Asymptotic Geometric Analysis IV

Polytopal Approximation in the dual Brunn–Minkowski Theory

Florian Besau*

base on joint work with Steven Hoehner and Gil Kur
(arXiv:1905.08862)

*Technische Universität Wien

TECHNISCHE UNIVERSITÄT WIEN

florian.besau@tuwien.ac.at

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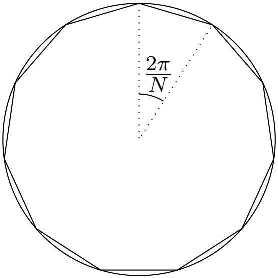
Volume Approximation of Convex Bodies

Classical Question: How well can we approximate the volume of a (smooth) convex body $K \subset \mathbb{R}^n$ by the volume of a convex polytope P_N with at most N vertices?

Optimization: Can we determine an optimal polytope P_N^b with at most N vertices inside of K such that $\mathrm{vol}_n(P_N^b)$ is maximal?

Stochastic: How does the volume $\mathrm{vol}_n(P_N)$ of a random polytope P_N generated as the convex hull of N random points inside of K behave?

Best Approximation in the Plane $\mathbb{R}^2$



$$\max_{P_N \in \mathcal{P}_N^i(B_2^2)} \operatorname{vol}_2(P_N) =$$

- $\mathcal{P}_N^i(K)$... convex polytopes $P \subset K$ with at most N vertices.

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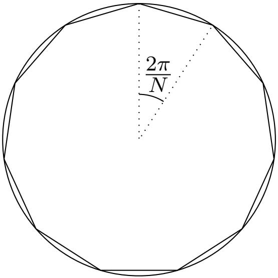

$$\max_{P_N \in \mathcal{P}_N^i(B_2^2)} \mathrm{vol}_2(P_N) = \frac{N}{2} \sin \frac{2\pi}{N}$$
$$= \pi - \frac{1}{12} \frac{(2\pi)^3}{N^2} + o(N^{-2})$$

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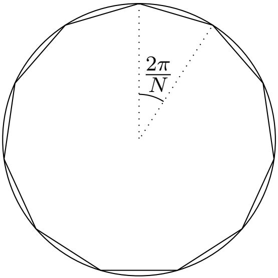

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Theorem (McClure & Vitale, 1975).
Let $K \in \mathcal{K}_0(\mathbb{R}^2)$ be of class C_+^2. Then

$$\max_{P_N \in \mathcal{P}_N^i(K)} \mathrm{vol}_2(P_N) = \mathrm{vol}_2(K) - \frac{1}{12} \frac{\Omega(K)^3}{N^2} + o(N^{-2})$$

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Best Approximation in n-dimensions

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Theorem (Gruber, 1988 + 1993).
Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ be of class C^2_+. Then

$$\max_{P_N \in \mathcal{P}_N^i(K)} \mathrm{vol}_n(P_N) = \mathrm{vol}_n(K) - \frac{\mathrm{del}_{n-1}}{2} \left(\frac{\Omega(K)^{n+1}}{N^2}\right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

- $\Omega(K) = \int_{\mathrm{bd}\, K} H_{n-1}^{\frac{1}{n+1}} \, \mathrm{d}\mu_K$... (equi-)affine surface area
- del_{n-1}... Delaunay (Делоне́) triangulation number, $\mathrm{del}_1 = \frac{1}{6}$, $\mathrm{del}_2 = \frac{1}{2\sqrt{3}}$

Weighted Best Approximation

Theorem (Gruber, 1988 + 1993).
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Weighted Best Approximation

Theorem (Gruber, 1988 + 1993).
Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ be of class C^2_+. Then

$$\max_{P_N \in \mathcal{P}^i_N(K)} \mathrm{vol}_n(P_N) = \mathrm{vol}_n(K) - \frac{\mathrm{del}_{n-1}}{2} \left(\frac{\Omega(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

Theorem (Ludwig, 1999).
Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ be of class[a] C^2_+ and let $\psi : \mathbb{R}^n \to (0, \infty)$ be a continuous weight function. Then

$$\max_{P_N \in \mathcal{P}^i_N(K)} \mathrm{vol}_\psi(P_N) = \mathrm{vol}_\psi(K) - \frac{\mathrm{del}_{n-1}}{2} \left(\frac{\Omega_\psi(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

[a] C^2: Böröczky, 2000

- $\mathrm{vol}_\psi(A) = \int_A \psi \, \mathrm{d}\lambda_n$ for all Borel A; λ_n Lebesgue measure
- $\Omega_\psi(K) = \int_{\mathrm{bd}\, K} \psi^{\frac{n-1}{n+1}} H_{n-1}^{\frac{1}{n+1}} \, \mathrm{d}\mu_K$

Random (Boundary) Polytopes and Approximation

We want to generate random polytopes that are as close to optimal as possible!

We therefore assume complete information about the boundary of our given convex body.

Random (Boundary) Polytopes and Approximation

Setup: Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ be of class C^2_+.

We may choose N points $X_1, \dots, X_N$ at random from $\operatorname{bd} K$ independently according to a continuous probability density $\varphi : \operatorname{bd} K \to (0, \infty)$ and set $P_N^\varphi = \operatorname{conv}\{X_1, \dots, X_N\}$.

Then $P_N^\varphi \in \mathcal{P}_N^i(K)$ and we may study the rand. variable $\operatorname{vol}_\psi(P_N^\varphi)$.

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Theorem (Schütt & Werner, 2003).
Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ of class[a] C^2_+. Then

$$\mathbb{E}\operatorname{vol}_\psi(P_N^\varphi) = \operatorname{vol}_\psi(K) - \frac{\beta_n}{2}\Omega_{\varphi,\psi}(K) N^{-\frac{2}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right).$$

[a] admitting a rolling ball: Böröczky, Fodor & Hug, 2013; Ziebarth, 2014

- $\Omega_{\varphi,\psi}(K) = \int_{\operatorname{bd} K} \psi \left(H_{n-1}/\varphi^2\right)^{\frac{1}{n-1}} \, d\mu_K$

Random vs. Best Approximation

Since $\int_{\mathrm{bd}\,K} \varphi \, \mathrm{d}\mu_K = 1$, by Hölder's inequality,

$$\int_{\mathrm{bd}\,K} \psi \, (H_{n-1}/\varphi^2)^{\frac{1}{n-1}} \, \mathrm{d}\mu_K \geq \left(\int_{\mathrm{bd}\,K} \psi^{\frac{n-1}{n+1}} \, H_{n-1}^{\frac{1}{n+1}} \, \mathrm{d}\mu_K \right)^{\frac{n+1}{n-1}}$$

Equality holds iff

$$\varphi = \psi^* := \Omega_\psi(K)^{-1} \psi^{\frac{n-1}{n+1}} \, H_{n-1}^{\frac{1}{n+1}},$$

i.e., $\Omega_{\varphi,\psi}(K) \geq \Omega_{\psi^*,\psi}(K) = \Omega_\psi(K)^{\frac{n+1}{n-1}}$.

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Recap:

$$\mathbb{E} \, \mathrm{vol}_\psi(P_N^{\psi^*}) = \mathrm{vol}_\psi(K) - \frac{\beta_n}{2} \left(\frac{\Omega_\psi(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

$$\max_{P_N \in \mathcal{P}_N^i(K)} \mathrm{vol}_\psi(P_N) = \mathrm{vol}_\psi(K) - \frac{\mathrm{del}_{n-1}}{2} \left(\frac{\Omega_\psi(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

Random vs. Best Approximation

Theorem (Müller, 1990; Affentranger, 1991).

$$\beta_n = \frac{1}{\pi}\frac{n-1}{n+1}\Gamma\left(\frac{n+1}{2}\right)^{\frac{2}{n-1}}\frac{\Gamma(n+1+\frac{2}{n-1})}{\Gamma(n+1)} = \frac{n}{2\pi e}\Big(1+O\Big(\frac{\ln n}{n}\Big)\Big)$$

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Theorem (Gordon, Reisner & Schütt 1997; Mankiewicz & Schütt, 2001).

$$\beta_n \geq \mathrm{del}_{n-1} \geq \frac{1}{\pi}\frac{n-1}{n+1}\Gamma\left(\frac{n+1}{2}\right)^{\frac{2}{n-1}}$$

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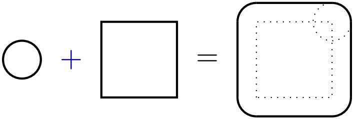
Corollary (Mankiewicz & Schütt, 2001).

$$1 \leq \lim_{N\to\infty} \frac{\mathrm{vol}_\psi(K) - \mathbb{E}\,\mathrm{vol}_\psi(P_N^{\psi^*})}{\mathrm{vol}_\psi(K) - \max\limits_{P_N \in \mathcal{P}_N^i(K)} \mathrm{vol}_\psi(P_N)} = \frac{\beta_n}{\mathrm{del}_{n-1}} = 1 + O\Big(\frac{\ln n}{n}\Big)$$

dual Brunn–Minkowski Theory (dBMT): the Basics

BMT

convex bodies, vol_n, $+$

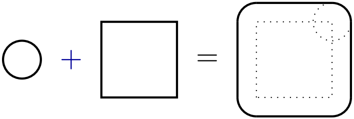


$h_K \quad + \quad h_L \quad = \quad h_{K+L}$

$$\mathrm{vol}_n(K+L)^{\frac{1}{n}} \geq \mathrm{vol}_n(K)^{\frac{1}{n}} + \mathrm{vol}_n(L)^{\frac{1}{n}}$$

BMT

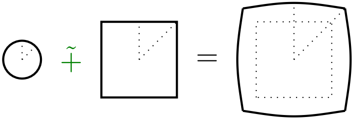
convex bodies, vol_n, $+$



$$h_K \quad + \quad h_L \quad = \quad h_{K+L}$$

dBMT

o-star bodies, vol_n, $\tilde{+}$



$$\rho_K \quad + \quad \rho_L \quad = \quad \rho_{K \tilde{+} L}$$

$$\mathrm{vol}_n(K+L)^{\frac{1}{n}} \geq \mathrm{vol}_n(K)^{\frac{1}{n}} + \mathrm{vol}_n(L)^{\frac{1}{n}} \geq \mathrm{vol}_n(K \tilde{+} L)^{\frac{1}{n}}$$

BMT

$$\mathrm{vol}_n(K + rB_2^n)$$

$$= \sum_{j=0}^{n} r^j \,\mathrm{vol}_j(B_2^j) V_{n-j}(K)$$

$$V_j(K) = \left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right] \int\limits_{\mathrm{Gr}_j(\mathbb{R}^n)} \mathrm{vol}_j(K|E)\,\mathrm{d}E$$

$$\frac{V_1(K)}{V_1(B_2^n)} \geq \left(\frac{V_j(K)}{V_j(B_2^n)}\right)^{\frac{1}{j}} \geq \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B_2^n)}\right)^{\frac{1}{n}}$$

$$\begin{bmatrix} n \\ j \end{bmatrix} := \binom{n}{j} \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_j(B_2^j)\,\mathrm{vol}_{n-j}(B_2^{n-j})}.$$

BMT

$$\mathrm{vol}_n(K + rB_2^n) = \sum_{j=0}^{n} r^j \,\mathrm{vol}_j(B_2^j) V_{n-j}(K)$$

$$V_j(K) = \left[{n \atop j}\right] \int_{\mathrm{Gr}_j(\mathbb{R}^n)} \mathrm{vol}_j(K|E) \,\mathrm{d}E$$

dBMT

$$\mathrm{vol}_n(K \tilde{+} rB_2^n) = \sum_{j=0}^{n} r^j \,\mathrm{vol}_j(B_2^j) \tilde{V}_{n-j}(K)$$

$$\tilde{V}_j(K) = \left[{n \atop j}\right] \int_{\mathrm{Gr}_j(\mathbb{R}^n)} \mathrm{vol}_j(K \cap E) \,\mathrm{d}E$$

$$\frac{V_1(K)}{V_1(B_2^n)} \geq \left(\frac{V_j(K)}{V_j(B_2^n)}\right)^{\frac{1}{j}} \geq \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B_2^n)}\right)^{\frac{1}{n}} \geq \left(\frac{\tilde{V}_j(K)}{\tilde{V}_j(B_2^n)}\right)^{\frac{1}{j}} \geq \frac{\tilde{V}_1(K)}{V_1(B_2^n)}$$

$$\begin{bmatrix} n \\ j \end{bmatrix} := \binom{n}{j} \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_j(B_2^j)\,\mathrm{vol}_{n-j}(B_2^{n-j})}.$$

Dual Volumes are Weighted Volumes

dual version of Kubota's formula (Lutwak, 1979):

$$\tilde{V}_j(K) = \begin{bmatrix} n \\ j \end{bmatrix} \int_{\mathrm{Gr}_j(\mathbb{R}^n)} \mathrm{vol}_j(K \cap E) \, \mathrm{d}E$$

- $\mathrm{Gr}_j(\mathbb{R}^n)$ ... Grassmannian of j-dimensional linear subspaces

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Lemma. Let $K \in \mathcal{K}_0(\mathbb{R}^n)$.

$$\tilde{V}_j(K) = V_j(B_2^n) \int_{\mathbb{S}^{n-1}} \rho_K^j \, \mathrm{d}\sigma = \frac{jV_j(B_2^n)}{nV_n(B_2^n)} \int_K \|x\|_2^{-(n-j)} \, \mathrm{d}x$$
$$= \mathrm{vol}_{\psi_j}(K).$$

- σ ... uniform probability measure on the unit sphere $\mathbb{S}^{n-1}$
- $\psi_j = \dfrac{jV_j(B_2^n)}{nV_n(B_2^n)} \|\cdot\|_2^{-(n-j)}$

Random Approximation of Intrinsic and Dual Volumes

Theorem (Reitzner, 2002). Let $1 \le j \le n$.

$$\mathbb{E}\, V_j(P_N^{\varphi_j}) = V_j(K) - \frac{\beta(n,j)}{2} \left(\frac{\Omega_j(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

where

$$\Omega_j(K) = \int_{\operatorname{bd} K} H_{n-j}^{\frac{n-1}{n+1}} \, H_{n-1}^{\frac{1}{n+1}} \, \mathrm{d}\mu_K$$

and $\varphi_j = \Omega_j(K)^{-1} H_{n-j}^{\frac{n-1}{n+1}} H_{n-1}^{\frac{1}{n+1}}$ is the optimal density.

- $\kappa_1(x), \dots, \kappa_{n-1}(x)$ ... principal curvatures of $\operatorname{bd} K$ at x
- $H_j = \binom{n-1}{j}^{-1} \sum_{1 \le i_1 < \cdots < i_j \le n-1} \kappa_{i_1} \cdots \kappa_{i_j}$

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'Theorem' (B., Hoehner & Kur 2019$^+$). Let $1 \leq j \leq n$.

$$\mathbb{E}\, \tilde{V}_j(P_N^{\tilde{\varphi}_j}) = \tilde{V}_j(K) - \frac{\tilde{\beta}(n,j)}{2} \left(\frac{\tilde{\Omega}_j(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

where

$$\tilde{\Omega}_j(K) = \int_{\operatorname{bd} K} \|\cdot\|_2^{-(n-j)\frac{n-1}{n+1}}\, H_{n-1}^{\frac{1}{n+1}}\, \mathrm{d}\mu_K$$

and $\tilde{\varphi}_j = \tilde{\Omega}_j(K)^{-1} \|\cdot\|_2^{-(n-j)\frac{n-1}{n+1}} H_{n-1}^{\frac{1}{n+1}}$ is the optimal density.

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Theorem (Glasauer & Gruber, 1997).
Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ be of class C^2_+. Then

$$\max_{P_N \in \mathcal{P}_N^i(K)} V_1(P_N) = V_1(K) - \frac{\alpha(n,1)}{2} \left(\frac{\Omega_1(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}} \right)$$

- $\alpha(n,1) = \mathrm{div}_{n-1} \frac{V_1(B_2^n)}{n V_n(B_2^n)}$
- div_{n-1} Dirichlet–Voroni (Вороно́й) tiling number,

  $\mathrm{div}_1 = \frac{1}{12}$, $\mathrm{div}_2 = \frac{5}{18\sqrt{3}}$

Theorem (Zador 1982; Mankiewicz & Schütt, 2000; Hoehner & Kur, 2018$^+$).

$$\mathrm{div}_{n-1} = \frac{n}{2\pi e} \left(1 + \frac{\ln n}{n} + O\left(\frac{1}{n} \right) \right)$$

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Open Question: Let $2 \leq j \leq n-1$. Do we also have

$$\max_{P_N \in \mathcal{P}^i_N(K)} V_j(P_N) \stackrel{?}{=} V_j(K) - \frac{\alpha(n,j)}{2}\left(\frac{\Omega_j(K)^{n+1}}{N^2}\right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

for some constant $0 < \alpha(n,j) \leq \beta(n,j)$?

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Theorem (B., Hoehner & Kur, 2019$^+$).

$$\liminf_{N\to\infty}\left(\frac{N^2}{\Omega_j(B_2^n)^{n+1}}\right)^{\frac{1}{n-1}}\left[V_j(B_2^n) - \max_{P_N \in \mathcal{P}_N^i(B_2^n)} V_j(P_N)\right] \geq \frac{\alpha_1(n,j)}{2}$$

$$\limsup_{N\to\infty}\left(\frac{N^2}{\Omega_j(B_2^n)^{n+1}}\right)^{\frac{1}{n-1}}\left[V_j(B_2^n) - \max_{P_N \in \mathcal{P}_N^i(B_2^n)} V_j(P_N)\right] \leq \frac{\alpha_2(n,j)}{2}$$

where

$$\alpha_1(n,j) = \operatorname{div}_{n-1}\frac{jV_j(B_2^n)}{nV_n(B_2^n)}, \quad \alpha_2(n,j) = \operatorname{del}_{n-1}\frac{jV_j(B_2^n)}{nV_n(B_2^n)}$$

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for some constant $\alpha(n,j) = \frac{n}{2\pi e} \frac{j V_j(B_2^n)}{n V_n(B_2^n)} (1 + O(\frac{\ln n}{n}))$?

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$$\max_{P_N \in \mathcal{P}^i_N(K)} V_1(P_N) = V_1(K) - \frac{\alpha(n,1)}{2} \left(\frac{\Omega_1(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

Open Question: Let $2 \le j \le n-1$. Do we also have

$$\max_{P_N \in \mathcal{P}^i_N(K)} V_j(P_N) \stackrel{?}{=} V_j(K) - \frac{\alpha(n,j)}{2} \left(\frac{\Omega_j(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

for some constant $\alpha(n,j) = \frac{n}{2\pi e} \frac{j V_j(B_2^n)}{n V_n(B_2^n)} (1 + O(\frac{\ln n}{n}))$?

'Theorem' (B., Hoehner & Kur, 2019^+). Let $1 \le j \le n$.

$$\max_{P_N \in \mathcal{P}^i_N(K)} \tilde{V}_j(P_N) = \tilde{V}_j(K) - \frac{\tilde{\alpha}(n,j)}{2} \left(\frac{\tilde{\Omega}_j(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

Best vs. Random Approximation of Dual Volumes

$$\mathbb{E}\,\tilde{V}_j(P_N^{\tilde{\varphi}_j}) = \tilde{V}_j(K) - \frac{\tilde{\beta}(n,j)}{2}\left(\frac{\tilde{\Omega}_j(K)^{n+1}}{N^2}\right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

$$\max_{P_N \in \mathcal{P}_N^i(K)} \tilde{V}_j(P_N) = \tilde{V}_j(K) - \frac{\tilde{\alpha}(n,j)}{2}\left(\frac{\tilde{\Omega}_j(K)^{n+1}}{N^2}\right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

$$1 \le \lim_{N\to\infty} \frac{\tilde{V}_j(K) - \mathbb{E}\,\tilde{V}_j(P_N^{\tilde{\varphi}_j})}{\tilde{V}_j(K) - \max\limits_{P_N \in \mathcal{P}_N^i(K)} \tilde{V}_j(P_N)} = \frac{\beta_n}{\mathrm{del}_{n-1}} = 1 + O\left(\frac{\ln n}{n}\right)$$

$$\tilde{\alpha}(n,j) = \mathrm{del}_{n-1}\frac{jV_j(B_2^n)}{nV_n(B_2^n)}, \qquad \tilde{\beta}(n,j) = \beta_n \frac{jV_j(B_2^n)}{nV_n(B_2^n)}$$

Actually there is much more...

- Extensions of $\tilde{V}_j(K) = V_j(B_2^n) \int_{\mathbb{S}^{n-1}} \rho_K^j \, \mathrm{d}\sigma$ for $j \in \{1, \dots, n\}$ to $\tilde{V}_q$ for $q \in \mathbb{R}$. In particular, for

$$\hat{V}_0(K) = \int_{\mathbb{S}^{n-1}} \ln \rho_K \, \mathrm{d}\sigma$$

we have

$$\max_{P_N \in \mathcal{P}_N^i(K)} \hat{V}_0(P_N) = \hat{V}_0(K) - \frac{\beta_n}{2n V_n(B_2^n)} \left(\frac{\tilde{\Omega}_0(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

$$\max_{P_N \in \mathcal{P}_N^i(K)} \hat{V}_0(P_N) = \hat{V}_0(K) - \frac{\mathrm{del}_{n-1}}{2n V_n(B_2^n)} \left(\frac{\tilde{\Omega}_0(K)^{n+1}}{N^2} \right)^{\frac{1}{n-1}} + o\left(N^{-\frac{2}{n-1}}\right)$$

- Best & random approximation of $\tilde{V}_j$ with respect to

$$\mathcal{P}_N = \{\#\text{vertices} \le N\} \qquad \mathcal{P}_N^i(K) = \{P \subset K\}$$
$$\mathcal{P}_{(N)} = \{\#\text{facets} \le N\} \qquad \mathcal{P}_N^o(K) = \{P \supset K\}$$