Estimates of norms of log-concave random matrices with dependent entries

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Operator norm of a random matrix with i.i.d. entries

For a deterministic $m \times n$ matrix $X = (x_{ij})$, with columns C_j and rows R_i we have

$$\|X\|_{p' \to q} := \|X\|_{\ell_{p'}^n \to \ell_q^m} \geq \max\left(\max_{1 \leq i \leq m} \|R_i\|_p, \max_{1 \leq j \leq n} \|C_j\|_q \right).$$

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In the case of the operator norm $\|\cdot\|_{2\to 2}$ this trivial estimate may be reversed (up to a universal constant) for some random matrices:

Theorem (Seginer, 2000)

If (x_{ij}) are **i.i.d.** *symmetric random variables, then*

$$\mathbb{E}\|X\|_{2\to 2} \asymp \mathbb{E} \max_i \|R_i\|_2 + \mathbb{E}\max_j \|C_j\|_2.$$

No i.i.d. assumption

For which random matrices we may generalize the result of Seginer:

$$\mathbb{E}\|X\|_{2\to 2} \asymp \mathbb{E} \max_i \|R_i\|_2 + \mathbb{E}\max_j \|C_j\|_2? \tag{1}$$

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Theorem (Seginer, 2000)

Let $X = (a_{ij}\varepsilon_{ij})$ where ε_{ij} are independent random signs. Then

$$\mathbb{E}\|X\|_{2\to 2} \lesssim \sqrt[4]{\log\min(m,n)} \left(\max_i \sqrt{\sum_j a_{ij}^2} + \max_j \sqrt{\sum_i a_{ij}^2} \right)$$

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Seginer moreover provided examples of square matrices (a_{ij}), for which constant $(\log n)^{1/4}$ is achieved. So (1) **cannot hold in general** without additional assumptions.

Weaker estimate in general independent case

Theorem (Latała, 2005)

If entries of X are symmetric independent, then

$$\mathbb{E}\|X\|_{2\to 2} \lesssim \max_i \Big(\sum_j \mathbb{E}X_{ij}^2\Big)^{1/2} + \max_j \Big(\sum_i \mathbb{E}X_{ij}^2\Big)^{1/2} + \Big(\sum_{i,j} \mathbb{E}X_{ij}^4\Big)^{1/4}.$$

Case of a structured Gaussian matrix

Let g_{ij} be i.i.d. standard Gaussian variables.

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Corollary

$$\mathbb{E}\big\|(a_{ij}g_{ij})\big\|_{2\to 2} \lesssim \max_i \Big(\sum_j a_{ij}^2\Big)^{1/2} + \max_j \Big(\sum_i a_{ij}^2\Big)^{1/2} + \Big(\sum_{i,j} a_{ij}^4\Big)^{1/4}.$$

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This bound is not always optimal.Van Handel noticed, that for symmetric Gaussian matrices $X = (a_{ij}g_{ij})$,

$$\mathbb{E}\max_i \sqrt{\sum_j X_{ij}^2} \asymp \max_i\Big(\sum_j a_{ij}^2\Big)^{1/2} + \max_{i,j}|a_{ij}^*|\sqrt{\log(i+1)},$$

where (a_{ij^*}) is obtained by permuting the rows of (a_{ij}) so that $\max_j|a_{1j}^*| \geq \ldots \max_j|a_{nj}^*|$.

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The authors used the moment method. In this case we cannot gain anything better using the moment method.

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Obtained by the Slepian-Fernique lemma. Moreover, if (a_{ij}^2) is positive definite, the author proved the aim.

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Decompose (a_{ij}) in a proper, clever way, and use the (weaker) bound due to Bandeira and van Handel "close" to the diagonal, and another (weaker) bound due to van Handel "far" from the diagonal.

What for other norms? Back to $m \times n$ matrices

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

For $p, q \geq 2$,

$$\mathbb{E}\|(a_{ij}g_{ij})\|_{p' \to q} \leq C(p,q)\Big[(\log m)^{1/q} \max_i \Big(\sum_{j=1}^{n} |a_{ij}|^p\Big)^{1/p} + \max_j \Big(\sum_{i=1}^{m} |a_{ij}|^q\Big)^{1/q} + (\log m)^{1/q}\mathbb{E}\max_{i,j}|a_{ij}g_{ij}|\Big].$$

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For a fixed (p,q) this bound is optimal up to logarithmic terms. This result may be generalized to log-concave random matrices:

Theorem (S., 2019)

Assume that rows of (y_{ij}) are **i.i.d. isotropic log-concave vectors***, and $p, q \ge 2$. Then $X = (a_{ij}y_{ij})$ satisfies*

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Under the assumptions of the previous theorem we have

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Corollary

If X is **unconditional** *(as an (mn)-dimensional vector), then*

$$\mathbb{E}\|X\|_{p'\to q} \leq C(p,q)\left((\log m)^{\frac{1}{2}+\frac{1}{q}}\mathbb{E}\max_{1\leq i\leq m}\|R_i\|_p + \sqrt{\log n}\,\mathbb{E}\max_{1\leq j\leq n}\|C_j\|_q\right).$$

X is **unconditional** if for every sequence of signs $\eta \in \{-1,1\}^n$, $X \sim (\eta_i X_i)_i$.

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$m, n \geq 2$, $\gamma \geq 0$, (a_{ij}) – deterministic $m \times n$ matrix, $G = (G_{ij})$ – i.i.d. standard normal entries, (Z_{ij}) – **log-concave isotropic matrix** *independent of G, $X := (a_{ij}|Z_{ij}|^{\gamma} G_{ij})$. Then for every $p, q \geq 2 \vee \frac{1}{\gamma}$ we have*

$$\mathbb{E}\|X\|_{p' \to q} \leq C(p,q,\gamma)\Big[(\log m)^{\frac{1}{q}+\gamma} \max_i \Big(\sum_j |a_{ij}|^p\Big)^{1/p} + (\log n)^{\gamma} \max_j \Big(\sum_i |a_{ij}|^q\Big)^{1/q} + (\log m)^{1/q} \mathbb{E} \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} |X_{ij}|\Big].$$

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$$\mathbb{E}\|X\|_{p'\to q} \le \left(\mathbb{E}\|X\|_{p'\to q}^q\right)^{1/q}$$

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$$\le \left[\mathbb{E}\sup_{t\in B_{p'}^n}\left|\sum_{i=1}^{m}\left(\left|\langle X_i, t\rangle\right|^q - \mathbb{E}\left|\langle X_i, t\rangle\right|^q\right)\right|\right]^{1/q} + \sup_{t\in B_{p'}^n}\left(\mathbb{E}\sum_{i=1}^{m}\left|\langle t, X_i\rangle\right|^q\right)^{1/q}$$

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Theorem (Guédon-Hinrichs-Litvak-Prochno 2017, Guédon–Rudelson 2007, Guédon–Mendelson–Pajor–Tomczak-Jaegerman 2008)

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$$\le C(p)\left(\sup_{t\in B_{p'}^n}\left(\sum_{i=1}^m \mathbb{E}\left|\langle X_i, t\rangle\right|^q\right)^{1/q} + \left(\log m\,\mathbb{E}\max_{1\le i\le m}\|X_i\|_p^q\right)^{1/q}\right).$$

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- comparison of weak and strong moments of $\|Z\|_p$ for log-concave Z in its tail form (Latała-S. 2016):

$$\mathbb{P}\Big(\|Z\|_p \ge C_1 p\big(u+\mathbb{E}\|Z\|_p\big)\Big) \le C_2 \sup_{t\in B^n_{p'}} \mathbb{P}\Big(\Big|\sum_{i=1}^n t_i Z_i\Big| \ge u\Big);$$

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- a weak analogue of Sudakov's minoration for log-concave Z:

$$\mathbb{E}\max_i |a_i Z_i| \ge \frac{1}{C}\max_{k\le m}\big(a_k^* \min_{i\le m}\|Z_i\|_{\log(k+1)}\big).$$

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- To prove the corollary for Gaussian mixtures we need the aforementioned comparison of weak and strong moment:

$$(\mathbb{E}\|Z\|_p^q)^{1/q} \le Cp\left[\mathbb{E}\|Z\|_p + \sup_{t\in B^n_{p'}}\left(\mathbb{E}\Big|\sum_{i=1}^n t_i Z_i\Big|^q\right)^{1/q}\right] \quad \text{for } q\ge 1.$$