Ellipsoids are the only local maximizers of the volume product

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Let $K \subset \mathbb{R}^n$ be a convex body (a compact and convex set with non-empty interior). For $z \in \mathrm{int}(K)$, the interior of K, let K^z be the polar of K with respect to z:

$$K^z = \{y \in \mathbb{R}^n; \langle y - z, x - z\rangle \leq 1 \text{ for every } x \in K\},$$

where $\langle .\,,\,.\rangle$ denotes the standard scalar product in $\mathbb{R}^n$. It is well known that K^z is also a convex body, that $z \in \mathrm{int}(K^z)$ and that $(K^z)^z = K$.

The *volume product* of K, $\Pi(K)$ is given by the following formula:

$$\Pi(K) := \min_{z \in \mathrm{int}(K)} |K|\, |K^z|,$$

where $|A|$ denotes the Lebesgue measure of a Borel subset A of $\mathbb{R}^n$. The unique point $z = s(K) \in K$, where this minimum is reached, is called the Santaló point of K. We denote $K^* = K^{s(K)}$.

Blaschke (1917) proved for dimensions $n = 2$ and $n = 3$ that

$$\Pi(K) = |K|\,|K^*| \leq \Pi(B_2^n)\,,$$

where $B_2^n = \{x \in \mathbb{R}^n; |x| \leq 1\}$ ($|x| = \sqrt{\langle x, x \rangle}$) is the Euclidean unit ball in $\mathbb{R}^n$. This was generalized to all dimensions by Santaló (1948).

The case of equality:

$$\Pi(K) = \Pi(B_2^n) \text{ if and only if } K \text{ is an ellipsoid.}$$

This was done by Saint-Raymond (1981), when K is centrally symmetric and by Petty (1982), in the general case. Another proof was given by Meyer and Pajor (1990), based on Steiner symmetrization.

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The following is a strengthened form of Blaschke-Santaló inequality:

Theorem 1

The convex bodies K in $\mathbb{R}^n$ which are local maximizers (with respect to the Hausdorff distance or to the Banach Mazur distance) of the volume product in $\mathbb{R}^n$, are the ellipsoids.

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A *shadow system* (K_t), along the direction u, is a family of convex sets (K_t) in $\mathbb{R}^n$, $t \in [a, b]$ such that

$$K_t = \mathrm{conv}\{x + t\alpha(x)u \,;\, x \in A\}$$

where A is a given bounded subset of $\mathbb{R}^n$ and $\alpha : A \to \mathbb{R}$ is a given bounded function, called the *speed* of the shadow system.

Shadow systems were introduced by Rogers and Shephard (1958) in order to treat extremal problems for convex bodies. They proved that $|K_t|$ is a convex function of t.

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An example The *Steiner symmetrization* of a convex body K with respect to the hyperplane $u^{\perp}$ orthogonal to $u \in S^{n-1}$:

If K is described as

$$K = \{y + su\,; y \in P_u K\,, s \in I(y)\}\,,$$

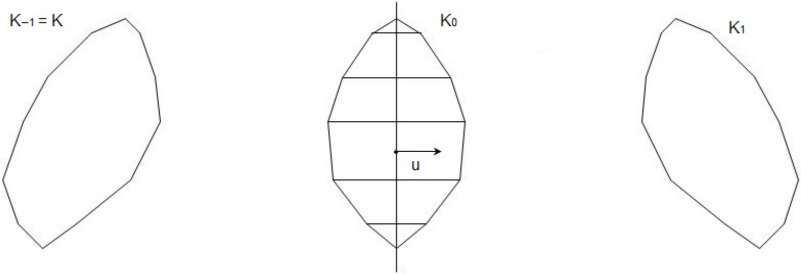
where P_u is the orthogonal projection onto $u^{\perp}$ and $I(y)$ is some nonempty closed interval depending on $y \in P_u K$. The Steiner symmetral $\mathrm{St}_u(K)$ is defined by

$$\mathrm{St}_u(K) := \left\{y + su\,; y \in P_u K, s \in \frac{I(y) - I(y)}{2}\right\}.$$

Steiner symmetrization can be obtained as the midpoint of a shadow system: Let $I(y)$, $y \in P_u K$ be the intervals from the previous slide and suppose $I(y) = [a(y), b(y)]$. Then

$$K_t = \left\{ z - t\frac{a(P_u z) + b(P_u z)}{2} u \, ; z \in \mathrm{St}_u(K) \right\}$$

is a shadow system with $A = K_0 = \mathrm{St}_u(K)$, $K_{-1} = K$, and K_1 the reflection of K with respect to $u^{\perp}$.


$K_{-1} = K$
K_0
u
K_1

A counterpart to the convexity of $|K_t|$ proved by Rogers and Shephard is:

Theorem 2

Let K_t, $t \in [a, b]$, be a shadow system in $\mathbb{R}^n$. Then $t \to |K_t^*|^{-1}$ is a convex function on $[a, b]$.

If $t \to |K_t|$ and $t \to |K_t^*|^{-1}$ are both affine functions in $[a, b]$ then, for all $t \in [a, b]$, K_t is an affine image of K_a, $K_t = A_{u,t}(K_a)$. Where $A_{u,t}$ is an affine transformation that satisfies $P_u A_{u,t} = P_u$.
More precisely: for some $v \in \mathbb{R}^n$ and some $c \in \mathbb{R}$, one has for all $t \in [a, b]$ and all $x \in \mathbb{R}^n$:

$$A_{u,t}(x) = x + (t - a)\big(\langle x, v\rangle + c\big)u\,.$$

The first part of Th. 2 (convexity of $|K_t^*|^{-1}$) was proved by Campi and Gronchi (2005) for centrally symmetric K.

In the form above, for the general case and including characterization of the case of affinity of $|K_t^*|^{-1}$, it was proved by Meyer and Reisner (2006).

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Suppose that K is a local maximizer and Let $u \in S^{n-1}$.

With the above notations we describe $\mathrm{St}_u(K)$ as K_0 of a shadow system K_t, $t \in [-1, 1]$, with $K_{-1} = K$ and K_1 being the mirror reflection of K about $u^\perp$.

It follows from the nature of this shadow system (parallel chord translation) that it preserves the volume of K: one has $|K_t| = |K|$ for all $t \in [-1, 1]$ and that K_t is the mirror reflection of K_{-t} with respect to $u^\perp$.

Thus $(K_t)^*$ is also the mirror reflection of $(K_{-t})^*$ with respect to $u^\perp$.

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Let

$$f(t) = (|K| \, |(K_t)^*|)^{-1} = \frac{1}{\Pi(K_t)} \, .$$

It is clear that the function $t \to K_t$ is continuous for both the Hausdorff and the Banach-Mazur distances. Thus such is also the function $t \to (K_t)^*$. It follows that f is continuous on $[-1, 1]$.

By theorem 2, f is convex on $[-1, 1]$ and by construction, it is even. Thus $f(t) \leq f(-1) = f(1)$ for all $t \in [-1, 1]$ and f has its absolute minimum at 0.

Since K is a local maximum of the volume product (i.e, a local minimum of f), one has: for some $-1 < c \leq 0$, $f(-1) \leq f(t)$ for all $t \in [-1, c]$. Hence f is constant on $[-1, c]$.

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It now follows from its convexity and the preceding observations, that f is actually constant on $[-1, 1]$, that is:

$$|(K_t)^*| = |K^*| \quad \text{for} \quad t \in [-1, 1]\,.$$

From the second part of Theorem 2 we conclude now that $K_0 = St_u(K)$ is an image of $K_{-1} = K$ under an affine transformation A_u that satisfies $P_u A_u = P_u$.

Since this fact is true for any $u \in S^{n-1}$, application of the next lemma completes the proof of Theorem 1.

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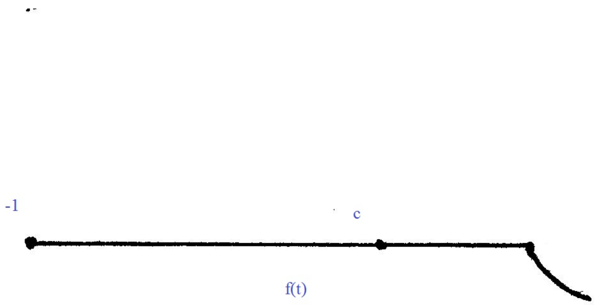

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Lemma 3

Let K be a convex body such that, for all $u \in S^{n-1}$, $\mathrm{St}_u(K)$ is an image of K, $St_u(K) = A_u(K)$ where A_u is an affine transformation that satisfies $P_u A_u = P_u$. Then (and only then) K is an ellipsoid.

Some remarks Lemma 3 can be formulated in an equivalent form as:

Let K be a convex body such that, for all $u \in S^{n-1}$, the centers of the chords of K that are parallel to u are located on a hyperplane. Then (and only then) K is an ellipsoid.

With this formulation (which is better known than the one in Lemma 3) the result, in dimension 2, was declared by Bertrand (1842). But his proof does not seem complete. The result was proved by Brunn (1889). Gruber (1974) proved the result under strongly relaxed assumptions.

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A number of proofs of the result appear in the literature. See e.g. Danzer, Laugwitz and Lenz (1957), that use the Löwner ellipsoid of K, or Grinberg (1991) that uses an infinite sequence of symmetrizations. In our paper we presented, for the sake of completeness, a proof that uses the uniqueness of the John ellipsoid of K.

We also point out a generalization by Meyer and Reisner (1989), that replaces the location of midpoints of chords by the location of centroids of sections of any fixed dimension k, $1 \leq k \leq n-1$.

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Additional Remarks

- Theorem 1 remains true if we restrict the attention to centrally symmetric bodies and local maximizers constrained by central symmetry.
- Bianchi and Kelly (2015), gave a Fourier-analytic proof of Blaschke-Santaló inequality, that, remarkably, does not use symmetrization. The generalization of M & R mentioned above, or rather its proof, was used by B & K to include the equality case in their proof.

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Supplement the following result that was observed by Shephard and used a few times by Campi and Gronchi explains the meaning of the term shadow system:

Proposition 4

Let K be a convex body in $\mathbb{R}^{n+1}$. Then, for $u, v \in S^n$, such that $\langle u, v \rangle = 0$, the family $L_t = P_{u+tv,u^\perp}K$, $t \in \mathbb{R}$, is a shadow system of convex bodies in $u^\perp$, in the direction v. Here $P_{u+tv,u^\perp}$ is the projection on $u^\perp$ parallel to the direction $u + tv$.

In fact, shadow systems can be characterized in this way: For every shadow system in $\mathbb{R}^n$ one can construct a convex body in $\mathbb{R}^{n+1}$ that produces the given shadow system in the form above. This was shown by Campi and Gronchi (2006).

THANK YOU!