Norms of weighted sums of log-concave random vectors

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$$\|\mathbf{t}\|_{\mathcal{C},K} = \frac{1}{\prod_{j=1}^s \mathrm{vol}_n(C_j)} \int_{C_1} \cdots \int_{C_s} \Big\| \sum_{j=1}^s t_j x_j \Big\|_K \, dx_s \cdots dx_1,$$

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Question (V. Milman)

To examine if, in the case $C = K$, one has that $\|\cdot\|_{K^s,K}$ is equivalent to the standard Euclidean norm up to a term which is logarithmic in the dimension, and in particular, if under some cotype condition on the norm induced by K to $\mathbb{R}^n$ one has equivalence between $\|\cdot\|_{K^s,K}$ and the Euclidean norm.

Lower bounds

- Bourgain, Meyer, V. Milman and Pajor (80's) obtained the lower bound

$$\|\mathbf{t}\|_{\mathcal{C},K} \geqslant c\sqrt{s}\Big(\prod_{j=1}^{s}|t_j|\Big)^{1/s}\Big(\prod_{j=1}^{s}\mathrm{vol}_n(C_j)\Big)^{\frac{1}{sn}}/\mathrm{vol}_n(K)^{1/n},$$

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Gluskin-Milman

Let $A_1,\dots,A_s$ be measurable sets in $\mathbb{R}^n$ and K be a star body in $\mathbb{R}^n$ with $0\in \mathrm{int}(K)$. Then, for all $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$,

$$\|\mathbf{t}\|_{\mathcal{A},K} := \frac{1}{\prod_{j=1}^{s}\mathrm{vol}_n(A_j)}\int_{A_1}\cdots\int_{A_s}\Big\|\sum_{j=1}^{s}t_jx_j\Big\|_K dx_s\cdots dx_1 \geqslant c\Big(\sum_{j=1}^{s}t_j^2\Big(\frac{\mathrm{vol}_n(A_j)}{\mathrm{vol}_n(K)}\Big)^{2/n}\Big)^{1/2},$$

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where $c>0$ is an absolute constant. Equivalently, if $\mathrm{vol}_n(A_j)=\mathrm{vol}_n(K)$ for all $1\leqslant j\leqslant s$ then

$$\|\mathbf{t}\|_{\mathcal{A},K}\geqslant c\,\|\mathbf{t}\|_2$$

for all $\mathbf{t}\in\mathbb{R}^s$.

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- Using the BLL-inequality, write

$$\int_{A_1} \cdots \int_{A_s} \mathbf{1}_K\Big(\sum_{i=1}^s t_i x_i\Big)\, dx_s \cdots dx_1 = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathbf{1}_K\Big(\sum_{i=1}^s t_i x_i\Big) \prod_{i=1}^s \mathbf{1}_{A_i}(x_i)\, dx_s \cdots dx_1$$

$$\leqslant \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathbf{1}_{B_2^n}\Big(\sum_{i=1}^s t_i x_i\Big) \prod_{i=1}^s \mathbf{1}_{B_2^n}(x_i)\, dx_s \cdots dx_1 = \int_{B_2^n} \cdots \int_{B_2^n} \mathbf{1}_{B_2^n}\Big(\sum_{i=1}^s t_i x_i\Big)\, dx_s \cdots dx_1.$$

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- Next, use the observation that

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- It follows that

$$\|t\|_{A_i,K} = \int_{A_1} \cdots \int_{A_s} \Big\| \sum_{i=1}^s t_i x_i \Big\|_K \frac{dx_s \cdots dx_1}{\prod \mathrm{vol}_n(A_i)}$$

$$\geqslant \int_{B_2^n} \cdots \int_{B_2^n} \Big\| \sum_{i=1}^s t_i x_i \Big\|_{B_2^n} \frac{dx_s \cdots dx_1}{\mathrm{vol}_n(B_2^n)^s}.$$

Lower bounds

- To give a lower bound for this quantity, write

$$\int_{B_2^n}\cdots\int_{B_2^n}\Big\|\sum_{i=1}^s t_ix_i\Big\|_{B_2^n}\frac{dx_s\cdots dx_1}{\mathrm{vol}_n(B_2^n)^s}=\int_{B_2^n}\cdots\int_{B_2^n}\mathrm{Ave}_{\varepsilon_i=\pm1}\Big\|\sum_{i=1}^s\varepsilon_it_ix_i\Big\|_{B_2^n}\frac{dx_s\cdots dx_1}{\mathrm{vol}_n(B_2^n)^s}$$
$$\geqslant\frac{1}{\sqrt{2}}\int_{B_2^n}\cdots\int_{B_2^n}\Big(\sum_{i=1}^s t_i^2|x_i|^2\Big)^{1/2}\frac{dx_s\cdots dx_1}{\mathrm{vol}_n(B_2^n)^s},$$

using the unconditionality of B_2^n and Khinthcine inequality.

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$$\geqslant\frac{1}{\sqrt{2}}\int_{B_2^n}\cdots\int_{B_2^n}\Big(\sum_{i=1}^s t_i^2|x_i|^2\Big)^{1/2}\frac{dx_s\cdots dx_1}{\mathrm{vol}_n(B_2^n)^s},$$

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- To finish the proof one may use the inequality

$$\|f\|_2\leqslant\|f\|_1^{1/3}\|f\|_4^{2/3}$$

for the function $f(x)=\big(\sum_{i=1}^s t_i^2|x_i|^2\big)^{1/2}$ defined on $\mathbb{R}^{ns}$ to estimate the last integral and get the result with

$$c=\frac{1}{\sqrt{2}}\Big(\frac{n}{n+2}\Big)^{3/2}\sqrt{\frac{n+4}{n}}\to\frac{1}{\sqrt{2}}\quad\text{as } n\to\infty.$$

G.-Chasapis-Skarmogiannis

Let $\mathcal{C} = (C_1, \ldots, C_s)$ be an s-tuple of symmetric convex bodies and K be a symmetric convex body in $\mathbb{R}^n$ with $\mathrm{vol}_n(C_j) = \mathrm{vol}_n(K) = 1$. Then, for any $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$,

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Lower bounds: alternative proof

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An identity

Let $X_1, \ldots, X_s$ be independent random vectors, uniformly distributed on $C_1, \ldots, C_s$ respectively. Given $\mathbf{t} = (t_1 \ldots, t_s) \in \mathbb{R}^s$, we write $\nu_{\mathbf{t}}$ for the distribution of the random vector $t_1X_1 + \cdots + t_sX_s$. Then,

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$$\|\mathbf{t}\|_{\mathcal{C},K} = \int_{\mathbb{R}^n} \|x\|_K \, d\nu_{\mathbf{t}}(x).$$

- Note that $\nu_{\mathbf{t}}$ is an even log-concave probability measure on $\mathbb{R}^n$ We write $g_{\mathbf{t}}$ for the density of $\nu_{\mathbf{t}}$.

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- Let $\mathbf{t} \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$ and $t_1, \ldots, t_s \geqslant 0$. Then, if $X_1, \ldots, X_s$ are independent random vectors with densities $g_1, \ldots, g_s$, by an equivalent form of the Shannon-Stam inequality, we have that $h(t_1 X_1 + \cdots + t_s X_s) \geqslant \sum_{j=1}^s t_j^2 h(X_j)$.

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- Since the density $g_{\mathbf{t}}$ of $t_1X_1 + \cdots + t_sX_s$ is also log-concave, we may write
$$\sum_{j=1}^s t_j^2 \log(\|g_j\|_\infty^{-1}) \leqslant \sum_{j=1}^s t_j^2 h(X_j) \leqslant h(t_1X_1 + \cdots + t_sX_s) \leqslant n + \log(\|g_{\mathbf{t}}\|_\infty^{-1}),$$
which implies that $\|g_{\mathbf{t}}\|_\infty \leqslant e^n \prod_{j=1}^s \|g_j\|_\infty^{t_j^2}$.

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which implies that $\|g_{\mathbf{t}}\|_\infty \leqslant e^n \prod_{j=1}^s \|g_j\|_\infty^{t_j^2}$.
- In our case, $g_j = \mathbf{1}_{C_j}$, therefore $\|g_j\|_\infty = 1$ and the lemma follows.

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If $\|\mathbf{t}\|_2 = 1$ then $\|g_{\mathbf{t}}\|_\infty \leqslant e^n$.

Lemma 2

Let f be a bounded positive density of a probability measure μ on $\mathbb{R}^n$. For any symmetric convex body K in $\mathbb{R}^n$ and any $p > 0$ one has

$$\left(\frac{n}{n+p}\right)^{1/p} \leqslant \left(\int_{\mathbb{R}^n} \|x\|_K^p f(x)\, dx\right)^{1/p} \|f\|_\infty^{1/n} \mathrm{vol}_n(K)^{1/n}.$$

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- We apply Lemma 2 for the log-concave probability measure $\nu_{\mathbf{t}}$. For any $\mathbf{t} \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$ we have $\|g_{\mathbf{t}}\|_\infty = g_{\mathbf{t}}(0) \leqslant e^n$, therefore

$$\frac{n}{n+1} \leqslant e\,\mathrm{vol}_n(K)^{1/n} \int_{\mathbb{R}^n} \|x\|_K \, d\nu_{\mathbf{t}}(x) = e\,\mathrm{vol}_n(K)^{1/n} \, \|\mathbf{t}\|_{\mathcal{C},K}.$$

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- This shows that if $\mathcal{C} = (C_1, \ldots, C_s)$ is an s-tuple of symmetric convex bodies of volume 1 and K is a symmetric convex body in $\mathbb{R}^n$ then, for any $s \geqslant 1$ and any $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$

$$\|\mathbf{t}\|_{\mathcal{C},K} \geqslant \frac{n}{e(n+1)} \mathrm{vol}_n(K)^{-1/n} \, \|\mathbf{t}\|_2.$$

- A convex body C in $\mathbb{R}^n$ is called isotropic if it has volume 1, it is centered, i.e. its barycenter is at the origin, and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_C > 0$ such that

$$\|\langle \cdot, \xi\rangle\|_{L_2(C)}^2 := \int_C \langle x, \xi\rangle^2 dx = L_C^2$$

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- We shall use the fact that if C is isotropic then $R(C) \leqslant cnL_C$ for some absolute constant $c > 0$.
- The hyperplane conjecture asks if there exists an absolute constant $A > 0$ such that

$$L_n := \max\{L_C : C \text{ is isotropic in } \mathbb{R}^n\} \leqslant A$$

  for all $n \geqslant 1$.

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$$\|\langle \cdot, \xi\rangle\|_{L_2(C)}^2 := \int_C \langle x, \xi\rangle^2 dx = L_C^2$$

  for all $\xi \in S^{n-1}$.
- We shall use the fact that if C is isotropic then $R(C) \leqslant cnL_C$ for some absolute constant $c > 0$.
- The hyperplane conjecture asks if there exists an absolute constant $A > 0$ such that

$$L_n := \max\{L_C : C \text{ is isotropic in } \mathbb{R}^n\} \leqslant A$$

  for all $n \geqslant 1$.
- Bourgain proved that $L_n \leqslant c\sqrt[4]{n}\log n$; later, Klartag improved this bound to $L_n \leqslant c\sqrt[4]{n}$.

Log-concave measures

- A Borel measure μ on $\mathbb{R}^n$ is called log-concave if $\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$ for any compact subsets A and B of $\mathbb{R}^n$ and any $\lambda \in (0,1)$.

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- If μ is a log-concave measure on $\mathbb{R}^n$ with density f_{μ}, we define the isotropic constant of μ by

$$L_{\mu} := \left(\frac{\sup_{x\in\mathbb{R}^n} f_{\mu}(x)}{\int_{\mathbb{R}^n} f_{\mu}(x)dx}\right)^{\frac{1}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}},$$

where $\mathrm{Cov}(\mu)$ is the covariance matrix of μ with entries

$$\mathrm{Cov}(\mu)_{ij} := \frac{\int_{\mathbb{R}^n} x_i x_j f_{\mu}(x)\,dx}{\int_{\mathbb{R}^n} f_{\mu}(x)\,dx} - \frac{\int_{\mathbb{R}^n} x_i f_{\mu}(x)\,dx}{\int_{\mathbb{R}^n} f_{\mu}(x)\,dx}\frac{\int_{\mathbb{R}^n} x_j f_{\mu}(x)\,dx}{\int_{\mathbb{R}^n} f_{\mu}(x)\,dx}.$$

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- We say that a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if it is centered, i.e. if

$$\int_{\mathbb{R}^n}\langle x,\xi\rangle d\mu(x)=\int_{\mathbb{R}^n}\langle x,\xi\rangle f_{\mu}(x)dx=0$$

for all $\xi\in S^{n-1}$, and $\mathrm{Cov}(\mu)$ is the identity matrix.

- If C is a centered convex body of volume 1 in $\mathbb{R}^n$ then we say that a direction $\xi \in S^{n-1}$ is a ψ_α-direction (where $1 \leqslant \alpha \leqslant 2$) for C with constant $\varrho > 0$ if
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- From log-concavity it follows that every $\xi \in S^{n-1}$ is a ψ_1-direction for any C or μ with an absolute constant ϱ: there exists $\varrho > 0$ such that

$$\|\langle \cdot, \xi\rangle\|_{L_{\psi_1}(\mu)} \leqslant \varrho \|\langle \cdot, \xi\rangle\|_{L_2(\mu)}$$

  for all $n \geqslant 1$, all centered log-concave probability measures μ on $\mathbb{R}^n$ and all $\xi \in S^{n-1}$.

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$$\int_{\mathbb{R}^n}f_{\mathbf{t}}(x)x_ix_j\,dx=L_C^n\int_{\mathbb{R}^n}g_{\mathbf{t}}(L_Cx)x_ix_j\,dx=L_C^{-2}\int_{\mathbb{R}^n}g_{\mathbf{t}}(y)y_iy_j\,dy=\delta_{ij}$$

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$$L_{\mu_{\mathbf{t}}}=\|f_{\mathbf{t}}\|_\infty^{\frac{1}{n}}=L_C\|g_{\mathbf{t}}\|_\infty^{\frac{1}{n}}\leqslant eL_C$$

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- We also have

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- It follows that $\int_{O(n)} \|\mathbf{t}\|_{U(C)^s,K} \approx (L_C \sqrt{n} M(K))\, \|\mathbf{t}\|_2$.
- Therefore, our goal is to obtain a constant of the order of $L_C \sqrt{n} M(K)$ in our upper estimate for $\|\mathbf{t}\|_{C^s,K}$.

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- There, it is also shown that in the case where K is a ψ_2-body with constant ϱ one has

$$M(K_{\rm iso}) \leqslant \frac{c\sqrt[3]{\varrho}(\log n)^{1/3}}{\sqrt[6]{n} L_K}.$$

A simple upper bound

Let C be an isotropic convex body in $\mathbb{R}^n$ and K be a symmetric convex body in $\mathbb{R}^n$. If $R(K^\circ)$ is the radius of K° then, for any $s \geqslant 1$ and $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$,

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For the proof we use the identity $\|\mathbf{t}\|_{C^s,K} = \|\mathbf{t}\|_2 L_C \, I_1(\mu_{\mathbf{t}}, K)$ and the simple inequality $\|y\|_K \leqslant b\|y\|_2$, where $b = b(K) = R(K^\circ)$. Note that

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An application: If K is a symmetric convex body in $\mathbb{R}^n$ then the modulus of convexity of K is the function $\delta_K : (0,2] \to \mathbb{R}$ defined by

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- Examples of 2-convex bodies are given by the unit balls of subspaces of L_p-spaces, $1 < p \leqslant 2$; one can check that the definition is satisfied with $\alpha \approx p - 1$.

2-convex bodies

Let C be an isotropic convex body in $\mathbb{R}^n$ and K be an isotropic symmetric convex body in $\mathbb{R}^n$ which is also 2-convex with constant α. Then for any $s \geqslant 1$ and $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$,

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Let C be an isotropic convex body in $\mathbb{R}^n$ and K be an isotropic symmetric convex body in $\mathbb{R}^n$ which is also 2-convex with constant α. Then for any $s \geqslant 1$ and $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$,

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Proof: The first claim follows from the fact that $R(K^\circ) \leqslant c_2^{-1}/(\sqrt{\alpha}\sqrt{n})$.

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Proof: The first claim follows from the fact that $R(K^\circ) \leqslant c_2^{-1}/(\sqrt{\alpha}\sqrt{n})$.
For the second assertion we may assume that K is isotropic. Since $L_K \leqslant c_1/\sqrt{\alpha}$ we see that

$$\mathbb{E}_{K^s}\Big\| \sum_{j=1}^{s} t_j x_j \Big\|_K \leqslant \frac{c_2^{-1} L_K}{\sqrt{\alpha}} \|\mathbf{t}\|_2 \leqslant \frac{c_3}{\alpha} \|\mathbf{t}\|_2.$$

G.-Chasapis-Skarmogiannis

Let C be an isotropic convex body in $\mathbb{R}^n$ and K be a symmetric convex body in $\mathbb{R}^n$. Then,

$$\|\mathbf{t}\|_{C^s,K} \leqslant c\left(L_C \max\left\{\sqrt[4]{n}, \sqrt{\log(1+s)}\right\}\right)\sqrt{n}M(K)\|\mathbf{t}\|_2$$

for every $\mathbf{t}=(t_1,\dots,t_s)\in\mathbb{R}^s$, where $c>0$ is an absolute constant.

- Assume that $\|\mathbf{t}\|_2 = 1$.

A general upper bound

G.-Chasapis-Skarmogiannis

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for every $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$, where $c > 0$ is an absolute constant.

- Assume that $\|\mathbf{t}\|_2 = 1$. Our starting point will be again

$$\|\mathbf{t}\|_{C^s,K} = L_C\, I_1(\mu_{\mathbf{t}}, K),$$

so we try to give an upper bound for $I_1(\mu_{\mathbf{t}}, K)$.

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If μ is an isotropic log-concave probability measure on $\mathbb{R}^n$, then

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 \geqslant c_1 r\sqrt{n}\}) \leqslant e^{-r\sqrt{n}}$$

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Support

Since $R(C) \leqslant c_2 n L_C$ and $\mathrm{supp}(\nu_{\mathbf{t}}) \subseteq sC$, we have that

$$\mathrm{supp}(\mu_{\mathbf{t}}) \subseteq \frac{s}{L_C} C \subseteq (c_2 ns) B_2^n$$

for any $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$.

A general upper bound

- We fix $r \geqslant 1$ and set $C_{\mathbf{t}}(r) = \mathrm{supp}(\mu_{\mathbf{t}}) \cap c_1 r\sqrt{n}B_2^n$. We may write

$$\begin{aligned}\int_{\mathbb{R}^n} \|x\|_K \, d\mu_{\mathbf{t}}(x) &= \int_{C_{\mathbf{t}}(r)} \|x\|_K \, d\mu_{\mathbf{t}}(x) + \int_{\mathrm{supp}(\mu_{\mathbf{t}})\setminus C_{\mathbf{t}}(r)} \|x\|_K \, d\mu_{\mathbf{t}}(x) \\ &\leqslant \int_{C_{\mathbf{t}}(r)} \|x\|_K \, d\mu_{\mathbf{t}}(x) + b(K) \int_{\mathrm{supp}(\mu_{\mathbf{t}})\setminus C_{\mathbf{t}}(r)} \|x\|_2 d\mu_{\mathbf{t}}(x) \\ &\leqslant \int_{C_{\mathbf{t}}(r)} \|x\|_K \, d\mu_{\mathbf{t}}(x) + b(K)\,(c_2 ns)\, e^{-r\sqrt{n}}.\end{aligned}$$

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- For the first term, we consider the log-concave probability measure $\mu_{\mathbf{t},r}$ with density

$$\frac{1}{\mu_{\mathbf{t}}(C_{\mathbf{t}}(r))} \mathbf{1}_{C_{\mathbf{t}}(r)} f_{\mathbf{t}}$$

and the stochastic process $(w_y)_{y\in K^\circ}$ on $(\mathbb{R}^n, \mu_{\mathbf{t},r})$, where $w_y(x) = \langle x, y\rangle$.

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- We consider a standard Gaussian random vector G in $\mathbb{R}^n$, and for $y \in K^\circ$ set $h_y(G) = \langle G, y\rangle$. Note that

$$\mathbb{E}\left(\max_{y\in K^\circ} h_y(G)\right) = \mathbb{E}\,\|G\|_K \approx \sqrt{n}M(K).$$

A general upper bound

To bound $\mathbb{E}(\max_{y\in K^\circ} w_y)$, we will use Talagrand's comparison theorem.

Talagrand

If $(Y_t)_{t\in T}$ is a Gaussian process and $(X_t)_{t\in T}$ is a stochastic process such that

$$\|X_s - X_t\|_{\psi_2} \leqslant \alpha \, \|Y_s - Y_t\|_2$$

for some $\alpha > 0$ and every $s, t \in T$, then

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- It is easily checked that $\|h_y - h_z\|_2 = \|y - z\|_2$ for all $y, z \in K^\circ$. To bound the ψ_2 norm of $w_y - w_z$, we use the inequality $\|h\|_{\psi_2} \leqslant \sqrt{\|h\|_{\psi_1}\|h\|_\infty}$. Note that

  $$\|w_y - w_z\|_{L^\infty(\mu_{\mathbf{t},r})} \leqslant R(C_{\mathbf{t}}(r))\|y - z\|_2 \leqslant c_1 r\sqrt{n}\|y - z\|_2$$

  and we also have

  $$\|w_y - w_z\|_{L^{\psi_1}(\mu_{\mathbf{t},r})} \leqslant c_3\|w_y - w_z\|_{L^2(\mu_{\mathbf{t},r})} \leqslant 2c_3\|y - z\|_2$$

  for some absolute constant $c_3 > 0$ (here we also use the fact that $\mu(C_{\mathbf{t}}(r)) \geqslant 1 - e^{-r\sqrt{n}} \geqslant 1/2$). It follows that

  $$\|w_y - w_z\|_{L^{\psi_2}(\mu_{\mathbf{t},r})} \leqslant c_4\sqrt{r}\,\sqrt[4]{n}\,\|h_y - h_z\|_2.$$

A general upper bound

- Then,

$$\int_{C_{\mathbf{t}}(r)} \|x\|_K \, d\mu_{\mathbf{t}}(x) = \mu_{\mathbf{t}}(C_{\mathbf{t}}(r))\, \mathbb{E}_{\mu_{\mathbf{t},r}} \left(\max_{y \in K^\circ} w_y \right) \leqslant c_5 \sqrt{r}\, \sqrt[4]{n}\, \mathbb{E} \left(\max_{y \in K^\circ} h_y \right)$$

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- Finally,

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- Since $b(K) \leqslant c_6 \sqrt{n} M(K)$ we have that

$$b(K)\, ns\, e^{-r\sqrt{n}} \leqslant c_6 nse^{-r\sqrt{n}}\, \sqrt{n}M(K) \leqslant \sqrt{r}\, \sqrt[4]{n}\, \sqrt{n}M(K)$$

if we choose

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- Therefore,

$$\|\mathbf{t}\|_{C^s,K} = L_C\, I_1(\mu_{\mathbf{t}}, K) \leqslant \left(c_2' L_C \max\left\{1, \frac{\sqrt{\log(1+s)}}{\sqrt[4]{n}}\right\} \sqrt[4]{n} \right) \sqrt{n} M(K)$$

  as claimed.

- Adapting the proof of the previous theorem one can show that if C is assumed a ψ_2-body with constant ϱ, which means that every direction ξ is a ψ_2-direction for C with constant ϱ, then a much better estimate is available.

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ψ_2-case

Let C be an isotropic convex body in $\mathbb{R}^n$, which is a ψ_2-body with constant ϱ, and K be a symmetric convex body in $\mathbb{R}^n$. Then for any $s \geqslant 1$ and every $\mathbf{t} = (t_1, \ldots, t_s) \in \mathbb{R}^s$,

$$\|\mathbf{t}\|_{C^s,K} \leqslant c\varrho^2\sqrt{n}M(K)\,\|\mathbf{t}\|_2$$

where $c > 0$ is an absolute constant.

- Let K be a symmetric convex body in $\mathbb{R}^n$. Recall that if X_K is the normed space with unit ball K, we write $C_{2,k}(X_K)$ for the best constant $C > 0$ such that

$$\left(\mathbb{E}_\epsilon \Big\| \sum_{i=1}^{k} \epsilon_i x_i \Big\|_K^2\right)^{1/2} \geqslant \frac{1}{C}\left(\sum_{i=1}^{k} \|x_i\|_K^2\right)^{1/2}$$

for all $x_1, \ldots, x_k \in X$. Then, the cotype-2 constant of X_K is defined as $C_2(X_K) := \sup_k C_{2,k}(X_K)$.

Cotype-2 case

- Let K be a symmetric convex body in $\mathbb{R}^n$. Recall that if X_K is the normed space with unit ball K, we write $C_{2,k}(X_K)$ for the best constant $C>0$ such that

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  for all $x_1,\dots,x_k\in X$. Then, the cotype-2 constant of X_K is defined as $C_2(X_K):=\sup_k C_{2,k}(X_K)$.

- Replacing the ϵ_j's by independent standard Gaussian random variables g_j in the definition above, one may define the Gaussian cotype-2 constant $\alpha_2(X_K)$ of X_K. One can check that $\alpha_2(X_K)\leqslant C_2(X_K)$.

Cotype-2 case

E. Milman

If μ is a finite, compactly supported isotropic measure on $\mathbb{R}^n$ then, for any symmetric convex body K in $\mathbb{R}^n$,

$$I_1(\mu, K) \leqslant c_1\alpha_2(X_K)\sqrt{n}M(K) \leqslant c_1C_2(X_K)\sqrt{n}M(K).$$

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$$\mathbb{E}_{C^s}\Big\| \sum_{j=1}^{s} t_j x_j \Big\|_K \leqslant \left(c_1 L_C C_2(X_K)\sqrt{n}M(K)\right) \|\mathbf{t}\|_2$$

where $c_1 > 0$ is an absolute constant.

- For the proof we combine the identity

$$\|\mathbf{t}\|_{\mathcal{C},K} = \int_{\mathbb{R}^n} \|x\|_K d\nu_{\mathbf{t}}(x) = L_C\, I_1(\mu_{\mathbf{t}}, K)$$

with the bound $I_1(\mu_{\mathbf{t}}, K) \leqslant c_1 C_2(X_K)\sqrt{n}M(K)$ to get

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for all $\mathbf{t} \in \mathbb{R}^s$ with $\|\mathbf{t}\|_2 = 1$.

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- In particular, for any symmetric convex body K of volume 1 in $\mathbb{R}^n$ we have that

$$\mathbb{E}_{K^s}\Big\| \sum_{j=1}^{s} t_j x_j \Big\|_K \leqslant \left(c_2 L_K\, C_2(X_K)\sqrt{n}M(K_{\rm iso})\right) \|\mathbf{t}\|_2,$$

where $K_{\rm iso}$ is an isotropic image of K.

Unconditional case

There exists an absolute constant $c > 0$ with the following property: if K and $C_1, \dots, C_s$ are isotropic unconditional convex bodies in $\mathbb{R}^n$ then, for every $q \geqslant 1$,

$$\Big(\int_{C_1} \cdots \int_{C_s} \Big\| \sum_{j=1}^{s} t_j x_j \Big\|_K^q \, dx_1 \dots dx_s \Big)^{1/q} \leqslant c n^{1/q} \sqrt{q} \cdot \max\{\|\mathbf{t}\|_2, \sqrt{q}\|\mathbf{t}\|_\infty\} \leqslant c n^{1/q} q \, \|\mathbf{t}\|_2,$$

for every $\mathbf{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$. In particular,

$$\|\mathbf{t}\|_{\mathcal{C},K} \leqslant c\sqrt{\log n} \cdot \max\{\|\mathbf{t}\|_2, \sqrt{\log n}\|\mathbf{t}\|_\infty\} \leqslant c \log n \, \|\mathbf{t}\|_2.$$

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- This is essentially proved in [G.-Hartzoulaki-Tsolomitis].
- The proof makes use of the comparison theorem of Bobkov and Nazarov.

- Let us first assume that $1 \leqslant p \leqslant 2$. Then, ℓ_p^n has cotype-2 constant bounded by an absolute (independent from p and n) constant.

ℓ_p^n-balls

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- It is also known that $M(B_p^n) \approx n^{\frac{1}{p}-\frac{1}{2}}$ and $\mathrm{vol}_n(B_p^n)^{1/n} \approx n^{-\frac{1}{p}}$.

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$$M(\overline{B_p^n}) = \mathrm{vol}_n(B_p^n)^{1/n} M(B_p^n) \approx 1/\sqrt{n}.$$

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- It follows that
$$M(\overline{B_p^n}) = \mathrm{vol}_n(B_p^n)^{1/n} M(B_p^n) \approx 1/\sqrt{n}.$$
- Since $\overline{B_p^n}$ is isotropic and its isotropic constant is also bounded by an absolute constant, the general estimate for the cotype-2 case gives
$$\|\mathbf{t}\|_{\overline{B_p^n}^s,\overline{B_p^n}} \leqslant c_1 \|\mathbf{t}\|_2$$
for every $s \geqslant 1$ and $\mathbf{t} \in \mathbb{R}^s$, where $c_1 > 0$ is an absolute constant.

- Next, let us assume that $2 \leqslant q \leqslant \infty$. It is then known that $\mathrm{vol}_n(B_q^n)^{1/n} \approx n^{-\frac{1}{q}}$ and

$$M(B_q^n) \approx \min\{\sqrt{q}, \sqrt{\log n}\} n^{\frac{1}{q}-\frac{1}{2}}.$$

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- It follows that

$$M(\overline{B_q^n}) = \mathrm{vol}_n(B_q^n)^{1/n} M(B_q^n) \approx \min\{\sqrt{q}, \sqrt{\log n}\}/\sqrt{n}.$$

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- Since $\overline{B_q^n}$ is an isotropic ψ_2-convex body with constant $\varrho \approx 1$ (independent from q and n), and its isotropic constant is also bounded by an absolute constant, the general estimate for the ψ_2-case gives

$$\|\mathbf{t}\|_{\overline{B_q^n}^s, \overline{B_q^n}} \leqslant c_2 \min\{\sqrt{q}, \sqrt{\log n}\} \, \|\mathbf{t}\|_2$$

for every $s \geqslant 1$ and $\mathbf{t} \in \mathbb{R}^s$, where $c_2 > 0$ is an absolute constant.