle x, \nabla\psi(x)\rangle} \ \det(\mathrm{Hess}\psi) dx$$

Generalizations: Starting point is transformation formula

$$\int e^{-\psi^*(y)} dy = \int e^{\psi(x) - \langle x, \nabla\psi(x)\rangle} \det(\text{Hess}\psi) dx$$

Let $F_1, F_2 : \mathbb{R} \cup \{+\infty\} \to \mathbb{R}_+$, such that $F_1(+\infty) = F_2(+\infty) = 0$.

Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex, $\lambda \in \mathbb{R}$.

Generalizations: Starting point is transformation formula

$$\int e^{-\psi^*(y)} dy = \int e^{\psi(x) - \langle x, \nabla\psi(x)\rangle} \det(\mathrm{Hess}\psi) dx$$

Let $F_1, F_2 \colon \mathbb{R} \cup \{+\infty\} \to \mathbb{R}_+$, such that $F_1(+\infty) = F_2(+\infty) = 0$.

Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex, $\lambda \in \mathbb{R}$.

Generalized affine surface areas

$$as_\lambda(F_1, F_2, \psi) = \int_{\mathbb{R}^n} (F_1(\psi(x)))^{1-\lambda} (F_2(\langle x, \nabla\psi(x)\rangle - \psi(x)))^\lambda \left(\det(\mathrm{Hess}\psi)\right)^\lambda dx$$

Generalizations: Starting point is transformation formula

$$\int e^{-\psi^*(y)} dy = \int e^{\psi(x) - \langle x, \nabla\psi(x)\rangle} \det(\mathrm{Hess}\psi) dx$$

Let $F_1, F_2 : \mathbb{R} \cup \{+\infty\} \to \mathbb{R}_+$, such that $F_1(+\infty) = F_2(+\infty) = 0$.

Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ convex, $\lambda \in \mathbb{R}$.

Generalized affine surface areas

$$as_\lambda(F_1, F_2, \psi) = \int_{\mathbb{R}^n} (F_1(\psi(x)))^{1-\lambda} (F_2(\langle x, \nabla\psi(x)\rangle - \psi(x)))^\lambda (\det(\mathrm{Hess}\psi))^\lambda \, dx$$

- $F_1(t) = F_2(t) = e^{-t}$ and $\lambda = 1$: RHS of transformation formula

- $\lambda = 0$: $as_0(F_1, F_2, \psi) = \int_{\mathbb{R}^n} F_1 \circ \psi \; dx$

- $\lambda = 0$: $as_0(F_1, F_2, \psi) = \int_{\mathbb{R}^n} F_1 \circ \psi \; dx$
- For any linear invertible map A on $\mathbb{R}^n$:

$$as_\lambda(F_1, F_2, \psi \circ A) = |\det A|^{2\lambda - 1} as_\lambda(F_1, F_2, \psi)$$

- $\lambda = 0$: $as_0(F_1, F_2, \psi) = \int_{\mathbb{R}^n} F_1 \circ \psi \; dx$
- For any linear invertible map A on $\mathbb{R}^n$:

$$as_\lambda(F_1, F_2, \psi \circ A) = |\det A|^{2\lambda - 1} as_\lambda(F_1, F_2, \psi)$$

- Duality formula

- $\lambda = 0$: $as_0(F_1, F_2, \psi) = \int_{\mathbb{R}^n} F_1 \circ \psi \; dx$
- For any linear invertible map A on $\mathbb{R}^n$:

$$as_\lambda(F_1, F_2, \psi \circ A) = |\det A|^{2\lambda - 1} as_\lambda(F_1, F_2, \psi)$$

- Duality formula

Theorem 2 Let F_1, F_2, λ, ψ as above. Then

$$as_\lambda(F_1, F_2, \psi) = as_{1-\lambda}(F_2, F_1, \psi^*)$$

- $\lambda = 0$: $as_0(F_1, F_2, \psi) = \int_{\mathbb{R}^n} F_1 \circ \psi \; dx$
- For any linear invertible map A on $\mathbb{R}^n$:

$$as_\lambda(F_1, F_2, \psi \circ A) = |\det A|^{2\lambda - 1} as_\lambda(F_1, F_2, \psi)$$

- Duality formula

Theorem 2 Let F_1, F_2, λ, ψ as above. Then

$$as_\lambda(F_1, F_2, \psi) = as_{1-\lambda}(F_2, F_1, \psi^*)$$

Theorem 2 $\implies$ affine isoperimetric inequalities for $as_\lambda(F_1, F_2, \psi)$

Special functions F_1, F_2

Special functions F_1, F_2 $F_1(t) = F_2(t) = e^{-t}$ gives

$$as_\lambda(\psi) \;=\; \int_{\mathbb{R}^n} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \mathrm{Hess}\ \psi(x)\right)^\lambda dx$$

Special functions F_1, F_2 $\quad F_1(t) = F_2(t) = e^{-t}$ gives

$$as_\lambda(\psi) = \int_{\mathbb{R}^n} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \mathrm{Hess}\ \psi(x)\right)^\lambda dx$$

$\varphi = e^{-\psi}$

$$= \int_{\mathbb{R}^n} \varphi \left(\frac{e^{\langle x,\nabla\varphi(x)\rangle}}{\varphi^2} \right)^\lambda \left(\det \mathrm{Hess}\ (-\log\varphi(x))\right)^\lambda dx$$

Special functions F_1, F_2 $F_1(t) = F_2(t) = e^{-t}$ gives

$$as_\lambda(\psi) = \int_{\mathbb{R}^n} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} (\text{det Hess } \psi(x))^\lambda \, dx$$

$\varphi = e^{-\psi}$

$$= \int_{\mathbb{R}^n} \varphi \left(\frac{e^{\langle x,\nabla\varphi(x)\rangle}}{\varphi^2} \right)^\lambda (\text{det Hess } (-\log\varphi(x)))^\lambda \, dx$$

L_λ-affine surface area for log concave function $\varphi = e^{-\psi}$

Special functions F_1, F_2 $\quad F_1(t) = F_2(t) = e^{-t}$ gives

$$as_\lambda(\psi) = \int_{\mathbb{R}^n} e^{(2\lambda-1)\psi(x)-\lambda\langle x,\nabla\psi(x)\rangle} \left(\det \text{Hess } \psi(x)\right)^\lambda dx$$

$\varphi = e^{-\psi}$

$$= \int_{\mathbb{R}^n} \varphi \left(\frac{e^{\langle x,\nabla\varphi(x)\rangle}}{\varphi^2}\right)^\lambda \left(\det \text{Hess } (-\log\varphi(x))\right)^\lambda dx$$

L_λ-affine surface area for log concave function $\varphi = e^{-\psi}$

$$as_\lambda\left(\frac{\|\cdot\|^2}{2}\right) = (2\pi)^{\frac{n}{2}}$$

Duality

$$as_{\lambda}(\psi) = as_{1-\lambda}(\psi^{*})$$

Duality

$$as_\lambda(\psi) = as_{1-\lambda}(\psi^*)$$

Theorem 3 (affine isoperimetric inequalities)

Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a C^2 strictly convex function such that $e^{-\psi}$ is integrable and $\int x e^{-\psi(x)} dx = 0$.

(i) Let $\lambda \in [0,1]$. Then

$$as_\lambda(\psi) \leq (2\pi)^{n\lambda} \left(\int e^{-\psi} \right)^{1-2\lambda}$$

Duality

$$as_\lambda(\psi) = as_{1-\lambda}(\psi^*)$$

Theorem 3 (affine isoperimetric inequalities)

Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a C^2 strictly convex function such that $e^{-\psi}$ is integrable and $\int x e^{-\psi(x)} dx = 0$.

(i) Let $\lambda \in [0,1]$. Then

$$as_\lambda(\psi) \leq (2\pi)^{n\lambda} \left(\int e^{-\psi} \right)^{1-2\lambda}$$

or

$$\frac{as_\lambda(\psi)}{as_\lambda\left(\frac{\|\cdot\|^2}{2}\right)} \leq \left(\frac{\int e^{-\psi}}{\int e^{-\frac{\|\cdot\|^2}{2}}} \right)^{1-2\lambda} .$$

Duality

$$as_\lambda(\psi) = as_{1-\lambda}(\psi^*)$$

Theorem 3 (affine isoperimetric inequalities)

Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a C^2 strictly convex function such that $e^{-\psi}$ is integrable and $\int x e^{-\psi(x)} dx = 0$.

(i) Let $\lambda \in [0, 1]$. Then

$$as_\lambda(\psi) \leq (2\pi)^{n\lambda} \left(\int e^{-\psi} \right)^{1-2\lambda}$$

or

$$\frac{as_\lambda(\psi)}{as_\lambda\left(\frac{\|\cdot\|^2}{2}\right)} \leq \left(\frac{\int e^{-\psi}}{\int e^{-\frac{\|\cdot\|^2}{2}}} \right)^{1-2\lambda}.$$

(ii) Let $\lambda \in (-\infty, 0]$. Then

$$as_\lambda(\psi) \geq (2\pi)^{n\lambda} \left(\int e^{-\psi} \right)^{1-2\lambda}$$

Equality holds in (i) and (ii) for $\lambda \neq 0$, iff there exists $c \in \mathbb{R}$ and a positive definite matrix A such that

$$\psi(x) = \langle Ax, x \rangle + c$$

Equality holds in (i) and (ii) for $\lambda \neq 0$, iff there exists $c \in \mathbb{R}$ and a positive definite matrix A such that

$$\psi(x) = \langle Ax, x\rangle + c$$

(iii) Let $\lambda > 1$. Then, there is a constant $c > 0$ such that

$$as_\lambda(\psi) \geq c^{n\lambda} \left(\int e^{-\psi} \right)^{1-2\lambda}$$

Applications to convex bodies

Suppose that ψ is, in addition, 2-homogeneous, that is for any $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$

$$\psi(\lambda x) = \lambda^2 \psi(x)$$

Applications to convex bodies

Suppose that ψ is, in addition, 2-homogeneous, that is for any $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$

$$\psi(\lambda x) = \lambda^2 \psi(x)$$

This is e.g. the case if ψ is the gauge function $\|\cdot\|_K$ of a convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$

$$\|x\|_K = \min\{\alpha \geq 0 : \ x \in \alpha K\} = \max_{y \in K^\circ} \langle x, y \rangle = h_{K^\circ}(x)$$

Applications to convex bodies

Suppose that ψ is, in addition, 2-homogeneous, that is for any $\lambda \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$

$$\psi(\lambda x) = \lambda^2 \psi(x)$$

This is e.g. the case if ψ is the gauge function $\|\cdot\|_K$ of a convex body K in $\mathbb{R}^n$ with $0 \in \mathrm{int}(K)$

$$\|x\|_K = \min\{\alpha \geq 0 : \; x \in \alpha K\} = \max_{y \in K^\circ} \langle x, y \rangle = h_{K^\circ}(x)$$

$$\psi(x) = \frac{1}{2}\|x\|_K^2$$

as_λ simplifies $\langle \nabla\psi, x\rangle = 2\psi$

$$as_\lambda(\psi) = \int_{\mathbb{R}^n} (\text{det Hess } \psi(x))^\lambda \, e^{-\psi(x)} dx$$

as_λ simplifies $\quad \langle \nabla \psi, x \rangle = 2\psi$

$$as_\lambda(\psi) = \int_{\mathbb{R}^n} (\text{det Hess } \psi(x))^\lambda \, e^{-\psi(x)} dx$$

K is a convex body in $\mathbb{R}^n$ with 0 in the interior

as_λ simplifies $\langle \nabla\psi, x\rangle = 2\psi$

$$as_\lambda(\psi) = \int_{\mathbb{R}^n} (\text{det Hess } \psi(x))^\lambda \, e^{-\psi(x)} dx$$

K is a convex body in $\mathbb{R}^n$ with 0 in the interior

L_p affine surface area of K $p \neq -n$

$$as_p(K) = \int_{\partial K} \frac{\kappa_K^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K$$

(Blaschke, Leichtweiss, Lutwak, Meyer-Werner, Schütt-Werner . . .)

Theorem 4 $p \neq -n$, $\lambda = \frac{p}{n+p}$

$$\frac{as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right)}{as_\lambda\left(\frac{\|\cdot\|^2}{2}\right)}$$

Theorem 4 $p \neq -n$, $\lambda = \frac{p}{n+p}$

$$\frac{as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right)}{as_\lambda\left(\frac{\|\cdot\|^2}{2}\right)} = \frac{as_p(K)}{as_p(B_2^n)}$$

Theorem 4 $p \neq -n$, $\lambda = \frac{p}{n+p}$

$$\frac{as_\lambda\left(\frac{\|\cdot\|_K^2}{2}\right)}{as_\lambda\left(\frac{\|\cdot\|^2}{2}\right)} = \frac{as_p(K)}{as_p(B_2^n)}$$

- $as_p(K) = as_{\frac{n^2}{p}}(K^\circ)$

  (Hug ($p > 1$), Werner+Ye (all p))

- L_p affine isoperimetric inequalities

(Deicke ($p = 1$), Lutwak ($p > 1$), Werner-Ye (all p))

$$\frac{as_p(K)}{as_p(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \qquad 0 \leq p \leq \infty$$

- L_p affine isoperimetric inequalities

(Deicke ($p = 1$), Lutwak ($p > 1$), Werner-Ye (all p))

$$\frac{as_p(K)}{as_p(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \qquad 0 \leq p \leq \infty$$

$$\frac{as_p(K)}{as_p(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \qquad -n < p \leq 0$$

Equality holds in both iff K is an ellipsoid.

- L_p affine isoperimetric inequalities

(Deicke ($p = 1$), Lutwak ($p > 1$), Werner-Ye (all p))

$$\frac{as_p(K)}{as_p(B_2^n)} \leq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \qquad 0 \leq p \leq \infty$$

$$\frac{as_p(K)}{as_p(B_2^n)} \geq \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \qquad -n < p \leq 0$$

Equality holds in both iff K is an ellipsoid.

$$\frac{as_p(K)}{as_p(B_2^n)} \geq c\left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \qquad -\infty \leq p < -n$$