Zhang's inequality for log-concave functions

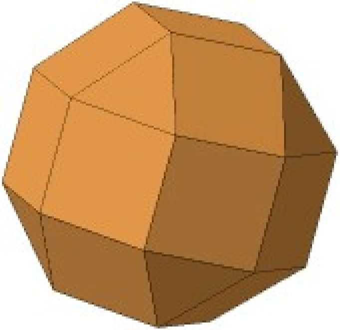
David Alonso Gutiérrez
Joint work with Julio Bernués and Bernardo González

Universidad de Zaragoza

5th of July, 2019

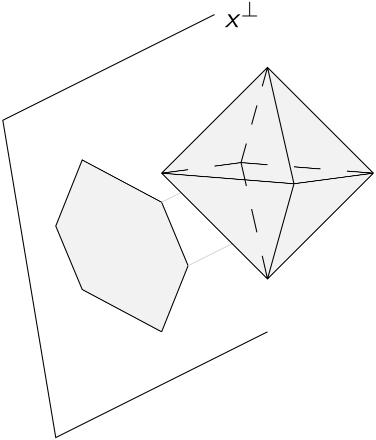
Convex bodies

- $K \subset \mathbb{R}^n$ is called a convex body if it is convex, compact and has non-empty interior.



The polar projection body

- The function $|x||P_{x^\perp}K|$ defines a norm in $\mathbb{R}^n$.



The polar projection body

- The function $|x||P_{x^\perp}K|$ defines a norm in $\mathbb{R}^n$.
- The unit ball of this norm is the polar projection body $\Pi^* K$.

The polar projection body

- The function $|x||P_{x^\perp}K|$ defines a norm in $\mathbb{R}^n$.
- The unit ball of this norm is the polar projection body Π^*K.
- The quantity $|K|^{n-1}|\Pi^*K|$ is affine invariant.

The polar projection body

- The function $|x||P_{x^\perp}K|$ defines a norm in $\mathbb{R}^n$.
- The unit ball of this norm is the polar projection body $\Pi^* K$.
- The quantity $|K|^{n-1}|\Pi^* K|$ is affine invariant.

Petty projection inequality (1971)

$$|K|^{\frac{n-1}{n}}|\Pi^* K|^{\frac{1}{n}} \leq |B_2^n|^{\frac{n-1}{n}}|\Pi^* B_2^n|^{\frac{1}{n}}.$$

The polar projection body

- The function $|x||P_{x^\perp}K|$ defines a norm in $\mathbb{R}^n$.
- The unit ball of this norm is the polar projection body Π^*K.
- The quantity $|K|^{n-1}|\Pi^*K|$ is affine invariant.

Petty projection inequality (1971)

$$|K|^{\frac{n-1}{n}}|\Pi^*K|^{\frac{1}{n}} \leq |B_2^n|^{\frac{n-1}{n}}|\Pi^*B_2^n|^{\frac{1}{n}}.$$

- Petty projection inequality implies the isoperimetric inequality.

Zhang's inequality

Petty projection inequality (1971)

For any convex body K

$$|K|^{\frac{n-1}{n}}|\Pi^*K|^{\frac{1}{n}} \leq |B_2^n|^{\frac{n-1}{n}}|\Pi^*B_2^n|^{\frac{1}{n}} = \frac{|B_2^n|}{|B_2^{n-1}|}$$

with equality if and only if K is an ellipsoid.

Zhang's inequality

Petty projection inequality (1971)

For any convex body K

$$|K|^{\frac{n-1}{n}}|\Pi^* K|^{\frac{1}{n}} \leq |B_2^n|^{\frac{n-1}{n}}|\Pi^* B_2^n|^{\frac{1}{n}} = \frac{|B_2^n|}{|B_2^{n-1}|}$$

with equality if and only if K is an ellipsoid.

Zhang's inequality (1991)

For any convex body K

$$|K|^{\frac{n-1}{n}}|\Pi^* K|^{\frac{1}{n}} \geq |\Delta_n|^{\frac{n-1}{n}}|\Pi^* \Delta_n|^{\frac{1}{n}} = \frac{\binom{2n}{n}^{\frac{1}{n}}}{n}$$

with equality if and only if K is a simplex.

The affine Sobolev inequality

Let $f \in W^{1,1}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \frac{\partial f}{\partial x_i} \in L^1(\mathbb{R}^n), i = 1, \ldots, n\}$.

$$\|x\|_{\Pi^*(f)} := \int_{\mathbb{R}^n} |\langle \nabla f(y), x \rangle| dy.$$

The affine Sobolev inequality

Let $f \in W^{1,1}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \frac{\partial f}{\partial x_i} \in L^1(\mathbb{R}^n), i = 1, \ldots, n\}$.

$$\|x\|_{\Pi^*(f)} := \int_{\mathbb{R}^n} |\langle \nabla f(y), x \rangle| dy.$$

Theorem (Affine Sobolev inequality, Zhang, 1999)

$$\|f\|_{\frac{n}{n-1}} |\Pi^*(f)|^{\frac{1}{n}} \leq \frac{|B_2^n|}{2|B_2^{n-1}|}.$$

The affine Sobolev inequality

Let $f \in W^{1,1}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \frac{\partial f}{\partial x_i} \in L^1(\mathbb{R}^n), i = 1, \dots, n\}$.

$$\|x\|_{\Pi^*(f)} := \int_{\mathbb{R}^n} |\langle \nabla f(y), x \rangle| dy.$$

Theorem (Affine Sobolev inequality, Zhang, 1999)

$$\|f\|_{\frac{n}{n-1}} |\Pi^*(f)|^{\frac{1}{n}} \leq \frac{|B_2^n|}{2|B_2^{n-1}|}.$$

We can take $f_k \to f = \chi_K$. Then

- $\Pi^*(f_k) \to \frac{1}{2}\Pi^*(K)$
- $\|f_k\|_{\frac{n}{n-1}} |\Pi^*(f_k)|^{\frac{1}{n}} \to \frac{1}{2}|K|^{\frac{n-1}{n}} |\Pi^* K|^{\frac{1}{n}}$

and we recover Petty projection inequality.

Log-concave functions

$f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$\frac{f(x)}{\|f\|_\infty} = e^{-u(x)} \text{ with } u : \mathbb{R}^n \to [0, \infty] \text{ convex.}$$

Log-concave functions

$f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$\frac{f(x)}{\|f\|_\infty} = e^{-u(x)} \text{ with } u : \mathbb{R}^n \to [0, \infty] \text{ convex.}$$

- The epigraph of u is convex

$$\begin{aligned} \mathrm{epi}(u) &= \{(x, t) \in \mathbb{R}^n \times [0, \infty) \ : \ u(x) \le t\} \\ &= \{(x, t) \in \mathbb{R}^n \times [0, \infty) \ : \ f(x) \ge e^{-t}\|f\|_\infty\}. \end{aligned}$$

Log-concave functions

$f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$\frac{f(x)}{\|f\|_\infty} = e^{-u(x)} \text{ with } u : \mathbb{R}^n \to [0, \infty] \text{ convex.}$$

- The epigraph of u is convex

$$\begin{aligned} \mathrm{epi}(u) &= \{(x, t) \in \mathbb{R}^n \times [0, \infty) \ : \ u(x) \le t\} \\ &= \{(x, t) \in \mathbb{R}^n \times [0, \infty) \ : \ f(x) \ge e^{-t}\|f\|_\infty\}. \end{aligned}$$

- $K_t(f) = \{x \in \mathbb{R}^n \ : \ f(x) \ge e^{-t}\|f\|_\infty\}$ is convex.

Log-concave functions

$f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$\frac{f(x)}{\|f\|_\infty} = e^{-u(x)} \text{ with } u : \mathbb{R}^n \to [0, \infty] \text{ convex.}$$

- The epigraph of u is convex

$$\begin{aligned} \mathrm{epi}(u) &= \{(x,t) \in \mathbb{R}^n \times [0,\infty) \ : \ u(x) \le t\} \\ &= \{(x,t) \in \mathbb{R}^n \times [0,\infty) \ : \ f(x) \ge e^{-t}\|f\|_\infty\}. \end{aligned}$$

- $K_t(f) = \{x \in \mathbb{R}^n \ : \ f(x) \ge e^{-t}\|f\|_\infty\}$ is convex.
- If K is a convex body then $f(x) = \chi_K(x)$ is log-concave and $\|f\|_1 = |K|$.

Log-concave functions

$f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$\frac{f(x)}{\|f\|_\infty} = e^{-u(x)} \text{ with } u : \mathbb{R}^n \to [0, \infty] \text{ convex.}$$

- The epigraph of u is convex

$$\begin{aligned} \mathrm{epi}(u) &= \{(x,t) \in \mathbb{R}^n \times [0,\infty) \; : \; u(x) \leq t\} \\ &= \{(x,t) \in \mathbb{R}^n \times [0,\infty) \; : \; f(x) \geq e^{-t}\|f\|_\infty\}. \end{aligned}$$

- $K_t(f) = \{x \in \mathbb{R}^n \; : \; f(x) \geq e^{-t}\|f\|_\infty\}$ is convex.
- If K is a convex body then $f(x) = \chi_K(x)$ is log-concave and $\|f\|_1 = |K|$.
- If K is a convex body containing the origin then $f(x) = e^{-\|x\|_K}$ is log-concave and $\|f\|_1 = n!|K|$.

Log-concave functions

$f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

$$\frac{f(x)}{\|f\|_\infty} = e^{-u(x)} \text{ with } u : \mathbb{R}^n \to [0, \infty] \text{ convex.}$$

- The epigraph of u is convex

$$\begin{aligned}\mathrm{epi}(u) &= \{(x, t) \in \mathbb{R}^n \times [0, \infty) \,:\, u(x) \le t\} \\ &= \{(x, t) \in \mathbb{R}^n \times [0, \infty) \,:\, f(x) \ge e^{-t}\|f\|_\infty\}.\end{aligned}$$

- $K_t(f) = \{x \in \mathbb{R}^n \,:\, f(x) \ge e^{-t}\|f\|_\infty\}$ is convex.
- If K is a convex body then $f(x) = \chi_K(x)$ is log-concave and $\|f\|_1 = |K|$.
- If K is a convex body containing the origin then $f(x) = e^{-\|x\|_K}$ is log-concave and $\|f\|_1 = n!|K|$.
- If $\frac{f(x)}{\|f\|_\infty} = e^{-u(x)}$ is integrable and log-concave then

$$\frac{\|f\|_1}{\|f\|_\infty} = \int_{\mathrm{epi}(u)} e^{-t}\, dx dt.$$

Projections of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function and $H \in G_{n,k}$. The projection of f onto H is

$$P_H f(x) = \sup_{y \in H^\perp} f(x + y), \quad x \in H.$$

Projections of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function and $H \in G_{n,k}$. The projection of f onto H is

$$P_H f(x) = \sup_{y \in H^\perp} f(x+y), \quad x \in H.$$

- $\frac{f(x)}{\|f\|_\infty} = e^{-u(x)} \Rightarrow \frac{P_H f(x)}{\|P_H f\|_\infty} = \frac{P_H f(x)}{\|f\|_\infty} = e^{-v(x)}$ with

$$\mathrm{epi}(v) = P_{\overline{H}} \mathrm{epi}(v),$$

  where $\overline{H} = \mathrm{span}\{H, e_{n+1}\}$.

Projections of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function and $H \in G_{n,k}$. The projection of f onto H is

$$P_H f(x) = \sup_{y \in H^\perp} f(x + y), \quad x \in H.$$

- $\frac{f(x)}{\|f\|_\infty} = e^{-u(x)} \Rightarrow \frac{P_H f(x)}{\|P_H f\|_\infty} = \frac{P_H f(x)}{\|f\|_\infty} = e^{-v(x)}$ with

$$\mathrm{epi}(v) = P_{\overline{H}} \mathrm{epi}(v),$$

  where $\overline{H} = \mathrm{span}\{H, e_{n+1}\}$.
- If K is a convex body then $P_H \chi_K(x) = \chi_{P_H K}(x)$.

Projections of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be a log-concave function and $H \in G_{n,k}$. The projection of f onto H is

$$P_H f(x) = \sup_{y \in H^\perp} f(x + y), \quad x \in H.$$

- $\frac{f(x)}{\|f\|_\infty} = e^{-u(x)} \Rightarrow \frac{P_H f(x)}{\|P_H f\|_\infty} = \frac{P_H f(x)}{\|f\|_\infty} = e^{-v(x)}$ with

  $$\mathrm{epi}(v) = P_{\overline{H}} \mathrm{epi}(v),$$

  where $\overline{H} = \mathrm{span}\{H, e_{n+1}\}$.
- If K is a convex body then $P_H \chi_K(x) = \chi_{P_H K}(x)$.
- If K is a convex body containing the origin then $P_H e^{-\|\cdot\|_K}(x) = e^{-\|x\|_{P_H K}}$.

The polar projection body of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its polar projection body $\Pi^*(f)$ is the unit ball of the norm

$$\|x\|_{\Pi^*(f)} := 2|x| \int_{x^\perp} P_{x^\perp} f(y) dy$$

The polar projection body of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its polar projection body $\Pi^*(f)$ is the unit ball of the norm

$$\begin{aligned} \|x\|_{\Pi^*(f)} &:= 2|x| \int_{x^\perp} P_{x^\perp} f(y) dy \\ &= 2|x| \|f\|_\infty \int_0^\infty |P_{x^\perp} K_t(f)| e^{-t} dt \end{aligned}$$

The polar projection body of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its polar projection body $\Pi^*(f)$ is the unit ball of the norm

$$\begin{aligned}
\|x\|_{\Pi^*(f)} &:= 2|x| \int_{x^\perp} P_{x^\perp} f(y) dy \\
&= 2|x| \|f\|_\infty \int_0^\infty |P_{x^\perp} K_t(f)| e^{-t} dt \\
&= 2\|f\|_\infty \int_0^\infty \|x\|_{\Pi^*(K_t(f))} e^{-t} dt
\end{aligned}$$

The polar projection body of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its polar projection body $\Pi^*(f)$ is the unit ball of the norm

$$\begin{aligned} \|x\|_{\Pi^*(f)} &:= 2|x| \int_{x^\perp} P_{x^\perp} f(y) dy \\ &= 2|x| \|f\|_\infty \int_0^\infty |P_{x^\perp} K_t(f)| e^{-t} dt \\ &= 2\|f\|_\infty \int_0^\infty \|x\|_{\Pi^*(K_t(f))} e^{-t} dt \end{aligned}$$

- If $f \in W^{1,1}(\mathbb{R}^n)$ then

$$\|x\|_{\Pi^*(f)} = \int_{\mathbb{R}^n} |\langle \nabla f(y), x \rangle| dy.$$

The polar projection body of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its polar projection body $\Pi^*(f)$ is the unit ball of the norm

$$\begin{aligned} \|x\|_{\Pi^*(f)} &:= 2|x| \int_{x^\perp} P_{x^\perp} f(y) dy \\ &= 2|x| \|f\|_\infty \int_0^\infty |P_{x^\perp} K_t(f)| e^{-t} dt \\ &= 2\|f\|_\infty \int_0^\infty \|x\|_{\Pi^*(K_t(f))} e^{-t} dt \end{aligned}$$

- If $f \in W^{1,1}(\mathbb{R}^n)$ then

$$\|x\|_{\Pi^*(f)} = \int_{\mathbb{R}^n} |\langle \nabla f(y), x \rangle| dy.$$

- If K is a convex body then $\Pi^*(\chi_K) = \frac{1}{2}\Pi^*(K)$.

The polar projection body of log-concave functions

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its polar projection body $\Pi^*(f)$ is the unit ball of the norm

$$\begin{aligned} \|x\|_{\Pi^*(f)} &:= 2|x| \int_{x^\perp} P_{x^\perp} f(y) dy \\ &= 2|x| \|f\|_\infty \int_0^\infty |P_{x^\perp} K_t(f)| e^{-t} dt \\ &= 2\|f\|_\infty \int_0^\infty \|x\|_{\Pi^*(K_t(f))} e^{-t} dt \end{aligned}$$

- If $f \in W^{1,1}(\mathbb{R}^n)$ then

$$\|x\|_{\Pi^*(f)} = \int_{\mathbb{R}^n} |\langle \nabla f(y), x \rangle| dy.$$

- If K is a convex body then $\Pi^*(\chi_K) = \frac{1}{2}\Pi^*(K)$.
- If K is a convex body containing the origin then $\Pi^*(e^{-\|\cdot\|_K}) = \frac{1}{2(n-1)!}\Pi^*(K)$.

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min\{f(x), f(y)\} dy dx \leq 2^n n! \|f\|_1^{n+1} |\Pi^*(f)|.$$

If $\|f\|_\infty = f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty} = e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min\{f(x), f(y)\} dy dx \leq 2^n n! \|f\|_1^{n+1} |\Pi^*(f)|.$$

If $\|f\|_\infty = f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty} = e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

If K is a convex body containing the origin and $f(x) = e^{-\|x\|_K}$ then

- $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min\{f(x), f(y)\} dy dx = \int_{\mathbb{R}^{2n}} e^{-\|(x,y)\|_{K \times K}} dy dx = (2n)! |K|^2$
- $\|f\|_1 = n! |K|$
- $|\Pi^*(f)| = \frac{1}{2^n((n-1)!)^n} |\Pi^*(K)|$

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f : \mathbb{R}^n \to [0,\infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \min\{f(x), f(y)\} dydx \leq 2^n n! \|f\|_1^{n+1} |\Pi^*(f)|.$$

If $\|f\|_\infty = f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty} = e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

If K is a convex body containing the origin and $f(x) = e^{-\|x\|_K}$ then

- $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \min\{f(x), f(y)\} dydx = \int_{\mathbb{R}^{2n}} e^{-\|(x,y)\|_{K\times K}} dydx = (2n)!|K|^2$
- $\|f\|_1 = n!|K|$
- $|\Pi^*(f)| = \frac{1}{2^n((n-1)!)^n} |\Pi^*(K)|$

$$\frac{\binom{2n}{n}}{n^n} \leq |K|^{n-1} |\Pi^*(K)|$$

The covariogram function

If K is a convex body, its covariogram function is

$$g_K(x) = |K \cap (x + K)|.$$

The covariogram function

If K is a convex body, its covariogram function is

$$g_K(x) = |K \cap (x + K)|.$$

- g_K is log-concave.

The covariogram function

If K is a convex body, its covariogram function is

$$g_K(x) = |K \cap (x + K)|.$$

- g_K is log-concave.
- $\|g_K\|_\infty = g_K(0) = |K|, \quad \|g_K\|_1 = |K|^2.$

The covariogram function

If K is a convex body, its covariogram function is

$$g_K(x) = |K \cap (x + K)|.$$

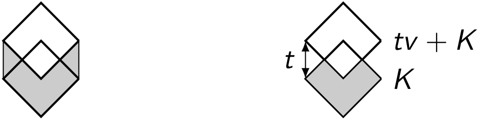
- g_K is log-concave.
- $\|g_K\|_\infty = g_K(0) = |K|, \quad \|g_K\|_1 = |K|^2.$
- g_K is even.

The covariogram function

If K is a convex body, its covariogram function is

$$g_K(x) = |K \cap (x + K)|.$$

- g_K is log-concave.
- $\|g_K\|_\infty = g_K(0) = |K|, \quad \|g_K\|_1 = |K|^2.$
- g_K is even.
- $-\frac{d^+}{dt} g_K(tv)\Big|_{t=0} = |P_{v^\perp} K| = \|v\|_{\Pi^* K}, \quad v \in S^{n-1}$



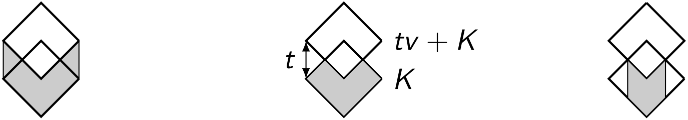
$t|P_{v^\perp} K|$ $\qquad |K| - |K \cap (tv + K)| \qquad$ $t|P_{v^\perp}(K \cap (tv + K))|$

The covariogram function

If K is a convex body, its covariogram function is

$$g_K(x) = |K \cap (x + K)|.$$

- g_K is log-concave.
- $\|g_K\|_\infty = g_K(0) = |K|, \quad \|g_K\|_1 = |K|^2.$
- g_K is even.
- $-\frac{d^+}{dt} g_K(tv)\Big|_{t=0} = |P_{v^\perp} K| = \|v\|_{\Pi^* K}, \quad v \in S^{n-1}$



$t|P_{v^\perp} K|$ $\qquad$ $|K| - |K \cap (tv + K)|$ $\qquad$ $t|P_{v^\perp}(K \cap (tv + K))|$

- Schmuckesläger proved (1992) that for any $0 < \lambda < 1$

$$\frac{K_{-\log(1-\lambda)}(g_K)}{-\log(1-\lambda)} \subseteq |K|\Pi^*(K) \subseteq \frac{K_{-\log(1-\lambda)}(g_K)}{\lambda}$$

The covariogram function

The left-hand side can be improved to

$$\frac{K_{-\log(1-\lambda)}(g_K)}{n(1-(1-\lambda)^{\frac{1}{n}})} \subseteq |K|\Pi^*(K), \text{for every } 0<\lambda<1.$$

The covariogram function

The left-hand side can be improved to

$$\frac{K_{-\log(1-\lambda)}(g_K)}{n(1-(1-\lambda)^{\frac{1}{n}})} \subseteq |K|\Pi^*(K), \text{for every } 0 < \lambda < 1.$$

Equivalently, for every $0 < \theta < 1$

$$\frac{K_{-\log\theta}(g_K)}{n(1-\theta^{\frac{1}{n}})} \subseteq |K|\Pi^*(K).$$

The covariogram function

The left-hand side can be improved to

$$\frac{K_{-\log(1-\lambda)}(g_K)}{n(1-(1-\lambda)^{\frac{1}{n}})} \subseteq |K|\Pi^*(K), \text{for every } 0<\lambda<1.$$

Equivalently, for every $0<\theta<1$

$$\frac{K_{-\log\theta}(g_K)}{n(1-\theta^{\frac{1}{n}})} \subseteq |K|\Pi^*(K).$$

$\frac{K_{-\log\theta}(g_K)}{1-\theta^{\frac{1}{n}}}$ is increasing in θ and then

$$(1-\theta^{\frac{1}{n}})(K-K) \subseteq K_{-\log\theta}(g_K) \subseteq n(1-\theta^{\frac{1}{n}})|K|\Pi^*(K).$$

The covariogram function

The left-hand side can be improved to

$$\frac{K_{-\log(1-\lambda)}(g_K)}{n(1-(1-\lambda)^{\frac{1}{n}})} \subseteq |K|\Pi^*(K), \text{for every } 0 < \lambda < 1.$$

Equivalently, for every $0 < \theta < 1$

$$\frac{K_{-\log\theta}(g_K)}{n(1-\theta^{\frac{1}{n}})} \subseteq |K|\Pi^*(K).$$

$\frac{K_{-\log\theta}(g_K)}{1-\theta^{\frac{1}{n}}}$ is increasing in θ and then

$$(1-\theta^{\frac{1}{n}})(K-K) \subseteq K_{-\log\theta}(g_K) \subseteq n(1-\theta^{\frac{1}{n}})|K|\Pi^*(K).$$

Taking volumes and integrating in $\theta \in (0,1)$

$$\binom{2n}{n}^{-1}|K-K| \leq |K| \leq \binom{2n}{n}^{-1} n^n|K|^n|\Pi^*(K)|.$$

The covariogram function

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its covariogram function is

$$g_f(x) = \int_0^\infty e^{-t} |K_t(f) \cap (x + K_t(f))| dt, \quad x \in \mathbb{R}^n$$

The covariogram function

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its covariogram function is

$$\begin{aligned} g_f(x) &= \int_0^\infty e^{-t} |K_t(f) \cap (x + K_t(f))| dt, \quad x \in \mathbb{R}^n \\ &= \int_{\mathrm{epi}(u) \cap (x + \mathrm{epi}(u))} e^{-t} dt dy \end{aligned}$$

if $\frac{f}{\|f\|_\infty} = e^{-u}$.

The covariogram function

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its covariogram function is

$$\begin{aligned} g_f(x) &= \int_0^\infty e^{-t} |K_t(f) \cap (x + K_t(f))| dt, \quad x \in \mathbb{R}^n \\ &= \int_{\mathrm{epi}(u) \cap (x + \mathrm{epi}(u))} e^{-t} dt dy \end{aligned}$$

if $\frac{f}{\|f\|_\infty} = e^{-u}$.

- g_f is log-concave.

The covariogram function

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its covariogram function is

$$\begin{aligned} g_f(x) &= \int_0^\infty e^{-t} |K_t(f) \cap (x + K_t(f))| dt, \quad x \in \mathbb{R}^n \\ &= \int_{\mathrm{epi}(u) \cap (x + \mathrm{epi}(u))} e^{-t} dt dy \end{aligned}$$

if $\frac{f}{\|f\|_\infty} = e^{-u}$.

- g_f is log-concave.
- $\|g_f\|_\infty = g_f(0) = \dfrac{\|f\|_1}{\|f\|_\infty}, \quad \|g_f\|_1 = \displaystyle\int_{\mathbb{R}^{2n}} \min\left\{ \frac{f(x)}{\|f\|_\infty}, \frac{f(y)}{\|f\|_\infty} \right\} dydx.$

The covariogram function

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Its covariogram function is

$$\begin{aligned} g_f(x) &= \int_0^\infty e^{-t} |K_t(f) \cap (x + K_t(f))| dt, \quad x \in \mathbb{R}^n \\ &= \int_{\mathrm{epi}(u) \cap (x + \mathrm{epi}(u))} e^{-t} dt dy \end{aligned}$$

if $\frac{f}{\|f\|_\infty} = e^{-u}$.

- g_f is log-concave.
- $\|g_f\|_\infty = g_f(0) = \frac{\|f\|_1}{\|f\|_\infty}, \quad \|g_f\|_1 = \int_{\mathbb{R}^{2n}} \min \left\{ \frac{f(x)}{\|f\|_\infty}, \frac{f(y)}{\|f\|_\infty} \right\} dydx.$
- g_f is even.

The covariogram function

Let $f:\mathbb{R}^n\to[0,\infty)$ be an integrable log-concave function. Its covariogram function is

$$\begin{aligned} g_f(x) &= \int_0^\infty e^{-t}|K_t(f)\cap(x+K_t(f))|dt, \quad x\in\mathbb{R}^n \\ &= \int_{\mathrm{epi}(u)\cap(x+\mathrm{epi}(u))} e^{-t}dtdy \end{aligned}$$

if $\frac{f}{\|f\|_\infty}=e^{-u}$.

- g_f is log-concave.
- $\|g_f\|_\infty=g_f(0)=\frac{\|f\|_1}{\|f\|_\infty}, \quad \|g_f\|_1=\int_{\mathbb{R}^{2n}}\min\left\{\frac{f(x)}{\|f\|_\infty},\frac{f(y)}{\|f\|_\infty}\right\}dydx.$
- g_f is even.
- For every $0<\lambda_0<1$

$$\bigcap_{0<\lambda<\lambda_0}\frac{K_{-\log(1-\lambda)}(g_f)}{\lambda}=2\|f\|_1\Pi^*(f).$$

Ball's bodies

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $g(0) > 0$ and $p > 0$.

$$\widetilde{K}_p(g) = \left\{ x \in \mathbb{R}^n \; : \; \int_0^\infty g(rx) r^{p-1} dr \geq \frac{g(0)}{p} \right\}.$$

Ball's bodies

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $g(0) > 0$ and $p > 0$.

$$\widetilde{K}_p(g) = \left\{ x \in \mathbb{R}^n \, : \, \int_0^\infty g(rx) r^{p-1} dr \geq \frac{g(0)}{p} \right\}.$$

- $\widetilde{K}_p(g)$ is a convex body.

Ball's bodies

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $g(0) > 0$ and $p > 0$.

$$\widetilde{K}_p(g) = \left\{ x \in \mathbb{R}^n \ : \ \int_0^\infty g(rx) r^{p-1} dr \geq \frac{g(0)}{p} \right\}.$$

- $\widetilde{K}_p(g)$ is a convex body.
- $\rho^p_{\widetilde{K}_p(g)}(u) = \dfrac{1}{g(0)} \displaystyle\int_0^\infty p r^{p-1} g(ru) dr.$

Ball's bodies

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $g(0) > 0$ and $p > 0$.

$$\widetilde{K}_p(g) = \left\{ x \in \mathbb{R}^n \, : \, \int_0^\infty g(rx) r^{p-1} dr \geq \frac{g(0)}{p} \right\}.$$

- $\widetilde{K}_p(g)$ is a convex body.
- $\rho^p_{\widetilde{K}_p(g)}(u) = \dfrac{1}{g(0)} \displaystyle\int_0^\infty p r^{p-1} g(ru) dr.$
- $|\widetilde{K}_n(g)| = \frac{\|g\|_1}{g(0)}.$

Ball's bodies and super-level sets

Lemma

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $\|g\|_\infty = g(0)$. Then for every $0 \le t \le \frac{n}{e}$

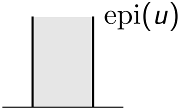
$$\frac{t}{(n!)^{\frac{1}{n}}} \widetilde{K}_n(g) \subseteq K_t(g).$$

Ball's bodies and super-level sets

Lemma

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $\|g\|_\infty = g(0)$. Then for every $0 \leq t \leq \frac{n}{e}$

$$\frac{t}{(n!)^{\frac{1}{n}}} \widetilde{K}_n(g) \subseteq K_t(g).$$

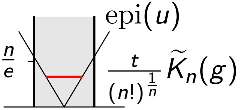


Ball's bodies and super-level sets

Lemma

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $\|g\|_\infty = g(0)$. Then for every $0 \le t \le \frac{n}{e}$

$$\frac{t}{(n!)^{\frac{1}{n}}} \widetilde{K}_n(g) \subseteq K_t(g).$$

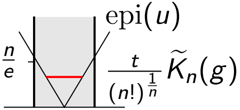


Ball's bodies and super-level sets

Lemma

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $\|g\|_\infty = g(0)$. Then for every $0 \le t \le \frac{n}{e}$

$$\frac{t}{(n!)^{\frac{1}{n}}} \widetilde{K}_n(g) \subseteq K_t(g).$$



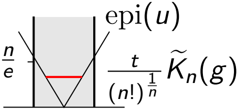
- For any $0 < t \le \frac{n}{e}$ there is equality if and only if g is log-linear on every ray emanating from 0.

Ball's bodies and super-level sets

Lemma

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $\|g\|_\infty = g(0)$. Then for every $0 \le t \le \frac{n}{e}$

$$\frac{t}{(n!)^{\frac{1}{n}}} \widetilde{K}_n(g) \subseteq K_t(g).$$



- Furthermore, if g is not log-linear on the ray emanating from 0 in the direction u the there exists $\varepsilon > 0$ such that for every $0 < t < \frac{n}{e}$

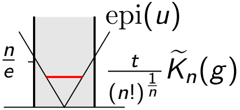
$$\frac{t}{n!}(\rho_{\widetilde{K}_n(g)}(u) + \varepsilon) \le \rho_{K_t(g)}(u).$$

Ball's bodies and super-level sets

Lemma

Let $g : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function with $\|g\|_\infty = g(0)$. Then for every $0 \leq t \leq \frac{n}{e}$

$$\frac{t}{(n!)^{\frac{1}{n}}} \widetilde{K}_n(g) \subseteq K_t(g).$$



- Furthermore, if g is not log-linear on the ray emanating from 0 in the direction u the there exists $\varepsilon > 0$ such that for every $0 < t < \frac{n}{e}$

$$\frac{t}{n!} (\rho_{\widetilde{K}_n(g)}(u) + \varepsilon) \leq \rho_{K_t(g)}(u).$$

- Follows ideas by Klartag-Milman, who proved the result (2005) for $t = cn$.

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min\{f(x), f(y)\} dy dx \leq 2^n n! \|f\|_1^{n+1} |\Pi^*(f)|.$$

If $\|f\|_\infty = f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty} = e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \min\{f(x), f(y)\} dydx \leq 2^n n! \|f\|_1^{n+1} |\Pi^*(f)|.$$

If $\|f\|_\infty = f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty} = e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

For any $0 < \lambda_0 < 1$

$$2\|f\|_1 \Pi^*(f) = \bigcap_{0<\lambda<\lambda_0} \frac{K_{-\log(1-\lambda)}(g_f)}{\lambda}$$

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \min\{f(x), f(y)\} dy dx \leq 2^n n! \|f\|_1^{n+1} |\Pi^*(f)|.$$

If $\|f\|_\infty = f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty} = e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

For any $0 < \lambda_0 < 1$

$$2\|f\|_1 \Pi^*(f) = \bigcap_{0<\lambda<\lambda_0} \frac{K_{-\log(1-\lambda)}(g_f)}{\lambda} \supseteq \bigcap_{0<\lambda<\lambda_0} \frac{-\log(1-\lambda)}{\lambda (n!)^{\frac{1}{n}}} \widetilde{K}_n(g_f)$$

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f:\mathbb{R}^n\to[0,\infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\min\{f(x),f(y)\}dydx\leq 2^n n!\|f\|_1^{n+1}|\Pi^*(f)|.$$

If $\|f\|_\infty=f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty}=e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

For any $0<\lambda_0<1$

$$2\|f\|_1\Pi^*(f)=\bigcap_{0<\lambda<\lambda_0}\frac{K_{-\log(1-\lambda)}(g_f)}{\lambda}\supseteq\bigcap_{0<\lambda<\lambda_0}\frac{-\log(1-\lambda)}{\lambda(n!)^{\frac{1}{n}}}\widetilde{K}_n(g_f)$$
$$=\frac{1}{(n!)^{\frac{1}{n}}}\widetilde{K}_n(g_f)$$

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f:\mathbb{R}^n\to[0,\infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\min\{f(x),f(y)\}dydx\le 2^n n!\|f\|_1^{n+1}|\Pi^*(f)|.$$

If $\|f\|_\infty=f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty}=e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

For any $0<\lambda_0<1$

$$2\|f\|_1\Pi^*(f)=\bigcap_{0<\lambda<\lambda_0}\frac{K_{-\log(1-\lambda)}(g_f)}{\lambda}\supseteq\bigcap_{0<\lambda<\lambda_0}\frac{-\log(1-\lambda)}{\lambda(n!)^{\frac{1}{n}}}\widetilde{K}_n(g_f)$$
$$=\frac{1}{(n!)^{\frac{1}{n}}}\widetilde{K}_n(g_f)$$

$$2^n\|f\|_1^n|\Pi^*(f)|\ge\frac{1}{n!}|\widetilde{K}_n(g_f)|$$

.

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f:\mathbb{R}^n\to[0,\infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\min\{f(x),f(y)\}dydx\leq 2^n n!\|f\|_1^{n+1}|\Pi^*(f)|.$$

If $\|f\|_\infty=f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty}=e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

For any $0<\lambda_0<1$

$$2\|f\|_1\Pi^*(f)=\bigcap_{0<\lambda<\lambda_0}\frac{K_{-\log(1-\lambda)}(g_f)}{\lambda}\supseteq\bigcap_{0<\lambda<\lambda_0}\frac{-\log(1-\lambda)}{\lambda(n!)^{\frac{1}{n}}}\widetilde{K}_n(g_f)$$
$$=\frac{1}{(n!)^{\frac{1}{n}}}\widetilde{K}_n(g_f)$$

$$2^n\|f\|_1^n|\Pi^*(f)|\geq\frac{1}{n!}\frac{\|g_f\|_1}{g_f(0)}.$$

Zhang's inequality for log-concave functions

Theorem (A., Bernués, González, to appear)

Let $f:\mathbb{R}^n\to[0,\infty)$ be an integrable log-concave function. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\min\{f(x),f(y)\}dydx\le 2^n n!\|f\|_1^{n+1}|\Pi^*(f)|.$$

If $\|f\|_\infty=f(0)$ then there is equality if and only if $\frac{f(x)}{\|f\|_\infty}=e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.

For any $0<\lambda_0<1$

$$2\|f\|_1\Pi^*(f)=\bigcap_{0<\lambda<\lambda_0}\frac{K_{-\log(1-\lambda)}(g_f)}{\lambda}\supseteq\bigcap_{0<\lambda<\lambda_0}\frac{-\log(1-\lambda)}{\lambda(n!)^{\frac{1}{n}}}\widetilde{K}_n(g_f)$$

$$=\frac{1}{(n!)^{\frac{1}{n}}}\widetilde{K}_n(g_f)$$

$$2^n\|f\|_1^n|\Pi^*(f)|\ge\frac{1}{n!}\frac{\|g_f\|_1}{g_f(0)}=\frac{1}{n!\|f\|_1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\min\{f(x),f(y)\}dydx.$$

Equality case

There is equality if and only if for every $x \in \mathbb{R}^n$ and every $\lambda \in [0, 1]$

$$g_f(\lambda x) = g_f(0)^{1-\lambda} g_f(x)^{\lambda}.$$

Equality case

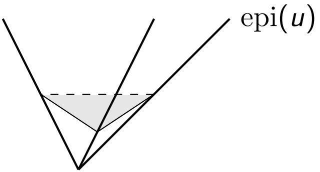
There is equality if and only if for every $x \in \mathbb{R}^n$ and every $\lambda \in [0,1]$

$$g_f(\lambda x) = g_f(0)^{1-\lambda} g_f(x)^{\lambda}.$$

Lemma

Let $f : \mathbb{R}^n \to [0,\infty)$ be an integrable log-concave function such that $\|f\|_\infty = f(0)$ and let g_f be its covariogram function. Then the following are equivalent:

- For every $x \in \mathbb{R}^n$ and every $\lambda \in [0,1]$ $g_f(\lambda x) = g_f(0)^{1-\lambda} g_f(x)^{\lambda}$
- $\frac{f(x)}{\|f\|_\infty} = e^{-\|x\|_{\Delta_n}}$ for some n-dimensional simplex containing the origin.



Another reverse Petty-projection inequality

Theorem (A., 2014)

For any convex body $K \subseteq \mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} |\Pi^* K|^{\frac{1}{n}} \geq \frac{1}{\mathrm{v.rat}(K)} |B_2^n|^{\frac{n-1}{n}} |\Pi^* B_2^n|^{\frac{1}{n}}.$$

$\mathrm{v.rat}(K) = \left(\frac{|K|}{|\mathcal{E}(K)|}\right)^{\frac{1}{n}}$, $\mathcal{E}(K)$ is the John's ellipsoid of K.

Another reverse Petty-projection inequality

Theorem (A., 2014)

For any convex body $K \subseteq \mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} |\Pi^* K|^{\frac{1}{n}} \geq \frac{1}{\mathrm{v.rat}(K)} |B_2^n|^{\frac{n-1}{n}} |\Pi^* B_2^n|^{\frac{1}{n}}.$$

$\mathrm{v.rat}(K) = \left(\frac{|K|}{|\mathcal{E}(K)|}\right)^{\frac{1}{n}}$, $\mathcal{E}(K)$ is the John's ellipsoid of K.

An ellipsoidal function is a function $\mathcal{E}^a(x) = a\chi_{\mathcal{E}}(x)$,with $\mathcal{E}$ an ellipsoid.

Another reverse Petty-projection inequality

Theorem (A., 2014)

For any convex body $K \subseteq \mathbb{R}^n$

$$|K|^{\frac{n-1}{n}}|\Pi^* K|^{\frac{1}{n}} \geq \frac{1}{\mathrm{v.rat}(K)}|B_2^n|^{\frac{n-1}{n}}|\Pi^* B_2^n|^{\frac{1}{n}}.$$

$\mathrm{v.rat}(K) = \left(\frac{|K|}{|\mathcal{E}(K)|}\right)^{\frac{1}{n}}$, $\mathcal{E}(K)$ is the John's ellipsoid of K.

An ellipsoidal function is a function $\mathcal{E}^a(x) = a\chi_{\mathcal{E}}(x)$,with $\mathcal{E}$ an ellipsoid.

Theorem (A., González, Jiménez, Villa, 2018)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable log-concave function. There exists a unique ellipsoidal function $\mathcal{E}(f)$ such that

- $\mathcal{E}(f) \leq f$
- $\int_{\mathbb{R}^n} \mathcal{E}(f)(x)dx = \max\left\{\int_{\mathbb{R}^n} \mathcal{E}^a(x)dx \ \mathcal{E}^a \leq f\right\}$

Another reverse affine Sobolev inequality

Theorem (A., González, Jiménez, Villa, 2018)

Let $f \in W^{1,1}(\mathbb{R}^n)$ be a log-concave function. Then

$$\frac{\|f\|_{\frac{n}{n-1}} |\Pi^*(f)|^{\frac{1}{n}}}{\left(\frac{|B_2^n|}{2|B_2^{n-1}|}\right)} \geq \frac{1}{e^{\frac{\int_{\mathbb{R}^n} f(x)\log\left(\frac{f(x)}{\|f\|_\infty}\right)dx}{n\int_{\mathbb{R}^n} f(x)dx}} \|f\|_\infty^{\frac{1}{n}} \left(\frac{\int_{\mathbb{R}^n} f(x)dx}{\int_{\mathbb{R}^n} f^{\frac{n}{n-1}}(x)dx}\right)^{\frac{n-1}{n}} \mathrm{I.rat}(f)}.$$

$$\mathrm{I.rat}(f) = \left(\frac{\|f\|_1}{\|\mathcal{E}(f)\|_1}\right)^{\frac{1}{n}}$$

Another reverse affine Sobolev inequality

Theorem (A., González, Jiménez, Villa, 2018)

Let $f \in W^{1,1}(\mathbb{R}^n)$ be a log-concave function. Then

$$\frac{\|f\|_{\frac{n}{n-1}} |\Pi^*(f)|^{\frac{1}{n}}}{\left(\frac{|B_2^n|}{2|B_2^{n-1}|}\right)} \geq \frac{1}{e^{\frac{\int_{\mathbb{R}^n} f(x)\log\left(\frac{f(x)}{\|f\|_\infty}\right)dx}{n\int_{\mathbb{R}^n} f(x)dx}} \|f\|_\infty^{\frac{1}{n}} \left(\frac{\int_{\mathbb{R}^n} f(x)dx}{\int_{\mathbb{R}^n} f^{\frac{n}{n-1}}(x)dx}\right)^{\frac{n-1}{n}} \mathrm{I.rat}(f)}.$$

$\mathrm{I.rat}(f) = \left(\frac{\|f\|_1}{\|\mathcal{E}(f)\|_1}\right)^{\frac{1}{n}}$

If $f = \chi_K$ for a convex body K then

$$|K|^{\frac{n-1}{n}} |\Pi^* K|^{\frac{1}{n}} \geq \frac{1}{\mathrm{v.rat}(K)} |B_2^n|^{\frac{n-1}{n}} |\Pi^* B_2^n|^{\frac{1}{n}}$$