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Logarithmic Minkowski problem and optimal transportation

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Minkowski problem

Given a probability measure μ on the unit sphere S^{n-1} find a convex body $K \subset \mathbb{R}^n$ such that μ is the push-forward image of the surface measure $\mathcal{H}^{n-1}|_{\partial K}$ under the **Gauss map**

$$x \to n_{\partial K}(x)$$

Analytically the problem is reduced to an equation of the **Monge-Ampère type**.

Variational solution:

The solution K minimizes the functional

$$L \to \int h_L d\mu, \ L \subset \mathbb{R}^n$$

under the constraint $Vol(L) = 1$. Here h_L is the **support functional** of L.

Uniqueness for the Minkowski problem: Brunn–Minkowski inequality

$A, B \subset \mathbb{R}^n$

$$Vol^{\frac{1}{n}}(A + B) \geq Vol^{\frac{1}{n}}(A) + Vol^{\frac{1}{n}}(B).$$

Equivalent form:

$$Vol(\lambda A + (1 - \lambda)B) \geq Vol^{\lambda}(A)Vol^{1-\lambda}(B), \ \forall \lambda \in [0, 1].$$

BM inequality implies uniqueness for the Minkowski problem

Logarithmic Minkowski problem

Given an (even) probability measure μ on the unit sphere S^{n-1} find a convex body $K \subset \mathbb{R}^n$ containing 0 such that μ is the push-forward measure of measure

$$m = \frac{1}{n}\langle x, n_{\partial K(x)}\rangle \mathcal{H}^{n-1}|_{\partial K}$$

under the **Gauss map**

$$x \to n_{\partial K}(x)$$

Geometrical meaning of m: m is the image of $Vol|_K$ under the mapping $x \to \frac{x}{\|x\|_K}$.

The push-forward of m under Gauss map is called the **cone measure** of K.

The corresponding **Monge-Ampère** equation (for probability measure $\mu = \rho_\mu \cdot \mathcal{H}^{n-1}|_{S^{n-1}}$)

$$\rho_\mu = \frac{1}{n} h \det D^2 h,$$

$D^2 h = h \cdot \mathrm{Id} + \nabla^2_{S^{n-1}} h.$

Why logarithmic?

Variational solution: Any minimizer of the functional

$$L \to \int \log h_L d\mu, \;\; L \subset \mathbb{R}^n$$

under the constraint $Vol(L) = 1$ solves the log-Minkowski problem.

Existence of solution (for even measures) : Böröczky K.J., Lutwak E., Yang D., Zhang G. The logarithmic Minkowski problem. J. Amer. Math. Soc., 26(3):831–852, 2013.

Uniqueness: open problem.

It is known that uniqueness follows from the conjectured **log-Brunn-Minkowski** inequality.

Log Brunn–Minkowski conjecture

$A, B \subset \mathbb{R}^n$ are **symmetric convex** bodies.

$$Vol(\lambda A +_0 (1-\lambda)B) \geq Vol^{\lambda}(A) Vol^{1-\lambda}(B), \ \forall \lambda \in [0,1].$$

Here $\lambda A +_0 (1-\lambda)B$ is the "logarithmic" Minkowski addition:

$$\lambda A +_0 (1-\lambda)B = \bigcap_{u \in \mathbb{R}^n} \left\{ x : \langle x, u \rangle \leq h_A^{\lambda}(u) h_B^{1-\lambda}(u) \right\}$$

It is the limiting case of "p-addition", $p \to 0+$

$$\lambda A +_p (1-\lambda)B = \bigcap_{u \in \mathbb{R}^n} \left\{ x : \langle x, u \rangle \leq \left[\lambda h_A^p(u) + (1-\lambda) h_B^p(u) \right]^{\frac{1}{p}} \right\}$$

Known results

- Böröczky–Lutwak–Yang–Zhang [2012] confirmed the conjecture in the plane $\mathbb{R}^2$.

- C. Saroglou [2015] verified the conjecture when $K_0, K_1 \subset \mathbb{R}^n$ are both simultaneously unconditional with respect to the same orthogonal basis, meaning that they are invariant under reflections with respect to the principle coordinate hyperplanes $x_i = 0$.

- L. Rotem [2014] : complex convex bodies

- A. Colesanti, G. Livshyts and A. Marsiglietti [2017] verified the conjecture locally for small enough C^2-perturbations of the Euclidean ball B_2^n.

- E. Milman and K. [2017] generalized this result to l^p-balls, $p > 2$ (dimension is large). Proved p–Minkowsky inequality locally for $p \geq 1 - \frac{C}{n^{3/2}}$.

- S. Chen, Y. Huang, Q.-R. Li, J. Liu (arXiv:1811.10181) Using PDE methods proved the corresponding **global** generalization of the result of K. and Milman.

Kantorovich problem

μ, ν are probability measures on $\mathbb{R}^n$, $\Pi(\mu,\nu)$ are probability measure on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals μ, ν

Quadratic transportation cost function / Kantorovich distance

$$W_2(\mu,\nu) = \left[\inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^2 d\pi\right]^{\frac{1}{2}}.$$

Kantorovich duality:

$$\frac{1}{2} W_2^2(\mu,\nu) = \sup_{\varphi(x)+\psi(y) \le \frac{1}{2}|x-y|^2} \left(\int \varphi d\mu + \int \psi d\nu\right)$$

Brenier theorem: Let $\pi \in \Pi(\mu,\nu)$ be the minimum point of

$$\pi \to \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^2 d\pi.$$

Then $\pi(\Gamma) = 1$, where

$$\Gamma = \{(x, x - \nabla\varphi(x))\} = \{(x, \nabla\Phi(x))\}.$$

In addition, $\Phi = \frac{1}{2}|x|^2 - \varphi$ is a convex function.

Optimal transportation

ν is the push-forward image of μ under the **optimal transportation** mapping

$$x \to \nabla\Phi(x)$$

The corresponding **Monge-Ampère** equation for $\mu = \rho_\mu dx, \nu = \rho_\nu dx$

$$\rho_\mu = \rho_\nu(\nabla\Phi) \det D^2\Phi$$

Kähler–Einstein equation

$$\varrho(\nabla\Phi)\,\det D^2\Phi = e^{-\Phi},$$

where

$$\nu = \varrho dx$$

is a probaility measure and Φ is a convex function. Assumption: $\int x d\varrho = 0$.

Well-posedness : **D. Cordero-Erausquin, B. Klartag, 2015**.

Approach: Φ is a maximum point of

$$J(f) = \log \int e^{-f^*} dx - \int f d\nu.$$

J is concave (Brunn-Minkowski inequality)

Another transportational functional for KE equation

F. Santambrogio, 2015 $\rho = e^{-\Phi}$ gives minimum to the functional

$$\mathcal{F}(\rho) = -\frac{1}{2}W_2^2(\nu, \rho dx) + \frac{1}{2}\int x^2 \rho \ dx + \int \rho \log \rho dx. \quad (1)$$

$\mathcal{F}$ is not convex, but **displacement convex**.

Gaussian version of this functional **(K., E. Kosov; [2017])**

$$\mathcal{F}_\gamma(\rho) = -\frac{1}{2}W_2^2(g\cdot\gamma, \rho\cdot\gamma) + \int \rho \log \rho \ d\gamma,$$

where

$$\gamma = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}|x|^2} dx,$$

$g\cdot\gamma, \rho\cdot\gamma$ are probability measures.

Spherical variational functional

Introduce the following Kantorovich functional on S^{n-1}:

$$K(\mu,\nu) = \min_{\pi \in \Pi(\mu,\nu)} \int_{(S^{n-1})^2} c(x,y)\, d\pi,$$

where

$$c(x,y) = \begin{cases} \log \frac{1}{\langle x,y \rangle}, & \langle x,y \rangle > 0 \\ +\infty, & \langle x,y \rangle \le 0. \end{cases}$$

Entropy functional (σ is the probablity uniform measure on S^{n-1}:)

$$Ent(m) = \begin{cases} \int \rho \log \rho\, d\sigma, & \text{if } m = \rho \cdot \sigma \\ +\infty, & \text{otherwise.} \end{cases}$$

Theorem (K. [2018]) The minimizers of the functional

$$F(\nu) = \frac{1}{n} Ent(\nu) - K(\mu,\nu), \qquad (2)$$

are solutions to the log-Minkowski problem for μ.

Displacement convexity

The strict displacement convexity of F on the space of measures would imply uniqueness of solution to the log-Minkowski problem.

Transportation inequalities

M. Talagrand (1999)

$$\frac{1}{2}W_2^2(\gamma, g\cdot\gamma) \le \mathrm{Ent}_\gamma(g),$$

whre $\gamma = \frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{|x|^2}{2}}dx$, $\mathrm{Ent}_\gamma(g) = \int g \log g d\gamma$.

M. Fathi (2018), strong transportation inequality

Assume that $f\cdot\gamma$ has zero mean. Then

$$\frac{1}{2}W_2^2(f\cdot\gamma, g\cdot\gamma) \le \mathrm{Ent}_\gamma(f) + \mathrm{Ent}_\gamma(g).$$

The proof of Fathi relies on the Kantorovich duality and the following result (functional Blaschke-Santaló inequality).

Theorem (S. Artstein, B. Klartag, V. Milman, 2006). Let $f(x) \geq 0, g(y) \geq 0$ satisfy

$$f(x)g(y) \leq e^{\langle x,y \rangle}.$$

Assume that $\int x f(x) = 0$. Then

$$\int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} g(y)dy \leq (2\pi)^n.$$

Strong transportation inequality for S^{n-1}

Let μ, ν — be even probability measures on S^{n-1}. Then

$$K(\mu,\nu) \le \frac{1}{n}\mathrm{Ent}(\mu) + \frac{1}{n}\mathrm{Ent}(\nu). \tag{3}$$

Proof:

Kantorovich-type duality for the functional K **(V. Oliker, 2007)**

$$K(\mu,\nu) = \sup_{h,r}\Big(\int \log r d\mu - \int \log h d\nu\Big),$$

where h, r are support and radial functional of a convex body Ω.

$$n \int \log r d\mu \le \int \Big(n \log r - \int r^n d\sigma\Big) d\mu + \log \int r^n d\sigma$$

By the Young inequality $xy \le e^x + y \log y - y$

$$\int \Big(n \log r - \int r^n d\sigma\Big) d\mu \le \mathrm{Ent}(\mu)$$

Apply the same arguments to $-\int \log h d\nu$.

Finally,

$$K(\mu,\nu) \leq \frac{1}{n}\mathrm{Ent}(\mu) + \frac{1}{n}\mathrm{Ent}(\nu) + \frac{1}{n}\log\Big(\int r^n d\sigma \int \frac{1}{h^n} d\sigma\Big)$$

Applying

$$\int_{\mathbb{S}^{n-1}} r^n d\mu = \frac{Vol(\Omega)}{Vol(B)},$$

we get

$$\int r^n d\sigma \int \frac{1}{h^n} d\sigma = \frac{Vol(\Omega)Vol(\Omega^\circ)}{Vol^2(B)},$$

where Ω° is the polar body to Ω. The result follows from the Blaschke-Santaló inequality

$$\frac{Vol(\Omega)Vol(\Omega^\circ)}{Vol^2(B)} \leq 1.$$