Convergence of symmetrization processes

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joint research with R.J. Gardner and P. Gronchi

Introduction

Let

- H be an hyperplane
- $\diamondsuit_H$ be Steiner or Minkowski symmetrization wrt H

It is well known that there exist sequences (H_m) of hyperplanes such that for each convex body K

$$\left(\diamondsuit_{H_m}\diamondsuit_{H_{m-1}}\dots\diamondsuit_{H_1}K\right)\to \text{ball.}$$

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- type of symmetrization $\diamondsuit_H$
- sequence of subspaces (H_m) (and their dimension)
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Interested in studying this process for different symmetrizations, classes of sets and to understand more about which sequences are ”rounding”

1) i-symmetrizations and classes of sets

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i-symmetrization

A map $\quad \diamondsuit_H : \mathcal{E} \to \mathcal{E}_H$

where

- $\mathcal{E} = \{$convex bodies$\}$ or $\mathcal{E} = \{$compact sets$\}$,
- $\mathcal{E}_H = \{$elements of $\mathcal{E}$ which are symmetric wrt $H\}$

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which symmetries

Let R_H denote reflection wrt H

- K is reflection symmetric wrt H if

$$R_H K = K$$

- K is rotationally symmetric wrt H if $\forall x \in H$

$$K \cap (H^{\perp} + x) = (n-i)\text{-dimensional ball centred at } x$$

2) *universal* sequences

Coupier, Davydov (2014)

(H_m) is called $\diamondsuit$-*universal sequence* in the class $\mathcal{E}$ if

$$\forall K \in \mathcal{E}, \qquad \forall j \in \mathbb{N} \qquad \left(\diamondsuit_{H_m}\diamondsuit_{H_{m-1}}\dots\diamondsuit_{H_j}K\right) \to \text{ball},$$

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We will deal only with universal sequences

literature

- results of probabilistic type: Mani-Levitska, Volčič, Van Shaftingen, Fortier, Burchard, Coupier and Davidov

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- speed of convergence to sphere: Bourgain, Lindestrauss, Milman, Klartag, Florentin and Segal

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  - Fiber symm. $F_H K = \bigcup_{x \in H} \left(\frac{1}{2}(K \cap (H^\perp + x)) + \frac{1}{2} R_H (K \cap (H^\perp + x)) \right)$

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- Any i; Rotationally symmetric
  - Schwarz symm. $S_H K$: sections of K orthogonal to H are transformed in balls centered in H and with same $(n - i)$-volume
  - Minkowski-Blaschke symm. $\overline{M}_H K$:

$$h_{\overline{M}_H K}(u) = \frac{1}{\mathcal{H}^{n-i}(S^{n-1} \cap (H^\perp + u))} \int_{S^{n-1} \cap (H^\perp + u)} h_K(v)dv$$

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- many more examples: i-symmetrization is a very general definition

A negative example

Let $\diamondsuit = \textit{Steiner}$.
There exists a convex body $K \subset \mathbb{R}^2$ and a sequence (H_m) of lines, dense in S^1, such that

$$(\diamondsuit_{H_m} \diamondsuit_{H_{m-1}} \dots \diamondsuit_{H_1} K) \quad \text{does not converge at all,}$$

Bianchi, Klain, Lutwak, Yang and Zhang (2011) universal sequences are indeed universal

Coupier and Davidov (2014)

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A sequence is Minkowski-universal if and only if it is Steiner-universal.

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Theorem

Let $1 \leq i \leq n - 1$ and let the class be $\{$convex bodies$\}$. Then:

- A sequence is Minkowski-universal if and only if it is Fiber-universal
- A sequence is (Minkowski-Blaschke)-universal if and only if it is Schwarz-universal

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Theorem

Let $1 \leq i \leq n - 1$ and let the class be $\{$convex bodies$\}$.
Let $\diamondsuit_H$ be a i-symmetrization (with reflection symmetry) which is

- monotone wrt inclusion,
- identity on sets which are already H-symmetric,
- invariant wrt translations orthogonal to H of H-symmetric sets

Then, a sequence is $\diamondsuit$-universal if and only if it is Minkowski-universal

Is it more difficult to ”round” compact sets than convex bodies?

Rounding compact sets

Compact sets need not become convex

There exist compact set $C \subset \mathbb{R}^2$ and a sequence (H_m) of lines "very close to being dense in S^1" such that

$$(\diamondsuit_{H_m} \diamondsuit_{H_{m-1}} \ldots \diamondsuit_{H_1} C) \to \text{a non-convex set.}$$

Bianchi, Burchard, Gronchi and Volcic (2012)

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Theorem

Let $1 \leq i \leq n-1$ and let $\diamondsuit$ be Steiner or Minkowski or Schwarz.
A sequence is $\diamondsuit$-universal in {compact sets} IFF it is $\diamondsuit$-universal in {convex bodies}

constructing universal sequence / how to generate $O(n)$ via finitely many i-reflections

sequences built from a finite alphabet

- Assume that each subspace in the sequence (H_m) belongs to a finite set of subspaces $\mathcal{F} = \{F_1, \ldots, F_j\}$

Example: $(H_m) = F_3, F_3, F_1, F_4, F_2, F_3, F_1, F_3, F_1, F_1, F_4, \ldots$

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Klain (2012)

Let $\diamondsuit$ be Steiner and let K be a convex body.
The sequence $\left(\diamondsuit_{H_m}\diamondsuit_{H_{m-1}}\dots\diamondsuit_{H_1} K\right)$ always converges.
The limit body is symmetric wrt each F_r which appears infinitely many times in (H_m).

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Bianchi, Burchard, Gronchi and Volcic (2013),
Result extended to Minkowski symmetrization and to compacts sets

sequences built from a finite alphabet 2

Theorem

Same conclusion hold for all 5 symmetrizations: Fiber, Schwarz and Minkowski-Blaschke

sequences built from a finite alphabet 2

Theorem

Same conclusion hold for all 5 symmetrizations: Fiber, Schwarz and Minkowski-Blaschke

Theorem

Same conclusion holds for any $\diamondsuit$ (with reflection symmetry) which satisfies the following properties:

1. monotone wrt inclusion
2. identity on sets which are already H-symmetric
3. invariant wrt translations orthogonal to H of H-symmetric sets
4. continuous

Create universal sequences using finite alphabet

- Let $\mathcal{F} = \{F_1, \dots, F_j\}$ be a finite set of *i-dimensional* subspaces in $\mathbb{R}^n$
- Let K be a convex body

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Problem 1

For which choices of $\mathcal{F}$ does the reflection symmetry of K wrt each $F_r \in \mathcal{F}$ forces K to be a ball?

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For which choices of $\mathcal{F}$ does the reflection symmetry of K wrt each $F_r \in \mathcal{F}$ forces K to be a ball?

Problem 1 rephrased

For which choices of $\mathcal{F}$ the closure of the subgroup of $O(n)$ generated by $R_{F_1}, \dots, R_{F_j}$ acts transitively on S^{n-1}?

Create universal sequences using finite alphabet

Problem 2

For which choices of $\mathcal{F}$ does the rotational symmetry of K wrt each $F_r \in \mathcal{F}$ forces K to be a ball?

An IFF answer to Problem 2

Theorem

Let $K \subset \mathbb{R}^n$ be a convex body and let $F_1, \dots, F_j$ be subspaces in $\mathbb{R}^n$ of dimension $\leq n-2$.
Being radially symmetric wrt each F_r forces K to be a ball IFF the following conditions hold

1. $F_1^\perp + \cdots + F_j^\perp = \mathbb{R}^n$
2. $\{F_1^\perp, \dots, F_j^\perp\}$ cannot be partitioned into two mutually orthogonal nonempty subsets

$F_1, \dots, F_j$ need not have equal dimension

Partial constructive answers to Problem 1

(implicit) answer when $F_1, \ldots, F_j$ are hyperplanes:

Eaton and Perlman (1977)

Burchard, Chambers and Dranovski (2017),

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Theorem

Description of how to construct sets $\mathcal{F}$ of i-dimensional subspaces which "force full radial symmetry" and consist of

$$\left\lceil \frac{n}{\min\{i, n-i\}} \right\rceil + 1$$

elements.

Partial constructive answers to Problem 1

Theorem

Let $2 \leq i \leq n/2$, let $j \geq 3$, and let $F_m \in \mathcal{G}(n,i)$, $m = 1, \ldots, j$, be such that
(i) F_1, F_2, and F_3 “form irrational angles”;
(ii) $F_1 + \cdots + F_j = \mathbb{R}^n$.
(iii) for each $m = 3, \ldots, j-1$,

$$F_{m+1} \cap (F_1 + \cdots + F_m)^{\perp} = \{o\};$$

Then the reflection symmetries wrt these subspaces ”force full radial symmetry”