Mixed volume inequalities on a special class of convex bodies.

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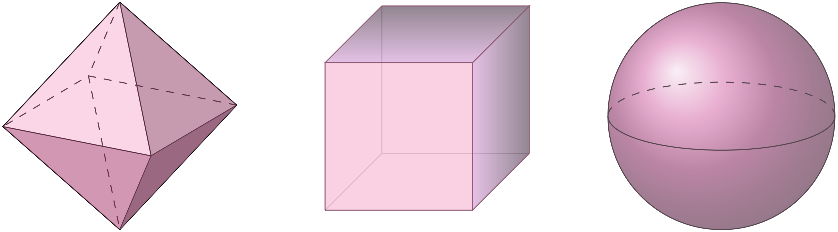
Motivation

A 1-unconditional convex body $K \subset \mathbb{R}^n$, associated with a 1-unconditional norm, i.e.

$$||(x_1, \ldots, x_n)||_K = ||(\pm x_1, \ldots, \pm x_n)||_K$$

is a simpler object than an arbitrary convex body.



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Mahler's conjecture states that for any centrally symmetric convex body K,

$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \geq \frac{4^n}{n!}.$$

The conjecture is known 1-unconditional bodies (J. Saint-Raymond (1980)).

One can give a proof using the following inequality in the positive orthant $O = \{x | x_i \geq 0 \forall i\}$, that

$$\mathrm{Vol}(K \cap O)\mathrm{Vol}(K^\circ \cap O) \geq \frac{1}{n!}.$$

This leads us to studying bodies which are convex in the positive orthants.

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Anti-blocking bodies

Definition (Anti-blocking body, Fulkerson (1971))

A convex body $K \subset \mathbb{R}^n_+$ is called **anti-blocking** if for any $x = (x_1, \dots, x_n) \in K$, if $y = (y_1, \dots, y_n) \in \mathbb{R}^n_+$ is such that $y \le x$ in the partial order on $\mathbb{R}^n$ (i.e. $y_i \le x_i$ for all $1 \le i \le n$) then $y \in K$.

I.e. it is order convex, containing the origin, and positive.

An important observation – for $K \subset \mathbb{R}^n$ anti blocking, and $E = \mathrm{sp}\{e_i\}_{i \in I}$ some **coordinate** subspace ($\mathcal{G}^c_n$),

$$K \cap E = P_E K.$$

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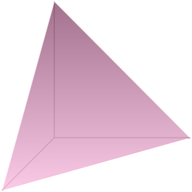

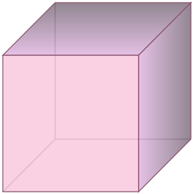

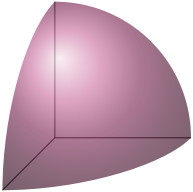

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Polarity

We should define an analogue of a polarity operation on this class, as it has 0 on the boundary.

Definition (Polarity for anti-blocking bodies)

Let $K \in \mathbb{R}^n_+$ be an anti-blocking body. Define

$$AK := \{x \in \mathbb{R}^n_+ : \sup_{y \in K} \langle x, y \rangle \leq 1\}.$$

The definition coincides with usual polarity for the associated 1-unconditional body, and also

$$AK = (K + \mathbb{R}^n_-)^\circ = K^\circ \cap \mathbb{R}^n_+.$$

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Decomposition of Difference

The main important property is the following formula for mixed volumes:

Lemma (Decomposition Lemma for difference of anti-blocking bodies, Chappell, Friedl, Sanyal (2017))

Let $K, T \subset \mathbb{R}^n_+$ be anti-blocking convex bodies, and let $\lambda \geq 0$, then

$$K - \lambda T = \bigcup_{E \in \mathcal{G}_n^c} P_E K \times P_{E^\perp}(-\lambda T),$$

and in particular, as this union is disjoint up to measure 0,

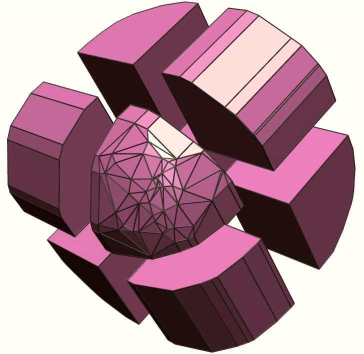
$$V(K[j], -T[n-j]) = \binom{n}{j}^{-1} \sum_{E \in \mathcal{G}_{n,j}^c} \mathrm{Vol}_j(P_E K) \cdot \mathrm{Vol}_{n-j}(P_{E^\perp} T).$$

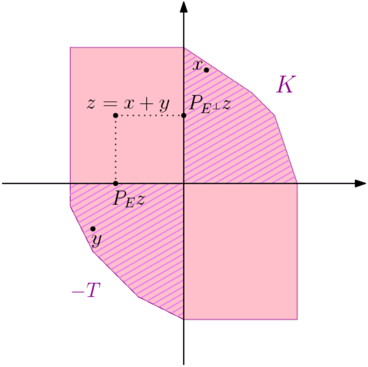
Where

$$V(K[j], T[n-j]) = V(\underbrace{K, \dots, K}_{j}, \underbrace{T, \dots, T}_{n-j}).$$

Sketch of proof

The idea of the proof is to note that for each point $x \in K - T$, its positive coordinates are in K and the negative are in $-T$.





Decomposition of convex hull

A similar lemma holds for the convex hull.

Lemma (Decomposition of the convex hull of anti-blocking bodies)

Let $K, T \subset \mathbb{R}^n_+$ be anti-blocking convex bodies, and let $\lambda > 0$, then:

$$\mathrm{conv}(K, -\lambda T) = \bigcup_{j=0}^{n} \bigcup_{E \in \mathcal{G}^c_{n,j}} \mathrm{conv}(P_E K, P_{E^\perp}(-\lambda T)),$$

and in particular, as this union disjoint up to measure 0,

$$\mathrm{Vol}(\mathrm{conv}(K, -\lambda T)) = \sum_{j=0}^{n} \lambda^j V(K[n-j], -T[j])$$

Godbersen's conjecture

Rogers and Shephard's inequality (1957) states that for $K \subset \mathbb{R}^n$ convex,

$$\mathrm{Vol}(K-K) \leq \binom{2n}{n}\mathrm{Vol}(K)$$

$$\sum_{j=1}^{n}\binom{n}{j}V(K[j],-K[n-j]) \leq \sum_{j=1}^{n}\binom{n}{j}^2\mathrm{Vol}(K)$$

Godbersen's conjecture (C. Godbersen (1938)) states that the inequality holds term by term, and that the only maximizers of this mixed volume are simplices.

Theorem (Godbersen holds for anti-blocking bodies)

Let $K \subset \mathbb{R}^n_+$ a convex anti-blocking body and $1 \leq j \leq n$, then

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with equality if and only if K is an anti-blocking simplex.

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Proof of the first part of the theorem is by an application of the Rogers-Shephard lemma for a product of section and projection.

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V(K[j], -K[n-j]) &= \binom{n}{j}^{-1} \sum_{E \in \mathcal{G}_{n,j}^c} \mathrm{Vol}_j(P_E K) \cdot \mathrm{Vol}_{n-j}(P_{E^\perp} K) \\
&\leq \binom{n}{j}^{-1} \sum_{E \in \mathcal{G}_{n,j}^c} \binom{n}{j} \mathrm{Vol}(K) \\
&= \binom{n}{j} \mathrm{Vol}(K).
\end{aligned}
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A Saint-Raymond type inequality

Theorem (A Saint-Raymond type inequality for Mixed Volumes)

Let

$$K_1, \dots, K_j, T_1, \dots, T_{n-j} \subset \mathbb{R}^n_+$$

be anti-blocking bodies. Then,

$$V(K_1, \dots K_j, -T_1, \dots - T_{n-j}) V(AK_1, \dots AK_j, -AT_1, \dots - AT_{n-j}) \geq \frac{1}{j!(n-j)!}.$$

In particular, for $K, T \subset \mathbb{R}^n_+$ anti-blocking,

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The theorem follows from

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using a repeated Alexandrov-Fenchel inequality.

We prove $(\star)$ using the **Decomposition Lemma**, applying Cauchy-Schwarz inequality and finally applying the Saint-Raymond inequality on the projections, i.e. for E a coordinate subspace of dimension j,

$$\mathrm{Vol}(P_E K)\mathrm{Vol}(P_E AK)\mathrm{Vol}(P_{E^\perp} T)\mathrm{Vol}(P_{E^\perp} AT) \geq \frac{1}{j!}\frac{1}{(n-j)!}.$$

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Nearly Mahler for C-bodies

C-bodies were first introduced in Rogers-Shephard (1958), as a means of associating to a convex body some centrally symmetric convex body.

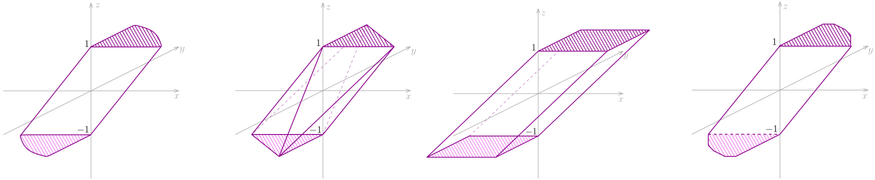
Definition

Define

$$C(K,T) = \mathrm{conv}\{K \times \{1\}, T \times \{-1\}\},$$

and mark $C(K) := C(K, -K)$.

We will next compute the volume product for these bodies.



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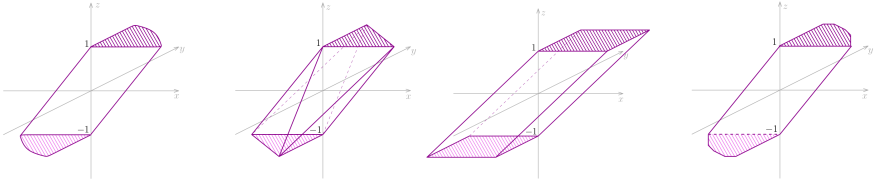

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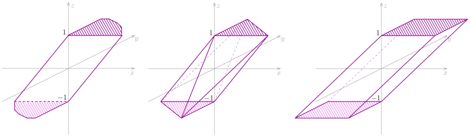
What is the polar of a C-body?



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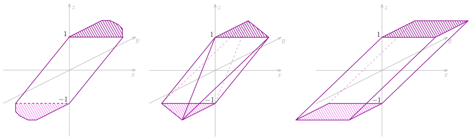

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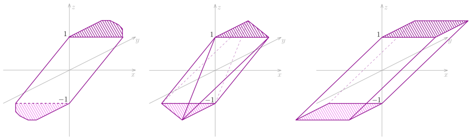

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The volume of a C-body

$$\begin{aligned}
C(K) &= \mathrm{conv}\{K\times\{1\}, -K\times\{-1\}\}\\
&= \bigcup_{s=-1}^{1}\left(\frac{1+s}{2}K-\frac{1-s}{2}K\right)\times\{s\}\\
&= \bigcup_{s=-1}^{1}\bigcup_{j=0}^{n}\bigcup_{E\in\mathcal{G}^c_{n,j}}\left(P_E(\frac{1+s}{2}K)\times P_{E^\perp}(-\frac{1-s}{2}K)\right)\times\{s\}\\
&= \bigcup_{E\in\mathcal{G}^c_n}\bigcup_{s=-1}^{1}\left(\frac{1+s}{2}P_E(K)\times\frac{1-s}{2}P_{E^\perp}(-K)\right)\times\{s\}\\
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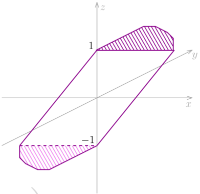

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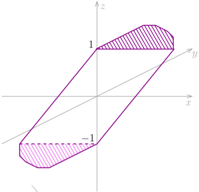

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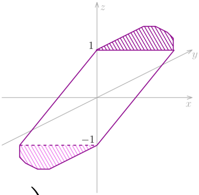

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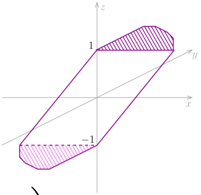

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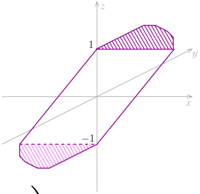

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What we got is disjoint up to measure zero, so have

$$\begin{aligned}
\mathrm{Vol}(C(K)) &= \sum_{j=0}^{n} \sum_{E \in \mathcal{G}^c_{n,j}} \mathrm{Vol}(C(P_E K, -P_{E^\perp} K)) \\
&= \sum_{j=0}^{n} \sum_{E \in \mathcal{G}^c_{n,j}} \mathrm{Vol}(P_E K)\mathrm{Vol}(P_{E^\perp} K) \frac{2}{(n+1)\binom{n}{j}} \\
&= \frac{2}{(n+1)} \sum_{j=0}^{n} V(K[j], -K[n-j]) \\
&= \frac{2}{(n+1)} \mathrm{Vol}(\mathrm{conv}(K, -K)).
\end{aligned}$$

The last equality is by the Decomposition Lemma for convex hull

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Mahler's Conjecture for C-bodies

Using the previous computations on $C(K)$ and $C(K)^\circ = C(-2AK)$, Mahler's conjecture for these bodies is equivalent to the following:

Conjecture

For an anti-blocking $K \subset \mathbb{R}^n_+$ we have that

$$\mathrm{Vol}(\mathrm{conv}(K,-K))\mathrm{Vol}(\mathrm{conv}(AK,-AK)) \geq 2^n \frac{n+1}{n!}.$$

Proposition

For an anti-blocking $K \subset \mathbb{R}^n_+$ we have that

$$\mathrm{Vol}(\mathrm{conv}(K,-K))\mathrm{Vol}(\mathrm{conv}(AK,-AK)) \geq \frac{1}{n!}\left(\sum_{j=0}^{n}\binom{n}{j}^{1/2}\right)^2 \approx \frac{2^n\sqrt{2\pi n}}{n!}.$$

Mahler's Conjecture for C-bodies

Using the previous computations on $C(K)$ and $C(K)^\circ = C(-2AK)$, Mahler's conjecture for these bodies is equivalent to the following:

Conjecture

For an anti-blocking $K \subset \mathbb{R}^n_+$ we have that

$$\mathrm{Vol}(\mathrm{conv}(K, -K))\mathrm{Vol}(\mathrm{conv}(AK, -AK)) \geq 2^n \frac{n+1}{n!}.$$

Proposition

For an anti-blocking $K \subset \mathbb{R}^n_+$ we have that

$$\mathrm{Vol}(\mathrm{conv}(K, -K))\mathrm{Vol}(\mathrm{conv}(AK, -AK)) \geq \frac{1}{n!} \left(\sum_{j=0}^{n} \binom{n}{j}^{1/2} \right)^2 \approx \frac{2^n \sqrt{2\pi n}}{n!}.$$

Elements from the proof

We use Cauchy-Schwartz and our Saint-Raymond type inequality:

$$V(K[j], -K[n-j])V(AK[j], -AK[n-j]) \geq \frac{1}{j!(n-j)!}.$$

We get

$$\begin{aligned}
&\mathrm{Vol}(\mathrm{conv}(K, -K))\mathrm{Vol}(\mathrm{conv}(AK, -AK)) \\
&= \left(\sum_{j=0}^{n} V(K[j], -K[n-j])\right)\left(\sum_{j=0}^{n} V(AK[j], -AK[n-j])\right) \\
&\geq \left(\sum_{j=0}^{n} (V(K[j], -K[n-j])V(AK[j], -AK[n-j]))^{1/2}\right)^2 \\
&\geq \frac{1}{n!}\left(\sum_{j=0}^{n} \binom{n}{j}^{1/2}\right)^2 \approx \frac{2^n\sqrt{2\pi n}}{n!}.
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\end{aligned}$$

A lower bound for mixed volume of difference

Recall the lower (trivial) bound on the volume of a difference body

$$2^n \mathrm{Vol}(K) \leq \mathrm{Vol}(K-K)$$

For anti-blocking bodies, a stronger fact is true:

Corollary

Let $K, T \subset \mathbb{R}^n_+$ anti-blocking bodies, then

$$\mathrm{Vol}(K+T) \leq \mathrm{Vol}(K-T).$$

This is a result of the Reveres Kleitman inequality (Bollobás, Leader, Radcliffe (1989)), which states that for an order-convex set $L \subset \mathbb{R}^n$,

$$\mathrm{Vol}((L-L) \cap \mathbb{R}^n_+ \leq \mathrm{Vol}(L).$$

We would like to show that the inequality holds term by term.

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A lower bound for mixed volume of difference

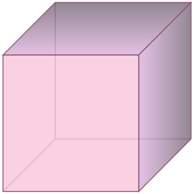
The mixed volume of two anti-blocking bodies depends on whether they are in the same orthant.

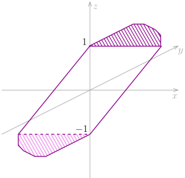
Theorem

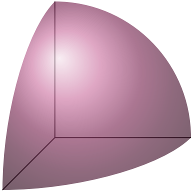
Given two anti-blocking bodies, $K, T \in \mathbb{R}^n_+$,

$$V(K[j], T[n-j]) \leq V(K[j], -T[n-j]).$$

The proof is achieved via a shadow system of Steiner symmetrizations for the right hand side.







Thank you for listening.