Appendix.

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July 6, 2019

Definition 0.0.1. (Diameter) For any subset $U \subset X$ of an n-dimensional normed space $(X, \|\dots\|)$ define its **diameter** as $\operatorname{diam}(U) = \sup\{\|x-y\| : x, y \in U\}$.

Definition 0.0.2. (Hausdorff measure) Let S be any subset of an n-dimensional normed space $(X, \|\dots\|)$ and let $\delta > 0$ be a real number. For an integer m such that $1 \le m \le n$ we define

$$\mathcal{H}^{m,\delta}_{\|\dots\|}(S) := \alpha_m \inf\left\{ \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam}(U_i)}{2}\right)^m : U_i \subset X \text{ are Borelian, } \bigcup_{i=1}^{\infty} U_i \supset S,\ \operatorname{diam}(U_i) \le \delta \right\},$$

where $\alpha_m := \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)}$. Note that $\mathcal{H}^{m,\delta}_{\|\dots\|}(S)$ is monotone decreasing in δ. Hence we are allowed to define

$$\mathcal{H}^{m}_{\|\dots\|}(S) := \sup_{\delta>0} \mathcal{H}^{m,\delta}_{\|\dots\|}(S) = \lim_{\delta\to 0} \mathcal{H}^{m,\delta}_{\|\dots\|}(S).$$

We call $\mathcal{H}^{m}_{\|\dots\|}(S)$ the m-dimensional **Hausdorff measure** of S.

Definition 0.0.3. (Polyhedral chains) Let $(X, \|\dots\|)$ be an n-dimensional normed space and let G be a complete normed Abelian group. We define $\mathcal{P}_m(X, G)$, the Abelian group of m **polyhedral chains**, as the group of equivalence classes of formal sums of the elements of type

$$P = \sum_{j=1}^{N} g_j[\sigma_j],$$

where $g_j \in G$ and σ_j are oriented m-simplexes.

Definition 0.0.4. (Top dimensional chains) Let $(X, \|\dots\|)$ be an n-dimensional normed space and let G be a complete normed Abelian group. The group $\mathcal{P}_n(X, G)$ is called the group of **top dimensional polyhedral chains** in X.

We adopt the following conventions: if $[\sigma]$ is a simplex endowed with the canonical orientation then we endow $-[\sigma]$ with the opposite orientation. Moreover $g(-[\sigma]) = (-g)[\sigma] = -g[\sigma]$; finally any summand whose coefficient is the neutral element of G may be omitted since it gives null contribution. It follows that $P = \sum g_i[\sigma_i]$ is the identity element if and only if every g_i is the neutral element of G, and the inverse element is obtained by $-P = \sum -g_i[\sigma_i]$, thus the group is well defined.

We say that two simplexes σ_1 and σ_2 are non overlapping if $\operatorname{int}(\sigma_1) \cap \operatorname{int}(\sigma_2) = \emptyset$. We define the support $\operatorname{spt}(P)$ of P as $\operatorname{spt}(P) = \bigcup_j \sigma_j$, once σ_i are non overlapping and the corresponding elements $g_i \neq 0$.

Definition 0.0.5. (Boundary of polyhedral chains) Define a group homeomorphism $\partial : \mathcal{P}_m(X, G) \to \mathcal{P}_{m-1}(X, G)$ called **boundary** as follows. Let $P = \sum_j g_j[\sigma_j]$. We first define its value on the basis elements i.e. on simplexes σ_j and then extend it by the formula

$$\partial P = \sum_j g_j \partial[\sigma_j].$$

We require $\partial[\sigma]$ to be the sum of all faces of σ each one endowed with an orientation compatible to the exterior normal vector of σ.

Definition 0.0.6. (Mass of polyhedral chains) Let $(X, \|\ldots\|)$ be an n-dimensional normed space and G be a complete normed Abelian group. Let $P \in \mathcal{P}_m(X,G)$ be a polyhedral m chain such that $P = \sum_{j=1}^{N} g_j[\sigma_j]$, where σ_j are non overlapping. We define **the mass** $\mathbb{M}_{\|\ldots\|}$ of P, as

$$\mathbb{M}_{\|\ldots\|}(P) = \sum_{j=1}^{N} |g_j| \mathcal{H}^m_{\|\ldots\|}(\sigma_j).$$

Remark 1. We shall often drop the sign $\|\ldots\|$ and simply write $\mathbb{M}$ for the mass instead of $\mathbb{M}_{\|\ldots\|}$.

Remark 2. The mass $\mathbb{M}$ is well defined.

Definition 0.0.7. (Flat norm) Define flat norm $\mathcal{F}(P)$ of a polyhedral chain $P \in \mathcal{P}_m(X,G)$ as follows

$$\mathcal{F}(P) = \inf\{\mathbb{M}(Q) + \mathbb{M}(R) : Q \in \mathcal{P}_m(X,G), R \in \mathcal{P}_{m+1}(X,G), P = Q + \partial R\}.$$

Definition 0.0.8. (Flat chains) Let $(X, \|\ldots\|)$ be an n-dimensional normed space and G be a complete normed Abelian group. Define the group of the m-dimensional flat chains denoted by $\mathcal{F}_m(X,G)$, as the $\mathcal{F}$ completion of $\mathcal{P}_m(X,G)$.

Definition 0.0.9. (Lipschitz chains) Let $(X, \|\ldots\|)$ be an n-dimensional normed space and G be a complete normed Abelian group. We define $\mathcal{L}_m(X,G)$, the Abelian group of m-**Lipschitz chains**, as the subgroup of $\mathcal{F}_m(X,G)$ formed by the elements of type

$$P = \sum_{j=1}^{N} g_j \gamma_{j\#}[\sigma_j],$$

where $g_j \in G, \sigma_j \subset \mathbb{R}^m$ are oriented m–simplexes and $\gamma_j : \sigma_j \to X$ are Lipschitz mappings.

Definition 0.0.10. (Rectifiable chains) Let $(X, \|\ldots\|)$ be an n-dimensional normed space and G be a complete normed Abelian group. Define the group of the m-dimensional rectifiable chains denoted by $\mathcal{R}_m(X,G)$, as the $\mathbb{M}$ completion of $\mathcal{L}_m(X,G)$.