Randomized Urysohn-type inequalities

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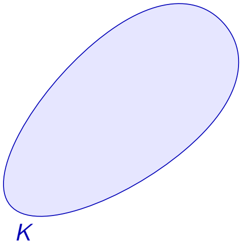
TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Intrinsic volumes in $\mathbb{R}^n$

$K \in \mathcal{K}(\mathbb{R}^n) = \{K \subseteq \mathbb{R}^n \mid K \text{ compact, convex, } \operatorname{int} K \neq \emptyset\}$:

$$V_i(K) = c_{i,n} \int_{\mathcal{E}^n_{n-i}} \chi(K \cap E)\, dE, \qquad i \in \{0, \ldots, n\}$$



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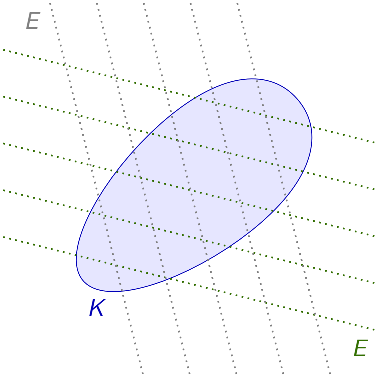

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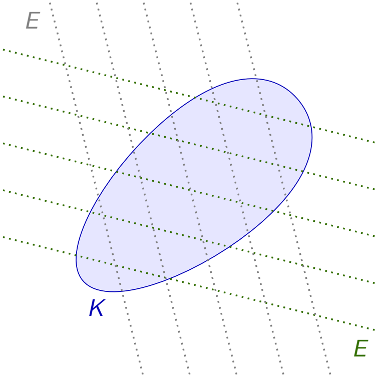
Isoperimetric inequalities:

$$V_i(K) \geq V_i(B_K),$$

B_K ball such that $|K| = |B_K|$.

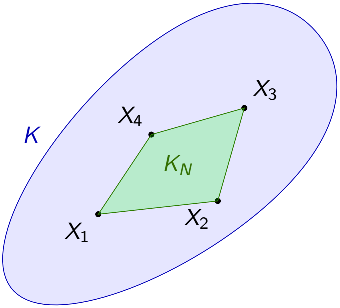
- $i = n - 1$: *the* isop. ineq.
- $i = 1$: Urysohn's inequality



Randomized isoperimetric inequalities

$X_1, \ldots, X_N \sim \frac{\mathbb{1}_K(x)}{|K|} dx$ independent random points, $N \in \mathbb{N}$,

$$K_N := \mathrm{conv}\{X_1, \ldots, X_N\}$$



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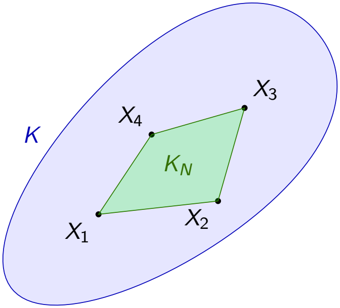

Empirical inequalities:

$$\mathbb{E}V_i(K_N) \geq \mathbb{E}V_i((B_K)_N),$$

$i \in \{0, \dots, n\}$

- Groemer '74
- Hartzoulaki, Paouris '03
- Paouris, Pivovarov '17

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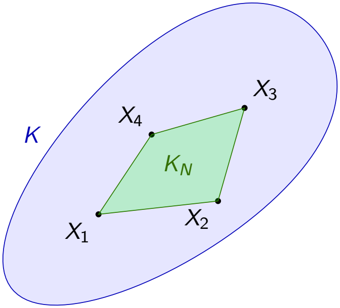

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$K_N \to K$ a.s. as $N \to \infty$ empirical $\Rightarrow$ deterministic

Spherical convex geometry

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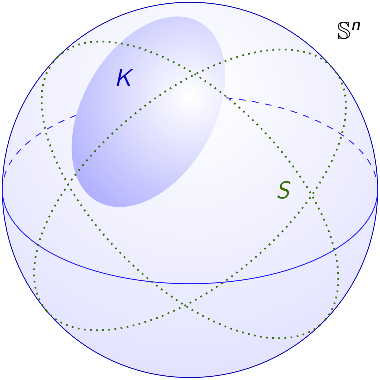

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Isoperimetric inequalities:

- Lévy '19, Schmidt '48:

$$U_{n-1}(K) \geq U_{n-1}(C_K)$$

- Gao, Hug, Schneider '03:

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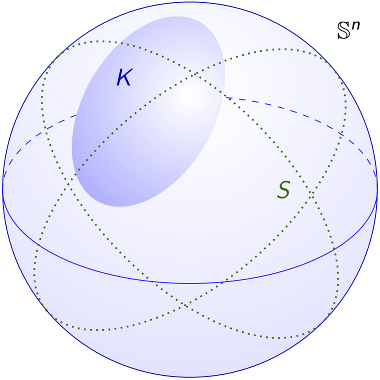
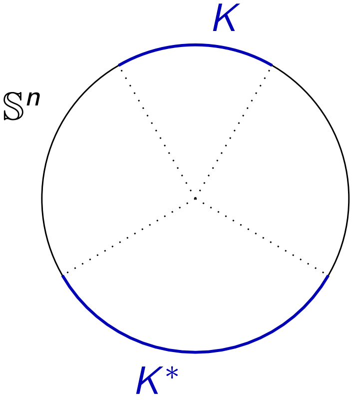
C_K cap s. t. $\sigma(K) = \sigma(C_K)$



Polarity on $\mathbb{S}^n$

- $K \in \mathcal{K}(\mathbb{S}^n)$, $K^* := \{u \in \mathbb{S}^n \,|\, u \cdot v \leq 0 \text{ for all } v \in K\}$



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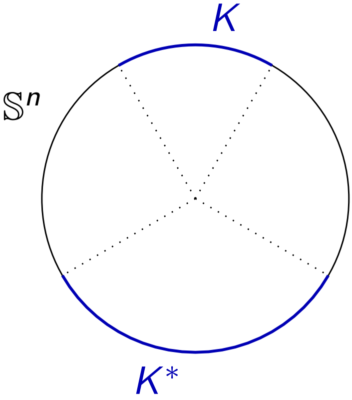

$$1 - U_1(K) = \frac{2}{\omega_n}\sigma(K^*)$$

on $\mathbb{S}^n$: Urysohn's inequality

$\Leftrightarrow$

Blaschke–Santaló inequality

Randomized inequalities on $\mathbb{S}^n$

Theorem (Cordero–E., Fradelizi, Paouris, Pivovarov '15)

Let $K \in \mathcal{K}(\mathbb{R}^n)$, $N \in \mathbb{N}$, and μ be a measure on $\mathbb{R}^n$ with radially symmetric, decreasing density. If $X_1, \ldots, X_N \sim \frac{\mathbb{1}_K(x)}{\mu(K)}dx$ independent, then

$$\mathbb{E}\mu(K_{N,s}^\circ) \leq \mathbb{E}\mu((B_K^\mu)_{N,s}^\circ),$$

where $K_{N,s} := \mathrm{conv}\{\pm X_1, \ldots, \pm X_N\}$.

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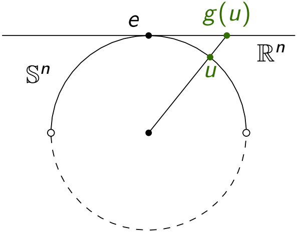
Gnomonic projection $g\colon \mathrm{int}\,\mathbb{S}_e^+ \to \mathbb{R}^n$

- $\mu := g\#\sigma \Rightarrow$ symmetric empirical Blaschke–Santaló inequality on $\mathbb{S}^n$



Main result

Theorem (H., Pivovarov)

Let $K \in \mathcal{K}(\mathbb{S}^n)$, $N \in \mathbb{N}$, *and* $X_1, \dots X_N \sim \frac{\mathbb{1}_K(x)}{\sigma(K)} d\sigma(x)$ *independent random points, then*

$$\mathbb{E} U_1(K_N) \geq \mathbb{E} U_1((C_K)_N),$$

where $K_N = \mathrm{conv}\{X_1, \dots, X_N\}$ *and* C_K *cap, s. t.* $\sigma(C_K) = \sigma(K)$.

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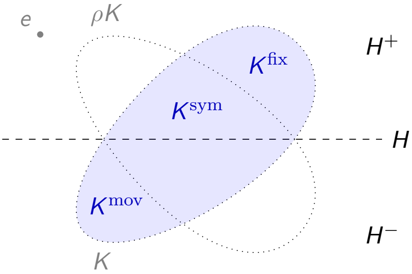
- As $N \to \infty$, we recover $U_1(K) \geq U_1(C_K)$.
- Empirical Blaschke–Santaló: $\mathbb{E}\sigma(K_N^*) \leq \mathbb{E}\sigma((C_K)_N^*)$

Two-point symmetrization

(H, ρ, T), $K \subseteq \mathbb{S}^n$:

$$TK = \underbrace{(K \cap \rho K)}_{K^{\mathrm{sym}}} \cup \underbrace{\left[(K \cap H^+) \setminus K^{\mathrm{sym}}\right]}_{K^{\mathrm{fix}}} \cup \rho \underbrace{\left[(K \cap H^-) \setminus K^{\mathrm{sym}}\right]}_{K^{\mathrm{mov}}}$$



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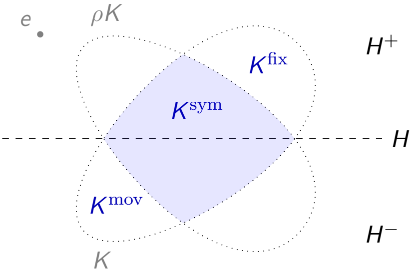

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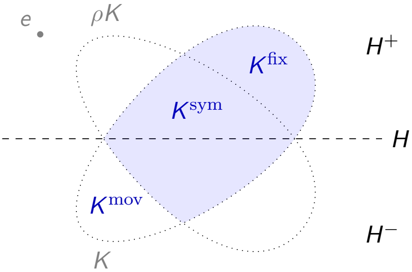

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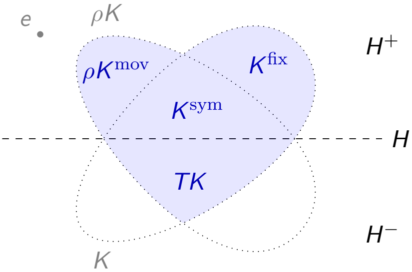

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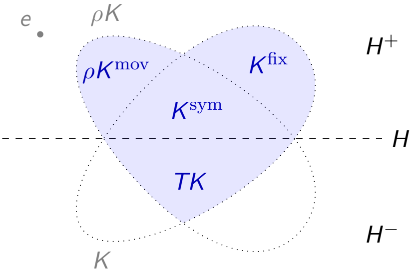
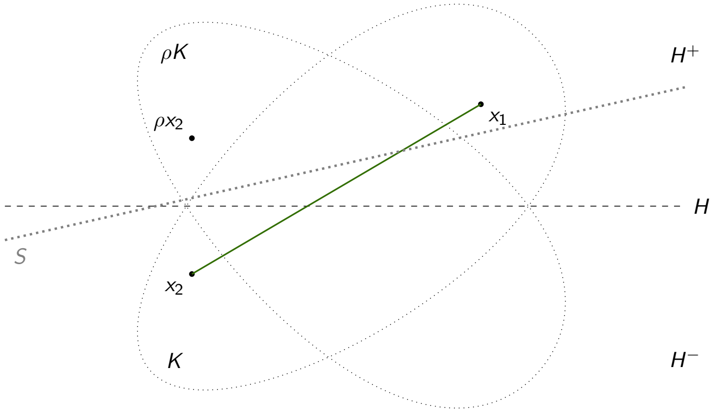


- $\sigma(TK) = \sigma(K)$
- similarly in Euclidean or hyperbolic space
- $\exists$ sequence $(T_{H_m})_{m \in \mathbb{N}}$, $T_{H_m} \cdots T_{H_1} K \to C_K$

Sketch of proof

$x_1, \ldots, x_N \in K$:

$$U_1(x_1, \ldots, x_N) = \int_{\mathcal{S}^n_{n-1}} \chi(\mathrm{conv}\{x_1, \ldots, x_N\} \cap S)\, dS$$



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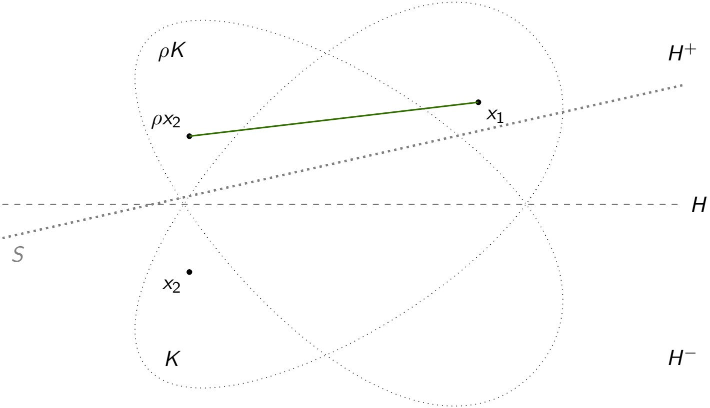

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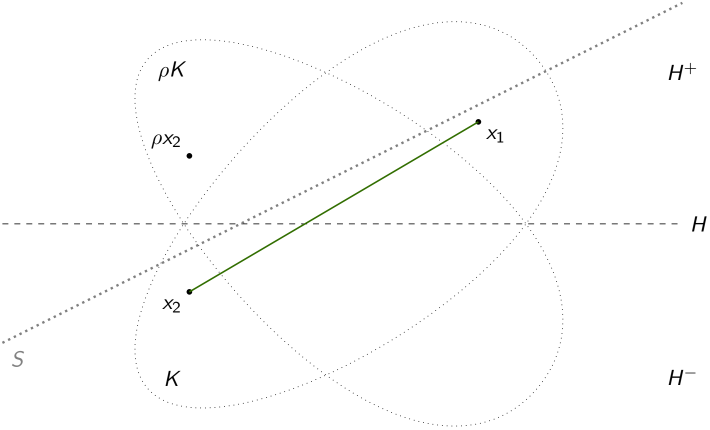

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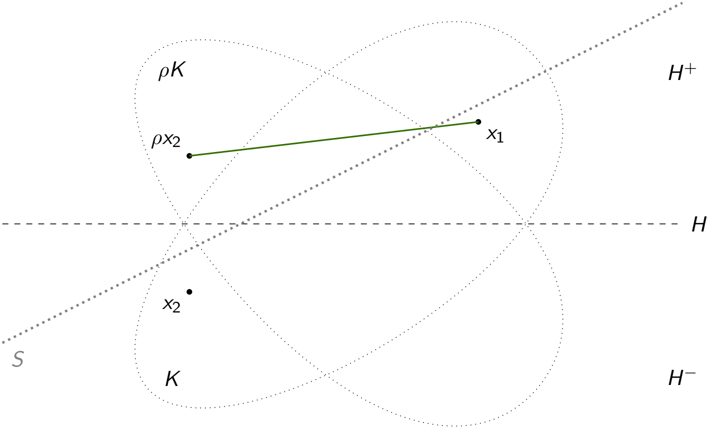

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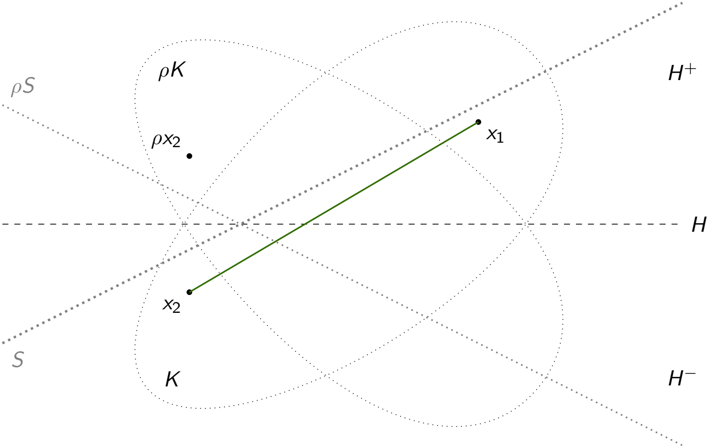

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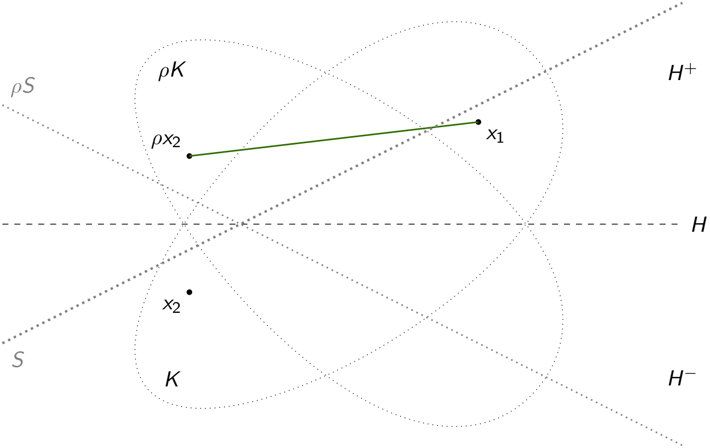

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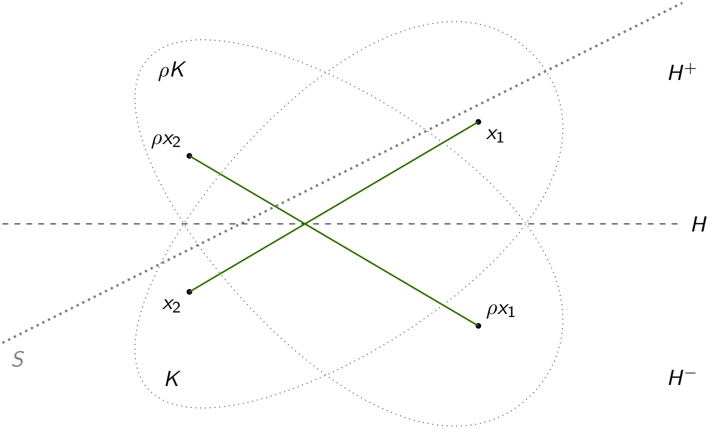

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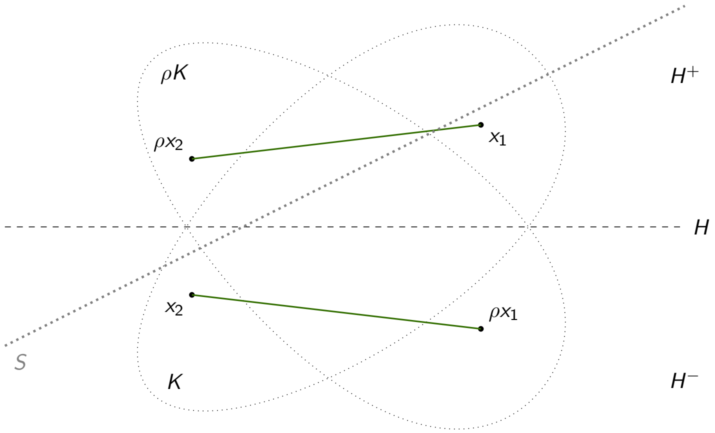

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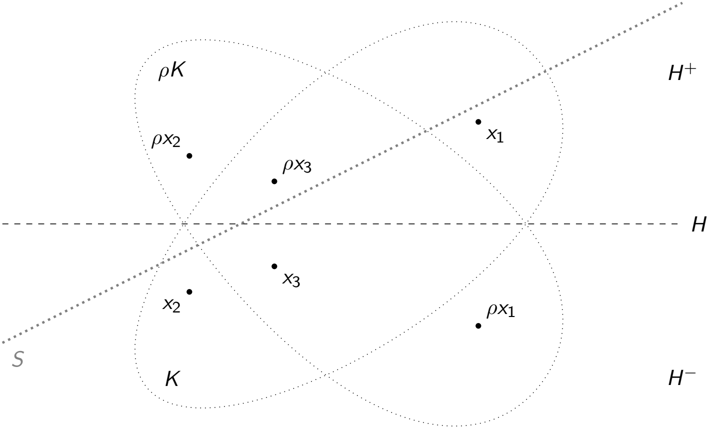

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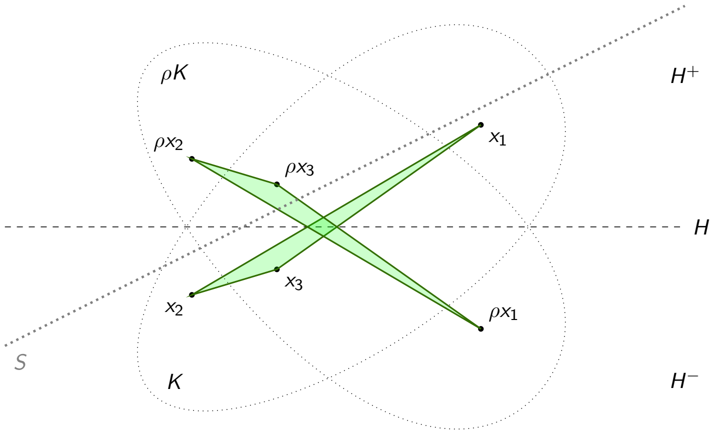

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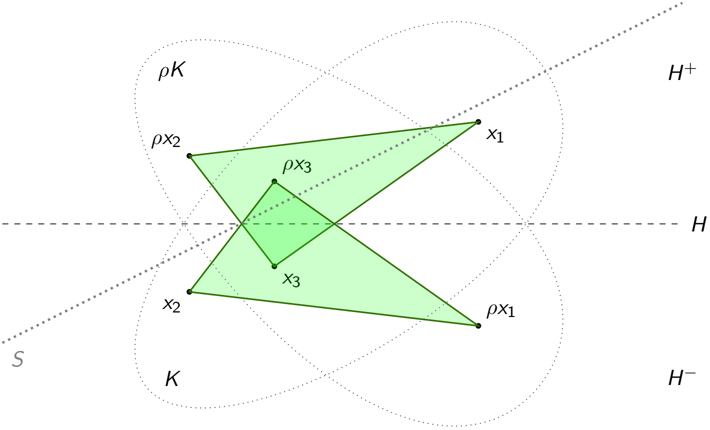

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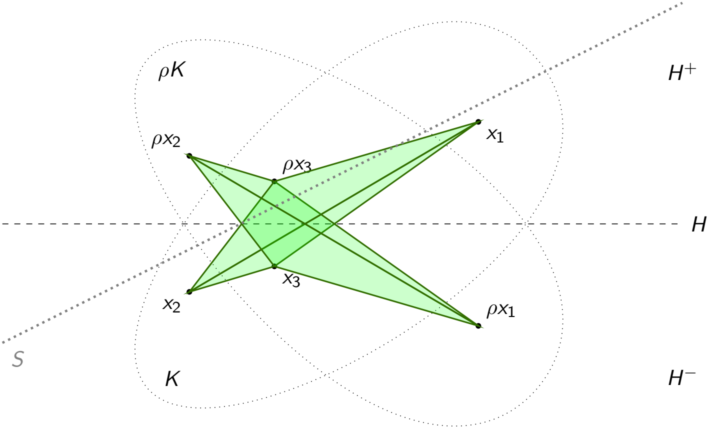

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