

# Intrinsic Volumes and Gaussian Processes

Dmitry Zaporozhets  
(joint work with Zakhar Kabluchko, Ulm University)

Russian Academy of Sciences

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# Notations

- 1  $H$  is a Hilbert space
- 2  $K$  is a convex set in  $\mathbb{R}^n$  or in  $H$
- 3  $B_k$  is the unit ball of dimension  $k$
- 4  $\kappa_k = \text{Vol}_k(B_K)$
- 5 Minkowski sum:  $A + B = \{x + y \mid x \in A, y \in B\}$

# Intrinsic volumes in $\mathbb{R}^n$

## Steiner's formula

$$\text{Vol}_n(K + rB_n) = \sum_{k=0}^n r^{n-k} \kappa_{n-k} V_k(K), \quad r > 0.$$

## Alternative definition

$$V_k(K) = \frac{\binom{n}{k} \kappa_n}{\kappa_k \kappa_{n-k}} \int_{L_k^n} \text{Vol}_k(K|L) \mu_k(dL).$$

In particular,  $V_0 = 1$ ,  $V_n = \text{Vol}$ ,  $2V_{n-1} =$  the surface area,  
 $(2\kappa_{n-1}/n\kappa_n)V_1 =$  the mean width.

# Intrinsic volumes in Hilbert space

Definition (Sudakov '71, Chevet '76)

The supremum of the intrinsic volumes of inscribed finitely dimensional convex sets.

# Intrinsic volumes and Gaussian Processes

## Theorem (Sudakov '71)

Let  $X_t, t \in T$ , be a centered Gaussian process and  $K = \text{conv}\{X_t \in H \mid t \in T\}$ . Then

$$V_1(K) = \sqrt{2\pi} \mathbb{E} \sup_{t \in T} X_t.$$

## Theorem (Tsirelson '85)

Let  $X_t^1, \dots, X_t^k$  be independent copies of  $X_t$ . Then

$$V_k(K) = \frac{(2\pi)^{k/2}}{k! \kappa_k} \mathbb{E} \text{Vol}_k(\text{conv}\{(X_t^1, \dots, X_t^k) \in \mathbb{R}^k \mid t \in T\})$$

## Sudakov-Tsirelson's result in Euclidian space

Let  $X, X_1, \dots, X_k \in \mathbb{R}^n$  be independent standard Gaussian vectors. For any convex  $K \subset \mathbb{R}^n$

$$V_k(K) = \frac{(2\pi)^{k/2}}{k! \kappa_k} \mathbb{E} \text{Vol}_k(\{\langle X_1, x \rangle, \dots, \langle X_k, x \rangle \mid x \in K\})$$

In particular,

$$V_1(K) = \sqrt{2\pi} \mathbb{E} \sup_{x \in K} \langle X, x \rangle.$$

# Mean width of a regular simplex

## Theorem (Schneider '92)

Let  $e_1, \dots, e_n$  be an orthonormal basis in  $\mathbb{R}^n$ . Let  $K$  be a regular simplex:

$$K = \text{conv}\{e_1, \dots, e_n\}.$$

Then

$$V_1(K) = \sqrt{2\pi} \sqrt{2 \ln n} \cdot (1 + o(1)), \quad n \rightarrow \infty.$$

## Theorem (very old)

Let  $\xi_1, \dots, \xi_n \in \mathbb{R}^1$  be independent standard Gaussian variables.

Then

$$\mathbb{E} \max_{i=1, \dots, n} \xi_i = \sqrt{2 \ln n} \cdot (1 + o(1)), \quad n \rightarrow \infty.$$

# Gaussian polytope

## Definition

Let  $X_1, \dots, X_n \in \mathbb{R}^k$  be independent standard Gaussian vectors. Their convex hull is called a Gaussian polytope:

$$P_{k,n} = \text{conv}\{X_1, \dots, X_n\}.$$

## Theorem (Affentranger '91)

$$\mathbb{E}\text{Vol}_k(P_{k,n}) = \kappa_k (\ln n)^{k/2} \cdot (1 + o(1)), \quad , n \rightarrow \infty.$$

## Corollary

If  $K$  is a regular simplex in  $\mathbb{R}^n$ , then

$$V_k(K) = \frac{(2\pi)^{k/2}}{k!} (\ln n)^{k/2} \cdot (1 + o(1)), \quad , n \rightarrow \infty.$$



# Regular crosspolytope

## Definition

Let  $X_1, \dots, X_n \in \mathbb{R}^k$  be independent standard Gaussian vectors. Their symmetrized convex hull is called a symmetric Gaussian polytope:

$$P_{k,n}^s = \text{conv}\{\pm X_1, \dots, \pm X_n\}.$$

## Theorem (Hug, Munsonius, Reitzner '04)

$$\mathbb{E}\text{Vol}_k(P_{k,n}^s) = C_{k,n}(\ln n)^{k/2} \cdot (1 + o(1)), \quad n \rightarrow \infty.$$

## Corollary (Finch '11 for $k=1$ )

If  $K$  is a regular crosspolytope in  $\mathbb{R}^n$  defined by  $\text{conv}\{\pm e_1, \dots, \pm e_n\}$ , then

$$V_k(K) = C'_{k,n}(\ln n)^{k/2} \cdot (1 + o(1)), \quad n \rightarrow \infty.$$

# Wiener spiral

## Definition (Kolmogorov '40)

Let  $H = L_2([0, 1])$ . Wiener spiral is a curve in  $H$  defined as

$$w = \{1_{[0,t]} \in H \mid t \in [0, 1]\}.$$

## Theorem (Gao, Vitale '03)

$$V_k(\text{conv}(w)) = \frac{\kappa_k}{k!}.$$

## Theorem (Eldan, '12)

Let  $W_t, t \in [0, 1]$ , be a Brownian motion in  $R^k$ . Then

$$\mathbb{E} \text{Vol}_k(\text{conv}\{W_t, t \in [0, 1]\}) = \left(\frac{\pi}{2}\right)^{k/2} \frac{1}{\Gamma(k/2 + 1)^2}.$$

# Spiral for Brownian bridge

## Notation

Consider a curve in  $H = L_2([0, 1])$

$$b = \{1_{[0,t]} - t \in H \mid t \in [0, 1]\}.$$

## Theorem (Gao '03)

$$V_k(\text{conv}(b)) = \frac{2\kappa_k^2}{k!\kappa_{k+1}}.$$

## Corollary (Randon-Furling, Majumdar, Comtet '09 for $k=2$ )

Let  $B_t, t \in [0, 1]$ , be a Brownian bridge in  $\mathbb{R}^k$ . Then

$$\mathbb{E} \text{Vol}_k(\text{conv}\{B_t, t \in [0, 1]\}) = \frac{2\kappa_k^3}{(2\pi)^{k/2}\kappa_{k+1}}.$$

THANK YOU!