Limit theorems for Poisson-Delaunay tessellation

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joint work with

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July 5, 2019

Motivation

General set-up:

- let $X_0, \ldots, X_k$ be some random n-dimensional vectors;
- consider a random polytope

$$P_{n,k} := \mathrm{conv}(X_0, \ldots, X_k);$$

Questions

Investigate the probabilistic behaviour of the volume $\mathrm{vol}(P_{n,k})$ of the random polytope $P_{n,k}$ as k or/and n tend to infinity, e.g.

- Does this random variable fulfils a central limit theorem?
- Does this random variable fulfils a Berry-Esseen bound?
- etc.

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Central limit theorem and Berry-Esseen bound

Let N be a standard Gaussian random variable.

Definition

We say that a sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$ satisfying $\mathbb{E}\,|X_n|^2 < \infty$ for all $n \in \mathbb{N}$ fulfils a **central limit theorem** if

$$\frac{X_n - \mathbb{E}\,X_n}{\sqrt{\operatorname{Var} X_n}} \xrightarrow{d} N, \quad n \to \infty.$$

Definition

We say that a sequence of real-valued random variables $(X_n)_{n\in\mathbb{N}}$ satisfying $\mathbb{E}\,|X_n|^2 < \infty$ for all $n \in \mathbb{N}$ fulfils **Berry-Esseen bound** with speed $(\epsilon_n)_{n\in\mathbb{N}}$ if

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left(\frac{X_n - \mathbb{E}\,X_n}{\sqrt{\operatorname{Var} X_n}} \le t \right) - \mathbb{P}\left(N \le t\right) \right| \le c\,\epsilon_n,$$

where $c > 0$ is a constant not depending on n.

Example of results

- Ruben (1977): let $X_0, \dots, X_k$ be i.i.d. and distributed uniformly inside in the n-dimensional unit ball. Then the random variable $k! \, \mathrm{vol}(P_{n,k})$ fulfils central limit theorem as $n \to \infty$.

- Vu (2006): let $X_0, \dots, X_k$ be i.i.d. and distributed uniformly inside some smooth convex body K. Then the random variable $\mathrm{vol}(P_{n,k})$ fulfils Berry-Esseen bound and, hence, central limit theorem as $k \to \infty$.

- Grote, Kabluchko, Thäle (2019): let $X_0, \dots, X_k$, $k \leq n$ be i.i.d. and distributed according to one of the three models (Gaussian distribution, Beta distribution, uniform on unit sphere). Let $k = k(n) \leq n$ be some arbitrary sequence of integers. Then the random variable $\log(n! \, \mathrm{vol}(P_{n,k}))$ fulfils Berry-Esseen bound and, hence, central limit theorem.

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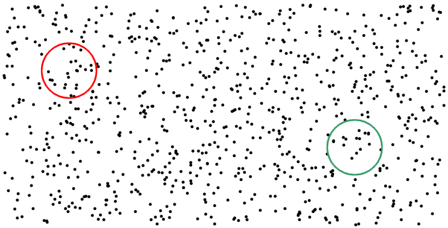
Stationary Poisson point process in $\mathbb{R}^n$

Definition

Stationary Poisson point process in $\mathbb{R}^n$ with intensity $\gamma \in (0, \infty)$ is a random counting measure η such that:

- for every Borel subset $A \in \mathbb{R}^n$ the distribution of $\eta(A)$ is Poisson with parameter $\gamma\lambda(A)$, where $\lambda(\cdot)$ is the Lebesgue measure;
- for every $m \in \mathbb{N}$ and pairwise disjoint Borel subsets $A_1, \ldots, A_m$ random variables $\eta(A_1), \ldots, \eta(A_m)$ are independent.



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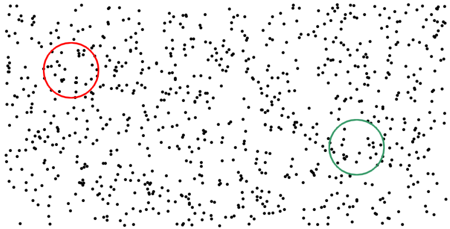

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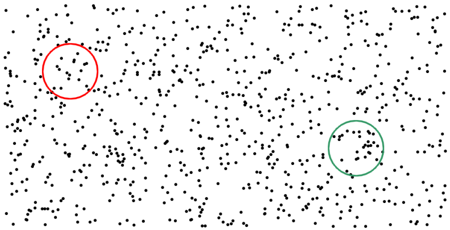

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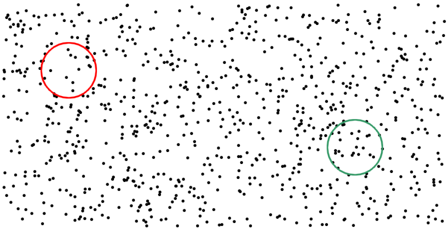

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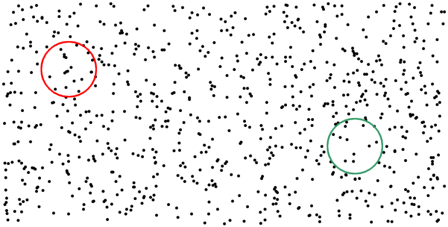
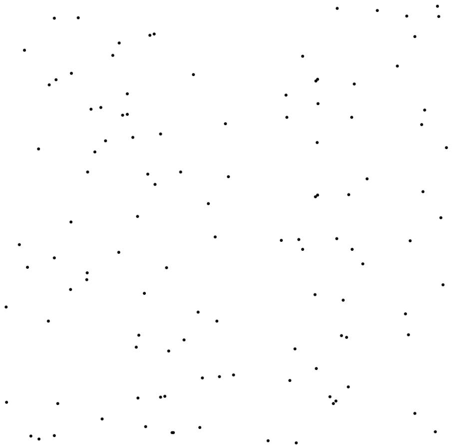


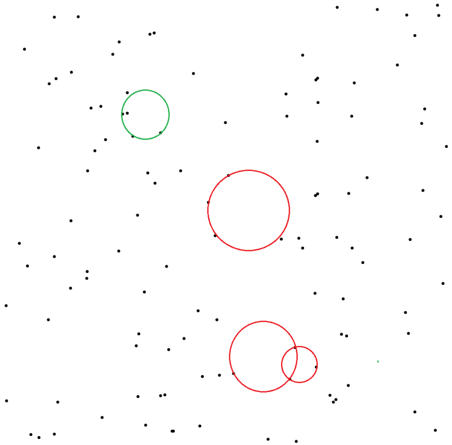
Poisson-Delaunay tessellation

Let η be a stationary Poisson point process in $\mathbb{R}^n$ with intensity $\gamma \in (0, \infty)$.



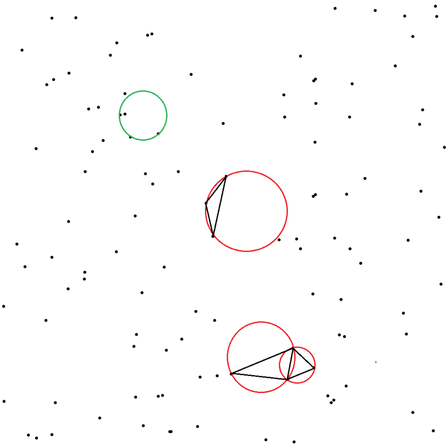
Poisson Delaunay Tesselation

For a $(n+1)$-tuple $(x_0, \dots, x_n)$ of distinct points of η we denote by $B(x_0, \dots, x_n)$ the almost surely uniquely determined ball having the points $x_0, \dots, x_n$ on its boundary.



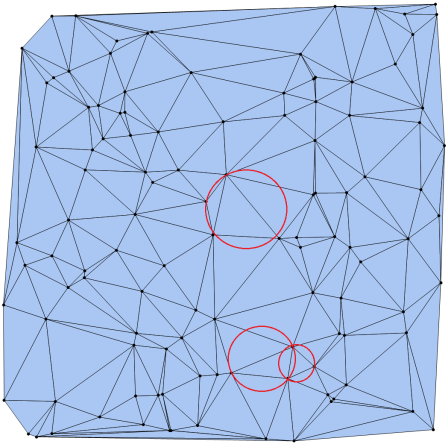
Poisson Delaunay Tesselation

The points $x_0, \ldots, x_n$ then form a Delaunay simplex $\mathrm{conv}(x_0, \ldots, x_n)$ whenever $B(x_0, \ldots, x_n) \cap \eta = \{x_0, \ldots, x_n\}$.



Poisson Delaunay tesselation

The collection $\mathscr{D}$ of all Delaunay simplices is the Poisson-Delaunay tessellation of $\mathbb{R}^n$.



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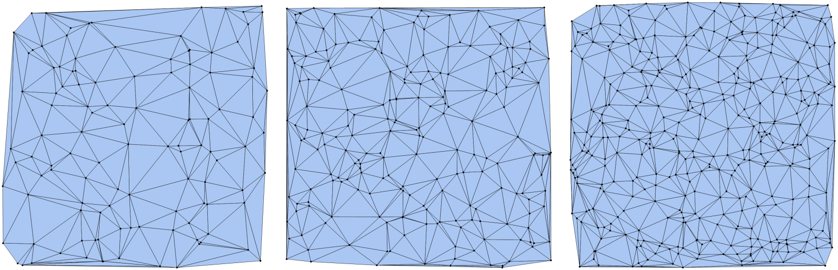

$\gamma = 0.3$ $\gamma = 0.5$ $\gamma = 1$

Weighted simplices in Poisson-Delaunay tessellation

- Consider a parameter $\mu \in (-2, \infty)$;
- Denote by $z(c)$ the midpoint of the circumsphere of a simplex c.
- Denote by Simpl_n the set of all simplices c in $\mathbb{R}^n$ with $z(c) = 0$.

Endowing Simpl_n with the usual Hausdorff distance, we can define on Simpl_n the Borel σ-field $\mathcal{B}(\mathrm{Simpl}_n)$. Then, we define a probability measure $\mathbb{P}_\mu^0$ as follows

$$\mathbb{P}_\mu^0(A) = \frac{1}{\gamma_\mu} \mathbb{E} \sum_{\substack{c \in \mathscr{D} \\ z(c) \in [0,1]^n}} \mathbf{1}\{c - z(c) \in A\} \operatorname{vol}(c)^{\mu+1}, \qquad A \in \mathcal{B}(\mathrm{Simpl}_n).$$

By $Z_{n,\mu}$ we denote a random simplex with distribution $\mathbb{P}_\mu^0$.

Interesting special cases: Z_{-1} is a **typical Delaunay simplex**; Z_0 is equal by distribution to the almost surely uniquely defined delaunay simplex, containing 0.

Aim

Investigate the probabilistic behaviour of the log-volume $Y_{n,\mu} := \log(\operatorname{vol}(Z_{n,\mu}))$ of the random simplex $Z_{n,\mu}$ as n or/and μ tend to infinity.

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Berry-Esseen bound and central limit theorem

We will consider the following cases (regimes):

- $n \to \infty$ and μ is fixed;
- $n \to \infty$ and $\mu = o(n)$;
- $n \to \infty$ and $\mu = \alpha n$ for some fixed $\alpha > 0$;
- $n \to \infty$ and $n - \mu = o(n)$;

Theorem

Suppose that n and μ are such that we are in one of the regimes described above. Then a sequence of random variables $(Y_{n,\mu})_{n\in\mathbb{N}}$ fulfils **Berry-Esseen bound** *with speed*

$$\epsilon_n = \frac{2}{(\mu+3)\sqrt{\log n}} : \mu = o(n) \text{ or } \mu \text{ is fixed},$$

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Asymptotic for mathematical expectation and variance

- $n \to \infty$ and μ is fixed:

$$\mathbb{E}\, Y_{n,\mu} = -\frac{n}{2}\log n + O(n); \qquad \mathrm{Var}\, Y_{n,\mu} = \frac{1}{2}\log n + O(1).$$

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- $n \to \infty$ and μ is fixed:

$$\mathbb{E}\, Y_{n,\mu} = -\frac{n}{2}\log n + O(n); \qquad \mathrm{Var}\, Y_{n,\mu} = \frac{1}{2}\log n + O(1).$$

- $n \to \infty$ and $\mu = o(n)$:

$$\mathbb{E}\, Y_{n,\mu} = -\frac{n}{2}\log n + o(n\log n); \qquad \mathrm{Var}\, Y_{n,\mu} = \frac{1}{2}\log n + o(\log n).$$

- $n \to \infty$ and $\mu = \alpha n$ for some fixed $\alpha > 0$:

$$\mathbb{E}\, Y_{n,\mu} = -\frac{n}{2}\log n + O(n); \qquad \mathrm{Var}\, Y_{n,\mu} = \frac{1}{2}\log\left(1+\frac{1}{\alpha}\right) - \frac{1}{2(1+\alpha)} + O\left(\frac{1}{n}\right).$$

- $n \to \infty$ and $n - \mu = o(n)$:

$$\mathbb{E}\, Y_{n,\mu} = -\frac{n}{2}\log n + O(n); \qquad \mathrm{Var}\, Y_{n,\mu} = \frac{1}{2}\log 2 - \frac{1}{4} + O\left(\frac{1}{n}\right).$$

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Probabilistic representation for the distribution of the volume of random simplex $Z_{n,\mu}$

Theorem

For any $\mu \in (-2, \infty)$ *we have*

$$\xi^n (1-\xi) \left[\gamma \kappa_n n! \operatorname{vol}(Z_{n,\mu})\right]^2 \stackrel{d}{=} \rho^2 \prod_{i=1}^{n} \xi_i,$$

where $\xi \sim \mathrm{Beta}\left(\frac{n^2+n+n\mu}{2}, \frac{\mu+2}{2}\right)$, $\xi_i \sim \mathrm{Beta}\left(\frac{i+\mu+1}{2}, \frac{n-i+1}{2}\right)$, $\rho \sim \mathrm{Gamma}(n+\mu+1, 1)$ *are independent random variables, independent of* $\operatorname{vol}(Z_{n,\mu})$.

Given a simplex $c \in \mathrm{Simpl}_n$ denote by $R(c)$ the radius of the circumsphere of c.

Lemma

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Probabilistic representation for the distribution of the volume of random simplex $Z_{n,\mu}$

Let $X_0, \dots, X_k$, $k \le n$ be i.i.d. and distributed uniformly on the unit sphere. Denote by

$$S_{n,k} := \mathrm{conv}(X_0, \dots, X_k)$$

and denote by $D_{n,k}$ the distance from the origin to the k-dimensional affine subspace spanned by $X_0, \dots, X_k$.

Corollary

For any integer $\mu \in (-2, \infty)$ we have

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- $\xi^n \stackrel{d}{=} D^{2n}_{n+\mu+2,n}$ (by Grote, Kabluchko, Thäle, 2019).

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Thank you for attention!