Kahane-Khinchin's inequality for quasi-norms.

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Abstract

We extend the recent results of R. Latała and O. Guédon about equivalence of L_q-norms of logconcave random variables (Kahane-Khinchin's inequality) to the quasi-convex case. We construct examples of quasi-convex bodies $K_n \subset \mathbb{R}^n$ which demonstrate that this equivalence fails for uniformly distributed vector on K_n (recall that the uniformly distributed vector on a convex body is logconcave). Our examples also show the lack of the exponential decay of the "tail" volume (for convex bodies such decay was proved by M. Gromov and V. Milman).

1 Introduction

It turned out that many crucial results of the asymptotic theory of finite dimensional spaces hold also in the quasi-convex case. It is somewhat surprising since the first proofs of most of theorems substantially used convexity and duality. Because using convexity and duality in the quasi-convex setting lead to the weak results, extensions to this case demand development of the new methods. As an example of one of the difficulties arising in dealing with quasi-convex bodies let us mention that, contrary to the convex case, intersection of the p-convex body with affine subspace (or any convex set) is not necessarily p-convex set and even is not necessarily connected set. This is an obvious remark, but it can be important when one works with the logconcave measure which is known to concentrate on some affine subspace.

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In this note we extend the recent results of R. Latała ([La]) and O. Guédon ([Gu]) about equivalence of L_q-norms of logconcave random variables (Kahane-Khinchin's inequality) to the quasi-convex case. Both theorems seem to be an important result of the asymptotic theory. See e.g. [MP], where the particular case of the theorem was proved and used. The Latała's theorem as well as our extension of it was already used in [LMS].

Of course not every result of the theory admits an extension to the quasi-convex case. In the last section we provide examples which illustrate why certain results are not possible to extend.

Let us introduce several definitions.

Recall that a set K is said to be quasi-convex if there is a constant C such that $K + K \subset CK$, and given a $p \in (0, 1]$, a body K is called p-convex if for any $\lambda, \mu > 0$ satisfying $\lambda^p + \mu^p = 1$ and for any points $x, y \in K$ the point $\lambda x + \mu y$ belongs to K. Note that for the gauge $\|\cdot\| = \|\cdot\|_K$ associated with the quasi-convex (resp. p-convex) body K one has $\|x + y\| \leq C \max\{\|x\|, \|y\|\}$ (resp. $\|x + y\|^p \leq \|x\|^p + \|y\|^p$) for all $x, y \in I\!\!R^n$ and this gauge is called the quasi-norm (resp. p-norm) if K is a compact centrally-symmetric body. In particular, every p-convex body K is also quasi-convex and $K + K \subset 2^{1/p}K$. A more delicate result is that for every quasi-convex body K, with the gauge $\|\cdot\|_K$ satisfying $\|x + y\|_K \leq C (\|x\|_K + \|y\|_K)$, there exists a q-convex body K_0 such that $K \subset K_0 \subset 2CK$, where $2^{1/q} = 2C$. This is the Aoki-Rolewicz theorem ([KPR], [R], see also [Kön], p.47). For an additional properties of p-convex sets see [KPR].

Let us recall that the definition of the seminorm (quasi-seminorm, p-seminorm) can be obtained from the definition of the norm (quasi-norm, p-norm) by omitting the condition: $\|x\| = 0$ implies $x = 0$.

Given body $K \subset I\!\!R^n$ we denote its n-dimensional volume by $|K|$.

Below we consider Borel measures only. A Borel measure μ on $I\!\!R^n$ is called a logconcave measure if for every Borel subsets B, K of $I\!\!R^n$ and all $0 < \lambda < 1$

$$\mu_* (\lambda B + (1 - \lambda)K) \geq \mu(B)^\lambda \mu(K)^{1-\lambda},$$

where

$$\mu_* (B) = \sup \{\mu (K) \mid K \subset B,\ K \text{ is compact}\}$$

for every $B \subset I\!\!R^n$. We refer to [Bor1], [Bor2], [Pr] for basic properties of logconcave measures.

We say that a random vector Y with values in $\mathbb{R}^n$ is logconcave if the distribution of Y is logconcave.

For a random vector Y on $\mathbb{R}^n$ and a quasi-seminorm $\|\cdot\|$ on $\mathbb{R}^n$ we denote $\|Y\|_q = (\mathbf{E}\,\|Y\|^q)^{1/q}$ for non-zero q, and $\|Y\|_0 = \lim_{q\to 0}\|Y\|_q = \exp(\mathbf{E}\ln\|Y\|)$.

The Kahane-Khinchin's inequality says that for every $q, s \in (0,\infty)$ there exists a constant $C_{q,s}$, depending on q, s only, such that $\|Y\|_q \le C_{q,s}\|Y\|_s$ for every seminorm $\|\cdot\|$ and every logconcave vector Y on $\mathbb{R}^n$. Recently, R. Latała ([La]) demonstrated that the constant in this inequality can be taken independent on s, i.e. there exists a constant C_q, depending on q only, such that $\|Y\|_q \le C_q\|Y\|_0$. Let us mention that for the Steinhaus random vector such equivalence was proved by Ullrich ([U]). Furthermore, using a different method, O. Guédon ([Gu]) has extended Latała's result to the negative exponent: $\|Y\|_1 \le C_q^{'}\|Y\|_q$ for every $q \in (-1,0]$, where $C_q^{'} = \frac{4e}{1+q}$. His paper helped us to realize that an extension to the negative exponent can be done also using Latała's method. We adapt Latała's methods to prove both inequalities for p-seminorm and for $q \in (-p,\infty)$.

Theorem 1.1 *Let $p \in (0,1]$ and $q_1 \ge 0 \ge q > -p$. Let $Y_1,\ldots,Y_k$ be independent logconcave random vectors on $\mathbb{R}^n$. Let $\|\cdot\|$ be p-seminorm on $\mathbb{R}^n$. Then*

$$\left\|\sum_{i=1}^k Y_i\right\|_{q_1} \le \max\{1,q_1\}\cdot C(p,q)\cdot C_p\cdot\left\|\sum_{i=1}^k Y_i\right\|_q,$$

where $C(p,q)=1$ for $q \ge -p/2$, $C(p,q)=(p+q)^{1/q}$ for $q < -p/2$, $C_p = (2/p)^{c/p}$ with an absolute constant c.

Applying this theorem to the uniformly distributed on a convex body K vector, i.e. for the vector Y with $\mathbf{Pr}(Y \in B) = |B\cap K|/|K|$, we get

Corollary 1.2 *Let $\|\cdot\|$ be a p-seminorm on $\mathbb{R}^n$. Let K be a convex body with non-zero volume $|K|$. Then for every $q_1 \ge 1$, $q \in (-p,0)$*

$$\frac{1}{q_1\cdot C_p}\left(\int_K \|x\|^{q_1}d\mu(x)\right)^{1/q_1} \le \int_K \|x\|d\mu(x) \le C_pC(p,q)\left(\int_K \|x\|^q d\mu(x)\right)^{1/q},$$

where $d\mu(x) = dx/|K|$, $C(p,q)$ and C_p as in Theorem 1.1.

Let us note, that for $\|\cdot\|$ being a linear functional and $q = 0$ the right hand side inequality was proved by V. Milman and A. Pajor ([MP]). In the last section we construct examples showing that the condition of convexity in this Corollary can not be replaced by the condition of p-convexity even in the case of linear functional. These examples also show the lack of the exponential decay of the "tail" volume (for convex bodies such decay was proved by M. Gromov and V. Milman in [GrM] (see also [Bou], [MS])).

2 Proof of the extension of Latała's theorem

Let C be some constant. We will use the term *C-quasi-seminorm* for the quasi-seminorm $\|\cdot\|$ if C is the constant of quasi-convexity of $\|\cdot\|$, i.e. $\|x+y\| \leq C(\|x\|+\|y\|)$ for every x and y. Analogously, a body K is said to be *C-quasi-convex* if $K + K \subset CK$.

Given body $B \subset I\!R^n$ we denote $I\!R^n \setminus B$ by B^c.

We follow Latała's scheme of the proof. First we prove a straightforward extension to the quasi-convex case of Borel's lemma ([Bor1], see also [MS], App. 3).

Lemma 2.1 *Let $t, C > 0$. Let μ be a logconcave probability measure. Let B be a C-quasi-convex symmetric Borel set. Then for every $\lambda \geq C$*

$$\mu\left((\lambda B)^c\right) \leq \mu(tB)\left(\frac{1-\mu(B)}{\mu(tB)}\right)^{\frac{\lambda+tC}{(1+t)C}}.$$

Proof: Set $\alpha = C\frac{1+t}{\lambda+tC} \leq 1$. Then by C-quasi-convexity of B one has $B - (1-\alpha)tB \subset C\left(1+t\left(1-\alpha\right)\right)B = \alpha\lambda B$. That means $\left(1-\alpha\right)tB + \alpha(\lambda B)^c \subset B^c$. Using logconcavity of μ we get $1-\mu\left(B\right) = \mu\left((B)^c\right) \geq \mu\left((\lambda B)^c\right)^{\alpha}\mu\left(tB\right)^{1-\alpha}$, which implies the lemma. □

The following lemma is the crucial step in the proof of Theorem 1.1. To prove it we adapt ideas of [La] to the p-convex case.

Lemma 2.2 *Let μ be a logconcave probability measure. Let B be a Borel p-convex, symmetric set such that $\mu(mB) \geq (1+\delta)\mu(B)$ for some $m > 1$ and $\delta > 0$. Then for every $\varepsilon \in (0,1)$*

$$\mu(\varepsilon B) \leq f(m^p/\delta)\cdot\varepsilon^p\cdot\mu(B),$$

where $f(x) = \max\left\{\frac{32}{p}\ln(x/2); \frac{64}{p}\ln(16/p)\right\}$.

Remark. In the case $p = 1$ one can take $f(x) = \max\{16\ln(x/2); 50\}$.

Proof of the lemma: Let us note first that when lemma is proved for some value δ_0 it automatically holds for all $\delta > \delta_0$. Therefore, it is enough to prove the lemma with fixed $m > 1$ and $\delta \leq \delta_0 := (8m^p/p) \cdot \ln(16/p)$.

Denote

$$z = \frac{2\delta}{m^p} \qquad \text{and} \qquad \alpha = \frac{\mu(\varepsilon B)}{\varepsilon^p \mu(B)}.$$

If $\alpha \leq 4z$ then $\mu(\varepsilon B) \leq 4z\varepsilon^p\mu(B) \leq 8(\delta_0/m^p)\,\varepsilon^p\mu(B) \leq f(m^p/\delta)\,\varepsilon^p\mu(B)$ and we are done. In the case $\varepsilon^p > p4^{p-1}/(2+p)$ one has $f(m^p/\delta)\varepsilon^p \geq 1$ and assertion of the lemma follows.

Therefore we may assume

$$\varepsilon^p \leq \frac{p4^p}{4(2+p)} \qquad \text{and} \qquad \alpha > 4z. \tag{1}$$

For $u > 0$, $w > 0$ denote $B(u;w) = \{x \mid u^p - w^p < \|x\|^p < u^p + w^p\}$, where $\|\cdot\|$ is the gauge of B.

Given ε satisfying (1) there is $A > 1$ such that

$$\mu(B(A;\varepsilon)) \geq z\varepsilon^p\mu(B). \tag{2}$$

Indeed, let

$$l = \left[\frac{m^p}{2\varepsilon^p}\right] \qquad \text{and} \qquad u_j^p = 1 + \frac{2j+1}{2l}(m^p - 1).$$

Clearly, $\varepsilon^p < 1/2$. So for every point $x \in ((mB) \setminus B)$ there is $0 \leq j \leq l-1$ such that $x \in B(u_j;\varepsilon)$, i.e. $\bigcup_{j=0}^{l-1} B(u_j;\varepsilon) \supset ((mB) \setminus B)$. Thus

$$\sum_{j=0}^{l-1} \mu(B(u_j;\varepsilon)) \geq \mu((mB) \setminus B) \geq \delta\mu(B)$$

and, by definitions of z and l, $\delta/l \geq z\varepsilon^p$ from which (2) follows.

Denote $\gamma = 2^{-1+1/p}$. Since for every $\lambda \in [0,1]$, $u > 0$, $w > 0$

$$\lambda B(u;w) + (1-\lambda)\,wB \subset B(\lambda u;\gamma w),$$

we get

$$(3) \qquad \mu\left(B(\lambda u;\gamma\varepsilon)\right) \geq \mu\left(B(u;\varepsilon)\right)^{\lambda}\mu\left(\varepsilon B\right)^{1-\lambda}.$$

Using (1)-(3) we obtain $\mu\left(B(1;\gamma\varepsilon)\right) \geq \mu\left(B(A;\varepsilon)\right)^{1/A}\mu\left(\varepsilon B\right)^{1-1/A} \geq z\varepsilon^p\mu\left(B\right)$. Choose

$$w=\gamma^2\varepsilon=4^{-1+1/p}\varepsilon, \quad v=\frac{2w^p\left(1-w^p\right)^{-1+1/p}}{p} \quad \text{and} \quad l=\left[\frac{p\left(1-w^p\right)}{2w^p}\right]\geq 1.$$

Denote $B_j=B(jv,w)$, $1\leq j\leq l$, and $\beta=(z/\alpha)^v$. Then the sets B_j are mutually disjoint and $B_j\subset B\setminus\varepsilon B$.

Since by (3): $\mu\left(B_j\right) \geq \mu\left(B(1;\gamma\varepsilon)\right)^{jv}\mu\left(\varepsilon B\right)^{1-jv} \geq \alpha\varepsilon^p\beta^j\mu\left(B\right)$, we obtain

$$\mu\left(B\right) \geq \sum_{j=1}^{l}\mu\left(B_j\right)+\mu\left(\varepsilon B\right) \geq \alpha\varepsilon^p\mu\left(B\right)\sum_{j=0}^{l}\beta^j.$$

Using (1) we get

$$\alpha \leq \varepsilon^{-p}\frac{1-\beta}{1-\beta^{1+l}} \leq 2\varepsilon^{-p}v\ln\frac{\alpha}{z} \leq \frac{16}{p4^p}\ln\frac{\alpha}{z}.$$

Hence $\alpha \leq \frac{32}{p}\ln\left(\max\left\{(16/p)^2;1/z\right\}\right)$, which concludes the proof. $\Box$

The following theorem follows from the lemma in a way similar to that in the convex case (cf. [La]).

Theorem 2.3 *Let μ be a logconcave probability measure. Let B be a Borel p-convex symmetric set such that $\mu(B)<1$. Then there is an absolute constant c such that for every $t\in[0,1]$*

$$\mu(tB) \leq \frac{c}{p}\cdot\ln\left(2/p\right)\cdot t^p\cdot\left(1-\ln\mu(B^c)\right)^p\cdot\mu(B).$$

Proof: Denote $\gamma:=\mu\left(B\right)$. The assertion of the corollary is obviously true for $\gamma=0$. Let us consider the case $\gamma\in\left(0,2-\sqrt{2}\right)$.

By Borel's theorem ([Bor1], [Bor2]) every logconcave probability measure on $I\!R^n$ is concentrated on some k-dimensional affine subspace E of $I\!R^n$. Moreover, on this subspace it is absolutely continuous with respect to the corresponding k-dimensional Lebesgue measure on E. So, for every Borel p-convex set B with $\mu(B) > 0$ we have $\sup_t \mu(tB) = 1$. Therefore we can choose $m \geq 1$ such that

$$\mu(mB) \leq 2\frac{1-\gamma}{2-\gamma} \quad \text{and} \quad \mu(2mB) > 2\frac{1-\gamma}{2-\gamma}.$$

By Lemma 2.1 and p-convexity of B, for every $\lambda \geq C = 2^{1/p}$

$$\mu\left((\lambda mB)^c\right) \leq \mu(2mB)\left(\frac{1-\mu(mB)}{\mu(2mB)}\right)^{(\lambda+2C)/(3C)} \leq \left(1-\frac{\gamma}{2}\right)^k,$$

where $k = \lambda/(3C)$. Choose $\lambda = 3C\left(\ln\frac{1-\gamma}{2}\right)/\left(\ln(1-\gamma/2)\right) < 6C\frac{\ln 6}{\gamma}$, then

$$\mu(\lambda mB) \geq \left(1-(1-\gamma/2)^k\right)\mu(mB)/\gamma \geq \left(1+\frac{1-\gamma}{2\gamma}\right)\mu(mB).$$

Let f be the function defined in Lemma 2.2 and $A := f\left(2\lambda^p\gamma/(1-\gamma)\right) = \frac{64}{p}\ln(32/p)$. Then Lemma 2.2 implies $\mu(tmB) \leq At^p\mu(mB)$ for every $t \leq 1$.

Therefore, if $\mu(mB) < 2\mu(B)$ then $\mu(tB) \leq \mu(tmB) \leq 2At^p\mu(B)$ for every $t \in (0,1)$. If $\mu(mB) \geq 2\mu(B)$ then

$$m^p \leq A\frac{\mu(mB)}{\mu(B)} \leq 2A\left(\frac{\mu(mB)}{\mu(B)} - 1\right).$$

Using Lemma 2.2 again, we obtain $\mu(tB) \leq f(2A)\cdot t^p \cdot \mu(B) \leq A \cdot t^p \cdot \mu(B)$. That proves the corollary for $\gamma \leq 2-\sqrt{2}$.

In the case $\gamma \geq 2-\sqrt{2}$, by Lemma 2.1, we have for $C = 2^{1/p}$

$$0 < 1-\gamma = \mu\left(\lambda(B/\lambda)^c\right) \leq \left(\frac{1-\mu(B/\lambda)}{\mu(B/\lambda)}\right)^{\lambda/(2C)}.$$

Thus for $\lambda = 2C\log_2(1/(1-\gamma))$ we get $\mu(B/\lambda) \leq 2/3$ and, by above, $\mu(tB) \leq A \cdot t^p \cdot \lambda^p \cdot \mu(B/\lambda)$, which completes the proof. □

Proof of Theorem 1.1: Since the convolution of logconcave measures is also logconcave ([Bor1], [Bor2], [Pr], see also [DKH] where corresponding result was proved for logconcave functions), it is enough to prove the theorem for the case $k = 1$. Let $Y = Y_1$ and μ is distribution of Y. Let B be the unit ball of $\|\cdot\|$. If $\mu(\{x \mid \|x\| = 0\}) = 1$ there is nothing to prove. Otherwise, by Lemma 2.1, $\mu(\{x \mid \|x\| = 0\}) = 0$. Therefore we can choose m such that $\mu(mB) \le 2/3$ and $\mu(2mB) > 2/3$.

Then, by Theorem 2.3, $\mu(smB) \le C_p s^p$ for $s \in (0,1)$ and $C_p = \frac{c_0}{p}\ln(2/p)$ with an absolute constant c_0. Thus for $q \in (-p, 0)$ one has

$$\mathbf{E}\,\|Y\|^q = -qm^q \int_0^\infty s^{q-1}\mu\left(\|Y\| < ms\right) ds \le m^q \left(-C_p \frac{q}{q+p} + 1\right).$$

Therefore there is an absolute constant c such that

$$\|Y\|_q \ge \begin{cases} (p/2)^{c/p}\, m & \text{for } q \in [-p/2, 0), \\ (p/2)^{c/p}\,(p+q)^{-1/q}\, m & \text{for } q \in (-p, -p/2). \end{cases}$$

On the other hand for $q > 0$: $\mathbf{E}\,\|Y\|^q = q\,(2m)^q \int_0^\infty s^{q-1}\mu\left((s2mB)^c\right) ds$. Since $\mu(2mB) \ge 2/3$, by Lemma 2.1 and p-convexity of $\|\cdot\|$ we obtain

$$\|Y\|_q^q \le q\,(2m)^q \left(\int_0^C s^{q-1} ds + \int_C^\infty s^{q-1} 2^{-\frac{s}{2C}} ds\right) \le 2^q C^q m^q \left(1 + q 2^q \Gamma(q)\right),$$

where $C = 2^{-1+1/p}$ and $\Gamma(\cdot)$ is the Gamma function. Thus $\|Y\|_q \le c\,C q\, m$ for $q \ge 1$ and $\|Y\|_q \le \|Y\|_1 \le c\,C\,m$ for $q < 1$, where c is an absolute constant. That proves the theorem. □

We end this section with another corollary of Theorem 1.1.

Corollary 2.4 *Let $\|\cdot\|$ be a p-seminorm on $\mathbb{R}^n$. Let C_p be as in the theorem. Denote the Euclidean sphere on $\mathbb{R}^n$ by S^{n-1} and the rotation invariant normalized measure on S^{n-1} by ν. Denote*

$$M_q = \left(\int_{S^{n-1}} \|x\|^q d\nu(x)\right)^{1/q} \quad \text{and} \quad M_0 = \exp\left(\int_{S^{n-1}} \ln\|x\| d\nu(x)\right).$$

Then for every $q_1 \ge 2$, $q \in (-p, -p/2)$

$$\frac{M_{q_1}}{q_1 \cdot C_p} \le M_2 \le C_p \cdot M_{-p/2} \le C_p\,(p+q)^{1/q}\, M_q.$$

This corollary immediately follows from Corollary 1.2 and integration over the Euclidean ball.

Let us note that for seminorms and $q \ge 1$ inequality $M_q \le c\sqrt{q} M_1$ is known and, moreover, $M_q/M_1 \approx \max\{1,\ b\sqrt{q/n}\}$ for $b = \max_{S^{n-1}} \|x\|$ and $q \in [1, n]$ (see Statement 3.1 of [LMS]).

3 Decay of “tail” volume

By “tail” volume of a body we mean the volume of difference of the body and some symmetric strip.

In the eighties M. Gromov and V. Milman investigated the law of decay of the “tail” volume when the width of the strip grows up. They proved the exponential decay of the “tail” volume ([GrM], see also [Bou], [MS]). For every $p \in (0,1)$ we construct examples of p-convex bodies without the exponential decay of the “tail” volume. Moreover, our examples show lack of any decay of the “tail” volume that is independent of the dimension. Thus, result of M. Gromov and V. Milman can not be extended to p-convex bodies in any sense. Our examples show also that the condition of the convexity in Corollary 1.2 is essential.

We need more definitions and notations.

Given set $K \subset I\!\!R^n$, the p-convex hull of K, p-conv K, is the intersection of all p-convex sets containing K.

It was shown by J.Bastero, J.Bernués, and A.Peña ([BBP]) that

$$\begin{aligned} p\text{-conv } K &= \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid m \in I\!\!N, x_i \in K, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i^p = 1 \right\} \\ &= \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid m \in I\!\!N, x_i \in K, \lambda_i \geq 0, 0 < \sum_{i=1}^{m} \lambda_i^p \leq 1 \right\}. \end{aligned}$$

In this section it will be more convenient for us to represent $I\!\!R^{n+1}$ as $I\!\!R \times I\!\!R^n = \{(x,y) \mid x \in I\!\!R,\ y \in I\!\!R^n\}$. So we fix one direction. Given $a < b$ by $S(a;b)$ we denote the strip $\{(x,y) \in I\!\!R^{n+1} \mid a \leq x \leq b\}$ and $S^{\pm}(a;b) = S(a;b) \cup S(-b;-a)$.

Given vector $y = \{y_i\}_{i=1}^n \in I\!\!R^n$ we denote the Euclidean norm $\sqrt{\sum y_i^2}$ of y by $|y|$.

Let v, w be positive numbers. Let $p \in (0,1)$ and $n > 0$ be an integer. Throughout this section by V_n we will denote the volume of the n-dimensional Euclidean ball, and by $B_p = B_p(v;w;n)$ we will denote the following p-convex body

$$B_p = p\text{-conv } \left\{ (x,y) \in I\!\!R^{n+1} \mid x \in I\!\!R, |x| \leq v, y \in I\!\!R^n, |y| \leq f(x) \right\} \subset I\!\!R^{n+1},$$

where

$$f(x) = \begin{cases} 1 & \text{for } |x| \leq 1, \\ w & \text{otherwise.} \end{cases}$$

The following lemma provides estimates of the volume of B_p.

Lemma 3.1 *Let* $p \in (0,1)$*,* $w \in (3/4,1)$ *and* $v > 3/2$*. There is an absolute constant* $c \in (0,1)$ *such that if* $1 - w^p < \beta_p := c^{1/(1-p)}$ *then for* $x_0 = \max\left\{3/2; 4\,(1-w^p)^{1/p}\,v\right\}$ *the following holds*

$$|B_p \cap S\,(3/2; x_0)| \leq 4 \cdot V_n \cdot \frac{v}{n^{1/p}},$$

and $B_p \cap S\,(x_0; v) = \{(x,y) \in I\!R^{n+1} \mid x_0 \leq x \leq v, |y| \leq w\}$ *(thus its volume* $|B_p \cap S\,(x_0; v)| = V_n \cdot (v - x_0) \cdot w^n$*).*

For reader's convenience we postpone the proof of this lemma.

Let us recall the result of M. Gromov and V. Milman ([GrM]).

Let K be a centrally-symmetric compact convex body in $I\!R^{n+1}$. Denote

$$V_t\,(K) = \left|K \cap S^{\pm}\,(t;\infty)\right| = |K| - |K \cap S\,(-t;t)|\,.$$

Let m be the median of K, i.e. a number which satisfies $V_m\,(K) = |K|\,/2$ (precisely speaking, m is the median of the function $f\,((x,y)) = |x|$ on the probability space $(I\!R \times I\!R^n,\ \mathbf{Pr})$ with $\mathbf{Pr}\,((x,y) \in B) = |B \cap K|/|K|$).

M. Gromov and V. Milman ([GrM]) proved that there is an absolute constant c such that

$$V_t\,(K) \leq \frac{1+e}{2} \exp\left(-\frac{c}{1+e}\frac{t}{m}\right)|K|$$

for every $t \geq (1+e)m$.

Lemma 3.1 yields the following corollary.

Corollary 3.2 *Let* $p \in (0,1)$*. Let* $A > 0$*. Then for large enough* $n \in I\!N$ *there exists a centrally-symmetric* p*-convex body* $B_p \subset I\!R^{n+1}$ *with median* m *such that there is* $t \geq Am$ *for which*

$$V_t\,(B_p) \geq \frac{1}{32}\,|B_p|\,.$$

Proof: Let β_p be as in Lemma 3.1. Let n be large enough to satisfy

$$n > (32A)^p\,,\ \alpha := \ln\frac{n^{1/p}}{4} < n\beta_p,\ \left(1 - \frac{\alpha}{n}\right)^n \geq \frac{2}{n^{1/p}}.$$

Put $v = \frac{n^{1/p}}{16}$ and $w = 1 - \frac{\alpha}{n}$. Let $B_p = B_p(v; w; n) \subset I\!R^{n+1}$ be as in Lemma 3.1. Then by choice of B_p and by the lemma, one has $|B_p \cap S(-1;1)| = 2V_n$, $|B_p \cap S^{\pm}(1;3/2)| \le V_n$, $|B_p \cap S^{\pm}(3/2;x_0)| \le V_n/2$, $|B_p \cap S^{\pm}(x_0,\infty)| \le V_n/2$, where x_0 was defined in Lemma 3.1. Hence $m \le 1$. Take $t = v/2$. Then $t/m > A$ and $t > x_0$, since $1 - w^p \le \frac{\alpha}{n} \le \frac{\ln n}{n}$. Thus

$$V_t(B_p) = \left|B_p \cap S^{\pm}(t;v)\right| = 2 \cdot V_n \cdot (v - t) \cdot w^n \ge V_n \cdot v \cdot \frac{2}{n^{1/p}} \ge \frac{1}{32}|B_p|,$$

which proves the corollary. □

Proof of Lemma 3.1: Let $L = \{(x,y) \in I\!R^2 \mid 0 \le x \le v,\ 0 \le y \le f(x)\}$. Let

$$G = \left\{\lambda(1,1) + (1-\lambda^p)^{1/p}(v,w) \mid \lambda \in [0,1]\right\} \subset I\!R^2$$

be p-convex curve between points $(1,1)$ and (v,w). By definition of p-convexity we have $K := p\text{-conv } L = L \cup \{\lambda z \mid z \in G, \lambda \in [0,1]\}$. Clearly, B_p is the rotation body of $K \cup -K$. We will show that

$$K \subset \left\{(x,y) \in I\!R^2 \mid 0 \le x \le v,\ 0 \le |y| \le g(x)\right\},$$

where

$$g(x) = \begin{cases} 1 & \text{for } x < 3/2, \\ \left(1 - \frac{x^p}{4^p v^p}\right)^{1/p} & \text{for } 3/2 \le x \le x_0, \\ w & \text{for } x_0 < x \le v, \end{cases}$$

and $x_0 = \max\left\{3/2; 4 \cdot (1-w^p)^{1/p} \cdot v\right\}$. Since

$$\int_{3/2}^{x_0} \left(1 - \frac{x^p}{4^p v^p}\right)^{n/p} dx \le \int_0^{\infty} \exp\left(-\frac{n x^p}{p 4^p v^p}\right) dx = \frac{4 \cdot v \cdot p^{1/p}}{n^{1/p}} \Gamma(1 + 1/p) \le 4\frac{v}{n^{1/p}},$$

where Γ is the Gamma function, the result follows.

Let $(x,y) \in G$. Then

$$\begin{cases} x = x(\lambda) = \lambda + (1-\lambda^p)^{1/p} v, \\ y = y(\lambda) = \lambda + (1-\lambda^p)^{1/p} w. \end{cases}$$

Hence $(v-w)^p = (vy - xw)^p + (x-y)^p$. Therefore,

$$y^p = y^p \left(\frac{v-w}{vy-xw}\right)^p - y^p \left(\frac{x-y}{vy-xw}\right)^p = \left(1 + \frac{xw - yw}{vy - xw}\right)^p - \frac{x^p}{v^p}\left(\frac{xyv - vy^2}{xyv - x^2 w}\right)^p.$$

Assume $(x,y) \in G$ be such that $y \geq w$ and $3/2 \leq x \leq x_0 = 4\,(1-w^p)^{1/p}\,v$ (if for every $x \geq 3/2$ one has $y \leq w$, we are done).

Then $y \geq 1/2$ and $vy \geq 2xw$ as long as $1 - w^p \leq 8^{-p}$. So, $\frac{x-y}{vy-xw} \leq \frac{4 \cdot x}{v}$. Since $x \geq \frac{3}{2}y$, we have $\frac{xyv - vy^2}{xyv - x^2 w} > 1/3$. Hence

$$y^p \leq \left(1 + 4\frac{xw}{v}\right)^p - \frac{1}{3^p}\frac{x^p}{v^p} \leq 1 + 4\frac{xw}{v} - \frac{1}{3^p}\frac{x^p}{v^p} \leq 1 - \frac{x^p}{4^p v^p}$$

if

$$\gamma_p := \left(\frac{1}{4}\left(\frac{1}{3^p} - \frac{1}{4^p}\right)\right)^{\frac{p}{1-p}} \cdot \frac{1}{4^p} > 1 - w^p.$$

Obviously, $\gamma_p \in (c^{1/(1-p)}, 8^{-p})$ for some absolute constant c. Therefore if $1 - w^p < c^{1/(1-p)}$ then for every $(x,y) \in G$ satisfying $y \geq w$, $3/2 \leq x \leq x_0 = 4\,(1-w^p)^{1/p}\,v$ we obtain $y \leq \left(1 - (x/(4v))^p\right)^{1/p}$.

Due to behavior of the function $y = y(\lambda)$ and since $w^p = 1 - \frac{x_0^p}{4^p v^p}$, we get that if $(x,y) \in G$ with $x \geq x_0$ then $y \leq w$. That proves the lemma. $\square$

The following corollary shows that the condition of convexity in Corollary 1.2 can not be replaced by the condition of p-convexity even if the seminorm is just the absolute value of the first coordinate.

Corollary 3.3 *Let $p \in (0,1)$. There are centrally-symmetric p-convex bodies $K = K(n) \subset I\!R^{n+1} = \{(x,y) \mid x \in I\!R,\ y \in I\!R^n\}$ such that for every $q \in (0,\infty)$ if n is sufficiently large then*

$$\frac{1}{2^{1/q}}\,(3/8)^{1+1/q}\left(\frac{1}{1+q}\right)^{1/q}\left(\frac{n}{\ln n}\right)^{1/p}(\ln n)^{-\frac{1}{qp}} \leq \left(\frac{1}{|K|}\int_K |x|^q d(x,y)\right)^{1/q} \leq$$

$$\leq 2^{1/q}\,(3/8)^{1+1/q}\left(\frac{1}{1+q}\right)^{1/q}\left(\frac{n}{\ln n}\right)^{1/p}(\ln n)^{-\frac{1}{qp}}.$$

Remark. This result demonstrates the existence of the centrally-symmetric p-convex bodies $K = K(n) \subset I\!R^{n+1}$ such that for every $s < q$ from $(0,\infty)$ if n is large enough then

$$\left(\frac{1}{|K|}\int_K |x|^q d(x,y)\right)^{1/q} \Big/ \left(\frac{1}{|K|}\int_K |x|^s d(x,y)\right)^{1/s} > C_{s,q}\,(\ln n)^{\frac{1}{p}\left(\frac{1}{s}-\frac{1}{q}\right)},$$

where $C_{s,q}$ depends on s, q only.

Proof: Let β_p be as in Lemma 3.1. Let n be large enough. Put

$$v = \frac{3}{8}\left(\frac{n}{\ln n}\right)^{1/p} \quad \text{and} \quad w = 1 - \frac{\ln n}{p \cdot n} > 1 - \beta_p.$$

Let $K = B_p = B_p(v; w; n) \subset I\!R^{n+1}$ be as in Lemma 3.1. Then, repeating the proof of the previous corollary, we obtain $2V_n < |K| < 4V_n$ and, since $1 - w > 1 - w^p > p(1-w)$, $x_0 = 4\left(1 - w^p\right)^{1/p} v \geq 3/2$ for large enough n. Therefore,

$$\frac{1}{|K|}\int_K |x|^q d(x,y) \geq \frac{2V_n}{|K|} w^n \int_0^v x^q dx > \frac{1}{2}\frac{v^{1+q}w^n}{1+q},$$

$$\frac{1}{|K|}\int_K |x|^q d(x,y) \leq \frac{2V_n}{|K|}\left(w^n \int_0^v x^q dx + \int_0^{x_0} x^q dx\right) < \frac{2v^{1+q}w^n}{1+q}$$

if $v^{1+q}w^n \geq x_0^{1+q} = 4^{1+q}\left(1 - w^p\right)^{(1+q)/p} v^{1+q}$, which is true for large enough n. Since

$$\gamma_n \frac{3}{8}\left(\frac{1}{\ln n}\right)^{1/p} < v \cdot w^n < \frac{3}{8}\left(\frac{1}{\ln n}\right)^{1/p}$$

for some $\gamma_n \longrightarrow 1$, we obtain the result. □

Remark. Let us fix a constant $C > 1/p$. Slightly different choice of v and w in the proof gives us

$$\left(\frac{1}{|K|}\int_K |x|^q d(x,y)\right)^{1/q} \approx (3/8)^{1+1/q}\left(\frac{1}{1+q}\right)^{1/q}\left(\frac{n}{\ln n}\right)^{1/p}(\ln n)^{-\frac{C}{q}}$$

up to the factor $2^{1/q}$. Moreover, if we restrict ourselves by the interval $[\alpha, \infty)$ for some $\alpha > 0$ then one can find bodies $K = K(n)$ such that for every $q \in [\alpha, \infty)$ and for large enough n one has

$$\left(\frac{1}{|K|}\int_K |x|^q d(x,y)\right)^{1/q} \approx \left(\frac{3}{8(1+\alpha)^{1/p}}\right)^{1+1/q}\left(\frac{1}{1+q}\right)^{1/q}\left(\frac{n}{\ln n}\right)^{1/p} n^{-\frac{\alpha}{qp}}$$

up to the factor $2^{1/q}$. Thus, for every $s < q$ from $[\alpha, \infty)$ there is a constant $C_{s,q}$ such that for large enough n

$$\left(\frac{1}{|K|}\int_K |x|^q d(x,y)\right)^{1/q} \Big/ \left(\frac{1}{|K|}\int_K |x|^s d(x,y)\right)^{1/s} > C_{s,q} n^{\frac{\alpha}{p}\left(\frac{1}{s}-\frac{1}{q}\right)}.$$

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