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PROPERTIES OF THE OPERATOR OF DISCRETE INTEGRATION AND SOME APPLICATIONS

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A theorem on the triangular projection, which was proved by S. Kwapien and A. Pelczynski, is generalized.

We say that a subset of naturals is solid , if it contains all natural numbers between its maximal and minimal elements or if it is empty. We say that a set $A \subset \{1, \dots, n\} \times \{1, \dots, n\}$ is k-admissible, if all its 'horisontal cross-sections' (or all its 'vertical cross-sections') consist of a union of no more than k solid sets.

Theorem. *Assume that M is the space of square matrices $a = \{a_{IJ}\}_{I,J=1}^n$, that α is an unconditional norm in M, $A \subset \{1, \dots, n\} \times \{1, \dots, n\}$ is K-admissible, that $P_A: M \to M$ is the projector, which corresponds to the set A. Then $\|P_A\|_\alpha \leqslant 2K\pi_1(\sigma_n)$, where π_1 is the absolutely summing norm of the operator $\sigma_n: l^1 \to l^\infty$, which is determined by the equality $\sigma_n(\{x_k\}_{k=1}^\infty) = \left\{\sum_{I=1}^k x_I\right\}_{k=1}^n$.*

In the work of L. Schwartz [1] a proof of the well-known Menshov-Rademacher Theorem was presented. It was based on properties of the so-called operator of discrete integration $\sigma: l^1 \to l^\infty$, which correlates the sequence of sums $\left\{\sum_{m=1}^k x_m\right\}_{k=1}^\infty \in l^\infty$ with each sequence $\{x_k\}_{k=1}^\infty \in l^1$. It turns out that the following statement, which was proved in [2], implies the Menshov-Rademacher Theorem.

Theorem. *The superposition of the diagonal operator $M_t: l^1 \to l^1$, which is generated by the sequence $t = \{t_k\}_{k=1}^\infty$, and of the operator σ is an absolutely summing operator, if $t_k = O(1/\ln k)$.*

Recall that an operator $S: X \to Y$, where X, Y are Banach spaces, is called absolutely summing, if there exists a constant c such that for any natural m and for any set $\{x_k\}_{k=1}^m$ from X the inequality

$$\sum_{k=1}^m \|Sx_k\| \leqslant c \sup_{x' \in X^*, \|x'\| \leqslant 1} \sum_{k=1}^m |(x_k, x')|$$

holds. The minimal such constant c, which is denoted by $\pi_1(S)$ is a norm in the space of absolutely summing operators.

Investigation of properties of the operator σ is based on evaluation of the norm of its finite-dimensional analogue, that is the operator $\sigma_n: l^1 \to l_n^\infty$, which correlates the sequence $\left\{\sum_{m=1}^k x_m\right\}_{k=1}^n$ with the sequences $\{x_k\}_{k=1}^\infty \in l^1$. In [3] with the aid of the theorem on the triangular projection, an estimate for $\pi_1(\sigma_n)$ is obtained in general. The following statement refines this estimate.

Lemma. $\pi_1(\sigma_n) \leqslant \pi/2 + \ln n \quad (n = 1, 2, \dots)$.

Applying this lemma we can obtain a theorem.

Theorem 1. *Assume that $t = \{t_k\}_{k=1}^{\infty}$ is a number sequence, $t_k \to 0$, assume that $M_t: l^1 \to l^1$ is the diagonal operator, which corresponds to the sequence t. If $\sum_{k=1}^{\infty} t_k^* \frac{\ln k}{k} < \infty$, where $(t_k^*)_{k=1}^{\infty}$ is a non-increasing transposition of the sequence $\{|t_k|\}_{k=1}^{\infty}$, then the operator σM_t is absolutely summing.*

Remark. It is easy to see that if the operator σM_t is absolutely summing and if the series $\sum_{k=1}^{\infty} f_k$ unconditionally converges in $L^1(X,\mu)$ then the series $\sum_{k=1}^{\infty} t_k f_k$ converges almost everywhere and its partial sums have a summable majorant. Along with this remark we obtain a corollary applying Theorem 1.

Corollary. If the series $\sum_{k=1}^{\infty} f_k$ unconditionally converges in $L^1(X,\mu)$ and if $\sum_{k=1}^{\infty} t_k^* \frac{\ln k}{k} < \infty$ for a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to 0$ then the series $\sum_{k=1}^{\infty} t_k f_k$ converges almost everywhere.

Particularly, this fact is true if $\sum_{k=1}^{\infty} |t_k|^p < \infty$ for some $p < \infty$ (see [3]).

We mentioned above that in [3] with the aid of the theorem on the triangular projection, an estimate is obtained, which is close to the statement of the lemma. It turns out that conversely, with the aid of this lemma we can obtain the theorem on the triangular projection and some its generalizations.

We need the following definitions and notation. We denote by M the linear space of square matrices $a = \{a_{IJ}\}_{I,J=1}^{n}$. We denote by u^{IJ} the matrix such that

$$u_{km}^{IJ} = \begin{cases} 1 & \text{for} \quad k = I, \ m = J, \\ 0 & \text{in other cases.} \end{cases}$$

For a set $A \subset \{1,\dots,n\} \times \{1,\dots,n\}$ we denote by P_A the projector onto the set A, that is the operator $P_A: M \to M$ such that $P_A(a) = \sum_{(I,J)\in A} a_{IJ} u^{IJ}$ for each matrix $a = (a_{IJ})_{I,J=1}^{n} \in M$. We say that the operator P_A is the triangular projection $T_n: M \to M$, if $A = \{(I,J): I + J \leqslant n+1\}$.

If α is a norm in M then we determine the conjugated norm by the equality $\alpha^*(a) = \sup_{b\in M, \alpha(b)\leqslant 1} \left|\sum_{I,J=1}^{n} a_{IJ} b_{JI}\right|$. Note that $\alpha^{**} = \alpha$.

Definition. The norm α in M is called unconditional if for any subsequences $\{s_I\}_{I=1}^{n}$ and $\{t_J\}_{J=1}^{n}$, where $|s_I| = 1$ $(1 \leqslant I \leqslant n)$, $|t_J| = 1$ $(1 \leqslant J \leqslant n)$ for each matrix $a = \{a_{IJ}\}_{I,J=1}^{n} \in M$ the equality $\alpha(a) = \alpha(\{s_I t_J a_{IJ}\}_{I,J=1}^{n})$ holds.

Definition. A subset of the naturals is called solid, if it contains all natural numbers between its maximal and minimal elements or if it is empty.

Definition. A set $A \subset \{1,\dots,n\} \times \{1,\dots,n\}$ is called admissible, if all its 'horisontal cross-sections' are solid, i.e. for all $I \in \{1,\dots,n\}$ either there exists n_I and m_I from $\{1,\dots,n\}$ such that $(I,K) \in A$ if and only if $K \in \{n_I,\dots,m_I\}$, or the cross-section with respect to I is empty.

Kwapien and Pelczynski have shown (we are interested in norms of projections P_A) [3] that under some conditions for the unconditional matrix norm α, the estimate $\|T_n\|_\alpha \leqslant \log_2 2n$ holds. We generalize their result for other projections.

Theorem 2. *Assume that $A \subset \{1,\dots,n\} \times \{1,\dots,n\}$ is an admissible set and that α is an unconditional norm in M. Then $\|P_A\|_\alpha \leqslant 2\pi_1(\sigma_n)$.*

Proof. For matrices $a = \{a_{IJ}\}_{I,J=1}^{n}$, $b = \{b_{IJ}\}_{I,J=1}^{n}$ and $k \leqslant n$ we denote by c_K the sequence $\{c_{KJ}\}_{J=1}^{\infty} \in l^1$ such that

$$c_{KJ} = \begin{cases} a_{KJ} b_{JK} & \text{for} \quad J \leqslant n, \\ 0 & \text{for} \quad J > n. \end{cases}$$

Then we obtain from the admissibility of the set A for all $a = \{a_{IJ}\}_{I,J=1}^{n} \in M$

$$\begin{aligned} \alpha(P_A(a)) &= \sup_{b\in M, \alpha^*(b)\leqslant 1} \left|\sum_{(I,J)\in A} a_{IJ} b_{IJ}\right| \\ &= \sup_{b\in M, \alpha^*(b)\leqslant 1} \left|\sum_{I=1}^{n}\sum_{J=n_I}^{m_I} c_{IJ}\right| \leqslant 2 \sup_{b\in M, \alpha^*(b)\leqslant 1} \sup_{k\leqslant n} \left|\sum_{I=1}^{n}\sum_{J=1}^{k} c_{IJ}\right| \\ &= 2 \sup_{b\in M, \alpha^*(b)\leqslant 1} \sum_{I=1}^{n} \|\sigma_n c_I\|_{l_n^\infty}. \end{aligned}$$

Applying the lemma we have

$$\begin{aligned}
\alpha(P_A(a)) &\leqslant 2 \sup_{b\in M,\alpha^*(b)\leqslant 1} \pi_1(\sigma_n) \sup_{x\in l^\infty,\|x\|\leqslant 1} \sum_{I=1}^{n} |\langle c_I, x\rangle| \\
&= 2\pi_1(\sigma_n) \sup_{b\in M,\alpha^*(b)\leqslant 1} \sup_{x\in l^\infty,\|x\|\leqslant 1} \sum_{I=1}^{n} \left| \sum_{J=1}^{n} c_{IJ} x_J \right| \\
&= 2\pi_1(\sigma_n) \sup_{b\in M,\alpha^*(b)\leqslant 1} \sup_{|x_J|=1, J\leqslant n} \sup_{|s_I|=1, I\leqslant n} \left| \sum_{I=1}^{n} s_I \sum_{J=1}^{n} c_{IJ} x_J \right| \\
&= 2\pi_1(\sigma_n) \sup_{\substack{b\in M,\alpha^*(b)\leqslant 1\\ |x_J|=1,|s_I|=1}} \left| \sum_{I,J=1}^{n} a_{IJ} b_{JI} s_I x_J \right| \\
&= 2\pi_1(\sigma_n)\alpha(a)
\end{aligned}$$

(the unconditionality of the norm α implies the last equality). Thus, $\|P_A\|_\alpha \leqslant 2\pi_1(\sigma_n) \leqslant \pi + 2\ln n$.

Remark 1. If we require in the definition of an admissible set A that all 'horizontal cross-sections' are solid and in this case for all $I \in \{1, \dots, n\}$ eigther $(I, 1) \in A$, or the cross-section with respect to I is empty then, as it is seen from the proof of Theorem 2, $\|P_A\|_\alpha \leqslant \pi_1(\sigma_n)$. Particularly, $\|T_n\|_\alpha \leqslant \pi/2 + \ln n$. The last estimate is better than the estimate of Kwapien and Pelczynski beginning from $n = 4$.

Remark 2. Obviously, if in the definition of an admissible set we take 'vertical cross-sections' then the estimate keeps.

Let us generalize Theorem 2.

Definition. We say that a set $A \subset \{1, \dots, n\} \times \{1, \dots, n\}$ is K-admissible, if all its 'horizontal cross-sections' (or all its 'vertical cross-sections') consist of a union of no more than K solid sets.

By repeating the proof of Theorem 2 with obvious modifications we obtain the following statement.

Theorem 3. *Assume that $A \subset \{1, \dots, n\} \times \{1, \dots, n\}$ is a K-admissible set and that α is an unconditional norm in M. Then $\|P_A\|_\alpha \leqslant 2K\pi_1(\sigma_n)$.*

References

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