Analyse fonctionnelle/Functional Analysis (Géométrie/Geometry)

Averages of norms and behavior of families of projective caps on the sphere *

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Abstract - We compute the number of summands in q-averages of norms needed to approximate an Euclidean ball. Surprisingly, for a large range of q-s, these numbers depend essentially only on the maximal ratio of the norm and the Euclidean norm. Some properties of a family of projective caps on the sphere are central in this study.

Moyennes de normes sur la sphère et comportement d'une famille de calottes projectives

Résumé - Nous calculons le nombre de termes d'une q-moyenne de normes qui sont nécéssaires pour approximer une norme euclidienne. Le résultat surprenant est l'existence d'un grand intervalle tel que si q appartienne à cet intervalle, ces nombres dépendent essentiellement seulement de la borne supérieure du rapport entre la norme donnée et la norme euclidienne. Certaines propriétés d'une famille de calottes projectives de la sphère sont centrales pour cette étude.

Version française abrégée - Soit $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$ un espace normé, muni d'une norme euclidienne, soit S^{n-1} une boule unité euclidienne munie de la mesure de probabilité invariante par rotation ν, et $G_{n,m}$ la grassmanienne des sous-espaces de dimension m de $\mathbb{R}^n$, munie de la mesure de Haar de probabilité μ determinée par la structure euclidienne dans $\mathbb{R}^n$. Soit a et b

*Note présentée par Michel Talagrand.

les meilleures constantes telles que $a^{-1}|x| \leq \|x\| \leq b|x|$ pour tout $x \in I\!\!R^n$. Soit

$$M_q = M_q(X) = \left(\int_{S^{n-1}} \|x\|^q d\nu(x) \right)^{1/q},$$

$k(X)$ le plus grand entier tel que

$$\mu\left(\left\{ E \in G_{n,m}\,;\ \frac{M_1}{2}|x| \leq \|x\| \leq 2M_1|x|, \quad \forall x \in E \right\} \right) \geq 1 - \frac{m}{n+m},$$

et $t_q(X)$ le plus petit t tel qu'il existe des opérateurs orthogonaux $u_1, ..., u_t$ tels que

$$\frac{1}{2}\, M_q\, |x| \leq \left(\frac{1}{t} \sum_{i=1}^{t} \|u_i x\|^q \right)^{1/q} \leq\ 2\, M_q\, |x| \quad \forall x \in I\!\!R^n.$$

On démontre le theorème:

Theorem.

(i) $t_q \approx t_1$, *for* $1 \leq q \leq 2$

(ii) $t_q^{2/q} \approx t_1 \approx \left(\frac{b}{M_1}\right)^2$, *for* $2 \leq q \leq k(X)$

(iii) $t_q^{2/q} \approx \frac{n}{q}$, *for* $k(X) \leq q \leq n$.

En outre, avec le bon choix de constantes (implicite dans la notation $\approx$), un choix aléatoire de transformations orthogonales $u_1, \ldots, u_{t_q}$ convient avec grande probabilité; alors un choix aléatoire de transformations orthogonales donne essentiellement le même résultat que le meilleur choix.

Nous étudions aussi le cas $q = \infty$ dans lequel la norme $\max_{1 \leq i \leq T} \|u_i^{-1} x\|$ correspond au corps $\cap_{i=1}^T u_i(K)$.

Outre que les techniques classiques de la théorie locale, nous utilisons dans la preuve certaines propriétés extrémales d'une famille de calottes projectives sur S^{n-1}. Nous cherchons un point commun pour une telle famille.

Comme application nous démontrons:

Application. *Soient* $\|\cdot\|_1, ..., \|\cdot\|_T$ *des normes. Alors*

$$\max_{S^{n-1}} \left(\|x\|_1 \cdot \|x\|_2 \cdot ... \cdot \|x\|_T \right) \geq \max_{S^{n-1}} \frac{\|x\|_1}{T} \cdot ... \cdot \max_{S^{n-1}} \frac{\|x\|_T}{T}.$$

Quelques résultats peuvent être généralisés au cas quasi-norme.

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1. Averages of norms.

Let $X = (I\!R^n, \|\cdot\|, |\cdot|)$ be a normed space equipped with the natural Euclidean norm $|\cdot|$, let S^{n-1} be the Euclidean sphere of radius one equipped with the probability rotation invariant measure ν and let $G_{n,m}$ be the Grassmanian of m-dimensional subspaces of $I\!R^n$ equipped with the Haar probability measure μ induced by ν. Let also a and b be the smallest constants such that $a^{-1}|x| \le \|x\| \le b|x|$ for every $x \in I\!R^n$ and let $0 < q < \infty$. We study in this paper the following parameters of the space X :

$$M_q = M_q(X) = \left(\int_{S^{n-1}} \|x\|^q d\nu(x) \right)^{1/q},$$

$k(X)$ - the maximal integer m for which

$$\mu\left(\left\{E \in G_{n,m}\,;\ \frac{M_1}{2}|x| \le \|x\| \le 2M_1|x|, \quad \forall x \in E\right\}\right) \ge 1 - \frac{m}{n+m},$$

and $t_q(X)$ - the smallest integer t such that there are orthogonal operators $u_1, ..., u_t$ such that

$$\frac{1}{2}\, M_q\, |x| \le \left(\frac{1}{t} \sum_{i=1}^{t} \|u_i x\|^q \right)^{1/q} \le 2\, M_q\, |x| \quad \forall x \in I\!R^n.$$

We write $f \approx g$ when $cf \le g \le Cf$ for some universal constants $0 < c, C < \infty$. The starting point of this paper is the following result from [6]:

$$k(X) \approx (M_1/b)^2 n, \quad t_1 \approx (b/M_1)^2.$$

(The facts $k(X) > cn(M_1/b)^2$ and $t_1 < C(b/M_1)^2$ were known before, see [5] and [1] correspondingly.)

Proposition 1.1.

(i) $M_q \approx M_1$, *for* $0 < q \le k(X)$

(ii) $M_q \approx b\sqrt{\frac{q}{n}}$, *for* $k(X) \le q \le n$

(iii) $M_q \approx b$, *for* $q > n$.

Theorem 1.2.

(i) $t_q \approx t_1$, *for* $1 \leq q \leq 2$

(ii) $t_q^{2/q} \approx t_1 \left(\frac{M_1}{M_q}\right)^2$, *for* $2 \leq q$.

Consequently,

(iii) $t_q^{2/q} \approx t_1 \approx \left(\frac{b}{M_1}\right)^2$, *for* $2 \leq q \leq k(X)$

(iv) $t_q^{2/q} \approx \frac{n}{q}$, *for* $k(X) \leq q \leq n$.

Moreover, with the appropriate choice of constants (implicit in the notation $\approx$*), a random choice of the orthogonal transformations* $u_1, \ldots, u_{t_q}$ *works with high probability. So, essentially, a random choice of orthogonal transformations gives the same result as the best choice.*

We now turn to the case $q = \infty$ in which case the norm $\max_{1 \leq i \leq T} \|u_i^{-1}x\|$ corresponds to the body $K_{\infty,T} = K_{\infty,T}(u_1, \ldots, u_T) = \cap_{i=1}^T u_i(K)$. Fix r with $b^{-1} < r < M_1^{-1}$ and let $T(r) = T(r, X)$ be the smallest T for which there are T orthogonal transformations $u_1, \ldots, u_T$ with $K_{\infty,T}(u_1, \ldots, u_T) \subseteq rD$.

In the following theorem M denotes either the median of the function $\|\cdot\|$ on the sphere S^{n-1} or its average (in which case $M = M_1$).

Theorem 1.3. *There are absolute constants* c_1 *and* c_2 *such that if* $b > 1/r > M$ *then*

$$c_1 \exp\left(\frac{n}{2}\frac{(1 - rM)^2}{(rb)^2}\right) \leq T(r) \leq c_2 n^{3/2} \log(1 + n)\left(1 - \frac{1}{(rb)^2}\right)^{-n/2}.$$

Corollary 1.4. *Under the non-degeneracy condition* $\frac{M}{b} > \sqrt{\frac{\log n}{n}}$, *for some universal constants* $0 < c < C < \infty$ *and for* r *in the interval* $[2b^{-1}, (2M_1)^{-1}]$,

$$\exp\left(\frac{cn}{r^2 b^2}\right) \leq T(r) \leq \exp\left(\frac{Cn}{r^2 b^2}\right).$$

Remarks. 1. The condition $M/b > \sqrt{\frac{\ln n}{n}}$ is necessary for this corollary. Indeed, let K be a strip $\{x \ ; \ |x_1| \leq 1\}$ (or a bounded approximation of the

strip). Then $M/b \approx 1/\sqrt{n}$ and we need at least n rotations to get a bounded K_∞.
2. It may be instructive to notice that, for any (fixed) C, the inequality in the previous corollary for r in the interval $[(CM_1)^{-1}, (2M_1)^{-1}]$ can be written as

$$(rM)^{-c_1 k(X)} \leq T(r) \leq (rM)^{-c_2 k(X)}.$$

The left hand side inequality continues to hold, also for r close (but smaller than) M^{-1} as long as we allow the constant c_1 to tend to zero when rM tends to 1; i.e., $T(r) \geq C(\theta)^{k(X)}$ if $1/r > (1+\theta)M$ where $C(\theta)$ depends only on θ and $C(\theta) \to 1^+$ as $\theta \to 0^+$.

2. Families of projective caps on S^{n-1}.

Besides classical techniques of Local Theory we are using in the proofs of the results from the previous section some extremal properties of families of projective caps on S^{n-1}.

Lemma 2.1. *Let $\{x_i\}_{i=1}^T$ be a set of vectors on S^{n-1}. Then there exists a $y \in S^{n-1}$ and a number $c_0 = \sqrt{2e^\gamma} < 2$ (where γ is the Euler constant) such that*

$$\left(\frac{1}{T}\sum_i |\langle y, x_i\rangle|^p\right)^{1/p} \geq \begin{cases} T^{-1/p} & \text{for } p \geq 2 \\ T^{-1/2} & \text{for } 1 \leq p < 2 \\ c_0^{-1} T^{-1/2} & \text{for } 0 < p < 1. \end{cases}$$

This lemma is the main new ingredient in the proof of the following theorem.

Theorem 2.2. *Let $u_1, ..., u_T$ be orthogonal operators on $\mathbb{R}^n$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and for some $q > 0$ put*

$$|||x||| = \left(\frac{1}{T}\sum_{i=1}^T \|u_i x\|^q\right)^{1/q}.$$

Assume $|||x||| \leq C|x|$ for every x in $\mathbb{R}^n$ and some constant C. Then

$$\|x\| \leq C(q)C|x| \cdot \begin{cases} T^{1/q} & \text{for } q \geq 2 \\ T^{1/2} & \text{for } q < 2, \end{cases}$$

where

$$C(q) = \begin{cases} 1 & \text{for } q \geq 1 \\ c_0 & \text{for } q < 1 \end{cases}$$

and c_0 is the same number as in the previous lemma.

Let us now see the pure geometric interpretation of Lemma 2.1 in the case $p = 1$.

Claim 2.3. *Denote*

$$A_i(t) \;=\; \left(tS^{n-1}\right) \cap \{x \mid p_i(x) \geq 1\}$$

and

$$A_i \;=\; A_i(1) \;=\; S^{n-1} \cap \{x \mid p_i(x) \geq 1\}, \quad 1 \leq i \leq T.$$

Then

(i) the projective caps $A_i(\lambda/b)$, $1 \leq i \leq T$, have a common point

(ii) for $b > 1$ at least

$$k \geq \frac{b\lambda - T}{b-1}$$

of the projective caps A_i have a common point.

Application. *Let $\|\cdot\|_1, ..., \|\cdot\|_T$ be norms. Then*

$$\max_{S^{n-1}} (\|x\|_1 \cdot \|x\|_2 \cdot ... \cdot \|x\|_T) \geq \max_{S^{n-1}} \frac{\|x\|_1}{T} \cdot ... \cdot \max_{S^{n-1}} \frac{\|x\|_T}{T}.$$

For $T > 3$ we have a better result, using Lemma 2.1 for $p \longrightarrow 0$.

Proposition 2.4. *Let $\|\cdot\|_1, ..., \|\cdot\|_T$ be norms on $\mathbb{R}^n$. Then*

$$\max_{S^{n-1}} (\|x\|_1 \cdot \|x\|_2 \cdot ... \cdot \|x\|_T) \geq \left(c_0\sqrt{\min\{T,n\}}\right)^{-T} \max_{S^{n-1}} \|x\|_1 \cdot \ldots \cdot \max_{S^{n-1}} \|x\|_T,$$

and c_0 is defined in Lemma 2.1.

3. A few results on quasi-norms.

Recall that a body K is said to be quasi-convex if there is a constant C such that $K + K \subset CK$, and given a $p \in (0,1)$, a body K is called p-convex if for any $\lambda, \mu > 0$ satisfying $\lambda^p + \mu^p = 1$ and any points $x, y \in K$ the point $\lambda x + \mu y$ belongs to K. Note that for the gauge $\|\cdot\| = \|\cdot\|_K$ associated with the quasi-convex (p-convex) body K the following inequality holds for all $x, y \in I\!R^n$

$$\|x+y\| \le C \max\{\|x\|, \|y\|\} \quad (\|x+y\|^p \le \|x\|^p + \|y\|^p \ )$$

and this gauge is called the quasi-norm (p-norm) if $K = -K$. In particular, every p-convex body K is also quasi-convex and $K + K \subset 2^{1/p}K$. A more delicate result is that for every quasi-convex body K (i.e. $K + K \subset CK$) there exists a q-convex body K_0 such that $K \subset K_0 \subset 2CK$, where $2^{1/q} = 2C$. This is the Aoki-Rolewicz theorem ([2], [8], see also [3], p.47).

Some of the previous results can be applied to the case of p-norms in a similar manner as they were applied for norms. However, the use of the p-triangle inequality, leads sometimes to gaps between the upper and the lower estimates. We still get some interesting versions of the convex case. The proofs are quite similar to the respective ones in the convex case. The real difference between the proofs here and those in the convex case is the use of a non-linear separation result, the following lemma, which can be viewed as a "non-linear" form of the Hahn-Banach theorem for p-convex sets.

Lemma 3.1. *Let $\|\cdot\|$ be a p-norm. Let $x_0 \in S^{n-1}$ be vector such that*

$$\|x_0\| = b = \max_{x \in S^{n-1}} \|x\|.$$

Then

$$\|x\| \ \ge \ \left(\frac{p}{2}\right)^{1/p} \cdot b \cdot \left(\frac{\langle x, x_0\rangle}{|x|}\right)^{-1+2/p} \cdot |x|$$

for any $x \in I\!R^n$.

This lemma allows us to extend Theorem 2.2.

Theorem 3.2. *Let $u_1, ..., u_T$ be orthogonal operators on $I\!R^n$. Let $\|\cdot\|$ be a p-norm on $I\!R^n$ and for some $q > 0$ put*

$$|||x||| = \left(\frac{1}{T}\sum_{i=1}^{T} \|u_i x\|^q\right)^{1/q}.$$

Assume $|||x||| \le C|x|$ for every x in $I\!R^n$ and some constant C. Then,

$$\|x\| \le C(p,q) C |x| \cdot \begin{cases} T^{1/q} & \text{for } q \ge \frac{2p}{2-p} \\ T^{1/p-1/2} & \text{for } q < \frac{2p}{2-p}, \end{cases}$$

where $C(p,q)$ depends on p and q only.

The standard concentration-phenomena methods ([7]) on the sphere imply that there exists a constant $C(p)$ depending on p only, such that for every p-convex space X we have $k(X) \geq C_1(p)n\,(M_1/b)^2$. Together with the theorem above this yields

Corollary 3.3. *Let $X = (I\!R^n, \|\cdot\|)$ be a p-normed space for some $p \in (0,1]$. Then there exists a constant $C(p)$, depending on p only, such that for $q \leq 2p/(2-p)$*

$$k(X) \cdot t_q^{(2-p)/p}(X) \;\geq c \cdot n \cdot C(p)$$

and for $q \geq 2p/(2-p)$

$$k(X) \cdot t_q^{2/q}(X) \;\geq C(p) \cdot n \cdot \left(\frac{M_1}{M_q}\right)^2 .$$

For the reverse inequality we get,

Corollary 3.4. *Let $X = (I\!R^n, \|\cdot\|)$ be a p-normed space for some $p \in (0,1]$. Let $q > 0$ and $\gamma = \frac{p}{2-p}$. Then for, $q > 2$,*

$$k(X) \cdot t_q^{2\gamma/q}(X) \leq c\,n \left(\frac{M_1}{M_q}\right)^{2\gamma}$$

and for $p \leq q \leq 2$

$$k(X) \cdot t_q^{\gamma}(X) \leq c\,n.$$

For the proofs see the manuscript [4].

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