On Calogero-Sutherland gases

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Asymptotic Geometric Analysis II
June 20–24, 2013
Euler International Mathematical Institute
Saint-Petersburg, Russia

First part

- Nathaël Gozlan and Pierre-André Zitt (Paris-Est)

  First order global asymptotics for Calogero-Sutherland gases (arXiv:1304.7569)

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- Configuration energy

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\begin{aligned}
H_N(x_1, \ldots, x_N) &= \sum_{i=1}^{N} \frac{1}{N} V(x_i) + \sum_{1 \leqslant i < j \leqslant N} \frac{1}{N^2} W(x_i, x_j) \\
&= \int V(x)\, d\mu_N(x) + \frac{1}{2} \iint_{\neq} W(x, y) d\mu_N(x) d\mu_N(y).
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- For $d \leqslant 2$, Random Normal Matrix Ensemble

$$M = U \mathrm{Diag}(x_1, \ldots, x_N) U^* \text{ with } U \text{ Haar independent of } x.$$

Randomness

- Boltzmann-Gibbs measure at inverse temperature $\beta_N > 0$

$$\frac{dP_N(x_1,\ldots,x_N)}{dx_1\cdots dx_N} = \frac{e^{-\beta_N H_N(x_1,\ldots,x_N)}}{Z_N} = \prod_{i=1}^{N} f_1(x_i) \prod_{1\leqslant i<j\leqslant N} f_2(x_i,x_j)$$

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- Stochastic Differential Equations (McKean-Vlasov)

$$dX_{t,i} = \sqrt{\frac{2}{\beta_N}}dB_{t,i} - \nabla V(X_{t,i})dt - \sum_{j\neq i} \nabla_1 W(X_{t,i},X_{t,j})dt$$

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- Infinitesimal generator ($\mathbb{R}^{dN}$): $L = \beta_N^{-1}\Delta - \nabla H_N \cdot \nabla$

Examples

- RMT (eigenvalues of random matrix ensembles):

	GUE	Complex Ginibre
Matrix law	$\propto \exp\left(-N\mathrm{Tr}(H^2)\right)$	$\propto \exp\left(-N\mathrm{Tr}(MM^*)\right)$
Spectrum dim.	$d = 1$	$d = 2$
Spectrum law	$\propto \prod_{i=1}^{N} e^{-N\|x_i\|^2} \prod_{i<j} \|x_i - x_j\|^2$	
Gas parameters	$\beta_N = N^2, V(x) = \|x\|^2, W(x,y) = 2\log\frac{1}{\|x-y\|}$	
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- [Khoruzhenko-Sommers], [Ameur-Hedenmalm-Makarof]

More general interactions

- **Coulomb interaction**

$$W(x,y) = k_\Delta(x-y) \quad \text{with} \quad k_\Delta(x) = \begin{cases} -|x| & \text{if } d=1 \\ \log\frac{1}{|x|} & \text{if } d=2 \\ \frac{1}{|x|^{d-2}} & \text{if } d \geqslant 3 \end{cases}$$

Fundamental solution for Laplacian:

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- **Riesz interaction** $0 < \alpha < d$ (Coulomb if $d \geqslant 3, \alpha = 2$)

$$d \geqslant 1, \quad W(x,y) = k_{\Delta_\alpha}(x-y), \quad \text{with} \quad k_{\Delta_\alpha}(x) = \frac{1}{|x|^{d-\alpha}}.$$

Fundamental solution for fractional Laplacian:

$$\Delta_\alpha k_{\Delta_\alpha} = -\delta_0$$

Motivation: physical control problem

- Encode system state with empirical measure

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- How to tune external field V and cooling scheme β_N s.t.

$$\lim_{N\to\infty} \mu_N = \mu_* \quad ?$$

General idea

- Approximate energy functional

$$H_N(x_1,\dots,x_N) = H_N(\mu_N) = \int V\,d\mu_N + \iint_{\neq} W\,d\mu_N^2 \approx I(\mu_N)$$

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- First order global asymptotics: under $P_N \propto e^{-\beta_N H_N}$

$$\mu_N \approx_{N\gg 1} \arg\inf I$$

Large Deviations Principle on $(\mathcal{M}_1, d_{FM})$

Theorem (Large Deviations Principle, CGZ 2013)

- *LDP Lower bound: for all* $A \subset \mathcal{M}_1(\mathbb{R}^d)$

$$\liminf_{N\to\infty} \frac{\log P_N(\mu_N \in A)}{\beta_N} \geqslant - \inf_{\mu\in\mathrm{int}(A)} (I - \inf I)(\mu)$$

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- *a.s.* $d_{FM}(\mu_N, \arg\inf I) \to 0$ *as* $N \to \infty$

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- **Cooling scheme.** $\beta_N \gg N\log(N)$ (RMT: $\beta_N = N^2$)

Comparison with Sanov theorem when $W \equiv 0$

$$P_N \propto \prod_{i=1}^{N} e^{-\frac{\beta_N}{N} V(x_i)}$$

	Sanov	Calogero-Sutherland
β_N	N	$\gg N \log N$
P_N	$\eta^{\otimes N}$	$\eta_N^{\otimes N}$
Rate	$\mu \mapsto \int \frac{d\mu}{d\eta} \log \frac{d\mu}{d\eta}\, d\eta$	$\mu \mapsto \int V\, d\mu - \inf V$
μ_*	η	$\operatorname{supp}(\mu_*) \subset \arg\inf V$

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- Laplace-Varadhan to pass from $W \equiv 0$ to W bounded
- Main problem is W unbounded!

Rate function analysis

- I is convex iff W is weakly positive
  $$\frac{tI(\mu)+(1-t)I(\nu)-I(t\mu+(1-t)\nu)}{t(1-t)}=\iint W\,d(\mu-\nu)^2$$
  Harmonic analysis: Bochner, Schoenberg, Bernstein
  Represent W as a mixture of weakly positive kernels

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- Lagrange variational analysis: gradient of I at point μ

If $W(x,y)=k_D(x-y)$ and $Dk_D=-\delta_0$ with D local op. then

$$U^\mu=k_D*\mu,\quad DU^\mu=-\mu \quad \text{and} \quad \mu_*\approx DV$$

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- *Characterization of μ_*:*

$$U^{\mu_*} + V = C_* \text{ on } \operatorname{supp}(\mu_*) \text{ and } \geqslant C_* \text{ outside}$$

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- *If Coulomb kernel $d \geqslant 3$ and radial external field V then μ_* is supported in a ring (uniform on ball if V quadratic) $\rightsquigarrow$ [Dragnev-Saff], [López García], [Bleher-Kuijlaars]*

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  - Behavior at the edge (support)

Second part

- Sandrine Péché (Paris-Diderot)

  At the edge of Coulomb gases on the plane

Edge behavior

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- Problem: nature and universality of fluctuations of

$$|x|_{(1)} := \max_{1 \leqslant i \leqslant N} |x_i|$$

when $d = 2$, $\beta_N = N^2$, $W(x, y) = 2\log \frac{1}{|x-y|}$, $V(x) = v(|x|)$

Layered structure

Theorem (Layered structure – CP 2012)

If $d = 2$, $\beta_N = N^2$, $V(x) = v(|x|)$ *and* $W(x,y) = 2\log\frac{1}{|x-y|}$ *then*

$$(|x|_{(1)}, \ldots, |x|_{(N)}) \stackrel{d}{=} (y_{(1)}, \ldots, y_{(N)})$$

where $y_1, \ldots, y_N$ *are indep. r.v. with* y_i *of density* $t^{2i-1}e^{-Nv(t)}$.

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- **Kostlan trick.** If $A_1, \ldots, A_N$ are indep. r.v. with densities $f_1, \ldots, f_N$ and if σ is a uniform random permutation of $\{1, \ldots, N\}$ then $(A_{\sigma_1}, \ldots, A_{\sigma_N})$ has density

$$(b_1, \ldots, b_N) \mapsto \frac{1}{N!}\mathrm{permanent}(f_i(b_j))_{1\leqslant i,j\leqslant N}.$$

Power potentials

Theorem (Power potentials and sums of i.i.d. r.v.)

If $d = 2$, $\beta_N = N^2$, $V(x) = |x|^{\alpha \geqslant 1}$, $W(x,y) = 2\log\frac{1}{|x-y|}$ *then*

$$N|x|^{\alpha}_{(k)} \stackrel{d}{=} Z_1 + \cdots + Z_k$$

where $Z_1, \ldots, Z_k$ *are i.i.d. of law* $\mathrm{Gamma}(2/\alpha, 1)$.

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- Exact edge tail representation

$$\mathbb{P}(\max_{1 \leqslant i \leqslant N} |x|_i^{\alpha} \leqslant x) = \prod_{k=1}^{N} \mathbb{P}\left(\frac{Z_1 + \cdots + Z_k}{N} \leqslant x\right)$$

- Competition between LLN and CLT $\rightsquigarrow$ [Rider]

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- Delta-method

Edge behavior

Theorem (Gumbel universality at the edge – CP 2012)

If $d = 2$, $\beta_N = N^2$, $V(x) = v(|x|)$, v convex, $W(x, y) = 2 \log \frac{1}{|x-y|}$ then there exists deterministic sequences (a_N) and (b_N) s.t.

$$a_N \max_{1 \leqslant i \leqslant N} |x_i| + b_N \xrightarrow[n\to\infty]{d} \mathrm{Gumbel}$$

- Gumbel: $F(t) = e^{-e^{-t}}$ $\rightsquigarrow$ [Fisher-Tippett-Gnedenko]

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- Universality: the limiting law does not depend on v
- Phenomenon: no Tracy-Widom on $\mathbb{C}$!

Recent related works

- Hedenmalm & Makarov 2011

  Coulomb gas ensembles and Laplacian growth
- Hardy 2012

  A note on large deviations for 2D Coulomb gas with weakly confining potential
- Sandier & Serfaty 2012

  2D Coulomb Gases and the Renormalized Energy
- Tao & Vu 2012

  Universality of local spectral statistics of non-Hermitian matrices
- Bourgade & Yau & Yin 2012

  Local Circular Law for Random Matrices
- Bleher & Kuijlaars 2012

  Orthogonal polynomials in the normal matrix model with a cubic potential