# On the symmetric average of a convex body

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**Abstract**

We introduce a new parameter, symmetric average, which measures the asymmetry of a given non-degenerated convex body $K$ in $\mathbb{R}^n$. Namely, $sav(K) = \inf_{a \in \text{int}K} \int_{K_a} \| - x\|_{K_a} \, dx/|K|$, where $|K|$ denotes the volume of $K$ and $K_a = K - a$. We show that for polytopes $sav(K) \leq C \ln N$, where $N$ is the number of facets of $K$. Moreover, in general $\frac{n}{n+1} \leq sav(K) < \sqrt{n}$ and equality in the lower bound holds if and only if $K$ is centrally symmetric. We apply these estimates to provide bounds for covering $K$ by homotets of $K \cap -K$.

## 1 Introduction

Let $K \subset \mathbb{R}^n$ be a convex body. A shift of $K$ by a vector $a \in \mathbb{R}^n$ is denoted by $K_a := K - a$. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}^n$ and $\lambda_K$ be defined by

$$\lambda_K(A) = \frac{\lambda(A \cap K)}{\lambda(K)}.$$

We consider the following parameter of $K$, which can be called *symmetric average of* $K$,

$$sav(K) := \inf_{a \in \text{int}K} \int_{K_a} \| - x\|_{K_a} \, d\lambda_{K_a},$$

where $\| \cdot \|_K$ denotes the Minkowski functional (or gauge) of a convex body (see the definitions below). Note that $sav(K)$ is an affine invariant, that is $sav(K) = sav(TK)$ for every affine invertible operator $T : \mathbb{R}^n \to \mathbb{R}^n$ and that $sav(K)$ measures the asymmetry of $K$. For other affine invariant measures of asymmetry and related discussions we refer to [Gr].

In Section 4, we provide general bounds for $sav(K)$. Namely, we show that $n/(n+1) \leq sav(K) < \sqrt{n}$. Moreover, we show that the equality in the lower bound holds if and only if $K$ is centrally symmetric. The upper bound is based on the study of a particular choice of a position of $K$, namely the one for which the Euclidean unit ball is the ellipsoid of maximal volume contained in $K$ and centered at the centroid of $K$. In Section 3, we show that the general estimate can be significantly improved if $K$ is a polytope with a small number of facets. More precisely, in Theorem 3.1 below we obtain that $sav(K) \leq C \ln N$, where $C$ is an absolute constant and $N$ is the number of facets of $K$. This means that although a convex body can be very far from being symmetric (say, in Banach-Mazur distance, or, equivalently, in the sense of the functional $\inf_{a \in K} \sup_{x \in K} \| - x\|_{K_a}$) like a simplex, in average it is not very asymmetric, i.e. for most points $\| - x\|_K$ is not very big. Here we take average in the sense of normalized Lebesgue measure on the body. This phenomenon should be compared with a recent result in [GL], where the opposite phenomenon was observed for polytopes and the measure uniformly distributed on vertices. In the last section, we provide some applications to the bounds of covering numbers.

# 2 Preliminaries and notation

By $B_2^n$, $|\cdot|$, and $\langle \cdot, \cdot \rangle$ we denote the standard Euclidean ball, the canonical Euclidean norm, and the canonical inner product on $\mathbb{R}^n$. Given points $x_1, \ldots, x_k$ in $\mathbb{R}^n$ we denote their convex hull by $\mathrm{conv}\,\{x_i\}_{i \leq k}$. By a body in $\mathbb{R}^n$ we always mean a connected compact set with non-empty interior. We denote by $\mathrm{int} A$ the interior of a set $A \subset \mathbb{R}^n$.

Let $K \subset \mathbb{R}^n$ be a convex body with 0 in its interior. The polar of $K$ is

$$K^0 = \{x \in \mathbb{R}^n \ \mid \ \langle x, y \rangle \leq 1 \ \text{ for every } \ y \in K\}.$$

The Minkowski functional of $K$ (or the gauge of $K$) is

$$\|x\|_K = \inf\{t > 0 \mid x \in tK\}.$$

As we mentioned in introduction, $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^n$ and $\lambda_K$ denotes the measure on $\mathbb{R}^n$ given by $\lambda_K(A) = \lambda(A \cap K)/\lambda(K)$. Note that by Brunn-Minkowski inequality $\lambda_K$ is a normalized log-concave measure on $\mathbb{R}^n$ (see [Bo] and, e.g., [MS], appendix III).

We will need two known lemmas. The first one is the Kahane-Khinchine inequality for linear functionals and it follows from the log-concavity of $\lambda_K$ (see [Bo]).

**Lemma** [Bo] *There exists an absolute positive constant $C$ such that for every $q \geq 2$, every $y \in \mathbb{R}^n$, and every convex body $K \subset \mathbb{R}^n$ with 0 in its interior one has*

$$\left( \int_K |\langle x, y \rangle|^q \ d\lambda_K(x) \right)^{1/q} \leq Cq \left( \int_K |\langle x, y \rangle|^2 \ d\lambda_K(x) \right)^{1/2}. \qquad (1)$$

The second lemma appeared in [KLS] (see also Chapter 3 in [F] for a simpler proof). In particular it was used to estimate the distance between a convex body and it's Legendre ellipsoid.

**Lemma** [KLS] *Let $K \subset \mathbb{R}^n$ be a convex body with centroid at 0. Then for every $y \in \mathbb{R}^n$ one has*

$$\frac{1}{\sqrt{n(n+2)}} \|y\|_{K^0} \leq \left( \int_K |\langle x, y \rangle|^2 \ d\lambda_K(x) \right)^{1/2} \leq \sqrt{\frac{n}{n+2}} \|y\|_{K^0}. \qquad (2)$$

*Moreover, there is equality in the right hand side if $K$ is a simplex in $\mathbb{R}^n$ and in the left hand side if $K$ is a Euclidean ball.*

# 3 The case of non-degenerated polytopes

In this section we prove that $sav(K)$ of a polytope with small number of facets cannot be large.

**Theorem 3.1** *Let $1 \leq n < N$. Let $K$ be a non-degenerated polytope in $\mathbb{R}^n$ with $N$ facets. Then*

$$sav(K) \leq C \ln N,$$

*where $C$ is an absolute positive constant.*

**Proof:** Since $sav(K)$ is an affine invariant, we can assume that 0 is the centroid of $K$. Since $K$ has $N$ facets, there exist $x_1, \ldots, x_N \in \mathbb{R}^n$ such that

$$\|x\|_K = \max_{i \leq N} \langle x, x_i \rangle .$$

Therefore for every $q \geq 2$ we have

$$sav(K) \leq \int_K \| -x\|_K \ d\lambda_K(x) = \int_K \max_{i \leq N} \langle -x, x_i \rangle \ d\lambda_K(x)$$

$$\leq \int_K \left( \sum_{i=1}^N |\langle x, x_i \rangle|^q \right)^{1/q} \ d\lambda_K(x) \leq \left( \int_K \sum_{i=1}^N |\langle x, x_i \rangle|^q \ d\lambda_K(x) \right)^{1/q} .$$

Applying Kahane-Khinchine inequality (1) we obtain

$$sav(K) \leq C_1 q \left( \sum_{i=1}^N \left( \int_K |\langle x, x_i \rangle|^2 \ d\lambda_K(x) \right)^{q/2} \right)^{1/q}$$

$$\leq C_1 q N^{1/q} \max_{i \leq N} \left( \int_K |\langle x, x_i \rangle|^2 \ d\lambda_K(x) \right)^{1/2} ,$$

where $C_1$ is an absolute positive constant.

Note that by the definition of $x_i$'s we have $K^0 = \operatorname{conv} \{x_i\}_{i \leq N}$ and $\|x_i\|_{K^0} = 1$. Therefore inequality (2) implies

$$sav(K) \leq C_1 q N^{1/q}.$$

The choice $q = \max\{2, \ln N\}$ completes the proof. □

# 4 The general case

Here we provide general estimates for $sav(K)$. Note that in general we have

$$\inf_{a \in \operatorname{int} K} \sup_{x \in K_a} \|-x\|_{K_a} \leq n$$

and that the bound $n$ here cannot be improved as the example of simplex shows. We show here that for every convex body $K \subset \mathbb{R}^n$ there is a choice of a center $a$ (in fact, the centroid works) such that in average $\|-x\|_{K_a}$ is much smaller than in the extremal case. Or, in other words, the measure of points $x \in K_a$ with large $\|-x\|_{K_a}$ is small. This should be compared with the recent paper [GL], where the opposite phenomenon was observed for polytopes and the discrete measure supported on vertices instead of the Lebesgue measure.

The bound $\sqrt{n}$ is an immediate consequence of the following theorem (see Theorem 2.3 of [G]).

**Theorem 4.1** *Let $K$ be a convex body in $\mathbb{R}^n$ such that $0$ is the centroid of $K$ and $B_2^n$ is the ellipsoid of maximal volume contained in $K \cap -K$, i.e. that*

*$B_2^n$ maximizes the volume of $TB_2^n$ over all linear operators on $\mathbb{R}^n$ satisfying $TB_2^n \subset K \cap -K$. Then*

$$\left( \int_K |x|_2^2 \, d\lambda_K \right)^{1/2} \leq \frac{n}{\sqrt{n+2}} = \left( \int_{\Delta_n} |x|_2^2 \, d\lambda_{\Delta_n} \right)^{1/2},$$

*where $\Delta_n$ is the regular simplex, circumscribed to the Euclidean unit ball.*

As an immediate consequence we obtain the general estimate for $sav(K)$.

**Theorem 4.2** *Let $K \subset \mathbb{R}^n$ be a convex body. Then*

$$sav(K) \leq \frac{n}{\sqrt{n+2}} < \sqrt{n}.$$

**Proof:** It is well known that for every convex body $K$ there exists a linear transformation $T$ such that $TK$ satisfies the condition of Theorem 4.1. Thus, since $sav(K)$ is an affine invariant, we can assume that 0 is the centroid of $K$ and that $B_2^n$ is the ellipsoid of maximal volume for $K \cap -K$. Then we have $\|x\| \leq |x|$ for every $x \in \mathbb{R}^n$ and therefore, applying Theorem 4.1,

$$sav(K) \leq \int_K \| -x\|_K \, d\lambda_K \leq \int_K |x| \, d\lambda_K \leq \left( \int_K |x|_2^2 \, d\lambda_K \right)^{1/2} \leq \frac{n}{\sqrt{n+2}}.$$

□

For the reader convenience, we provide the proof of Theorem 4.1.

**Proof of Theorem 4.1:** Let $K$ be a convex body in $\mathbb{R}^n$ such that 0 is the centroid of $K$ and $B_2^n$ is the ellipsoid of maximal volume contained in $K \cap -K$. By John's theorem [J] (see also [B]), there exist scalars $c_1, \ldots, c_m > 0$ and contact points $u_1, \ldots, u_m$ of $B_2^n$ and $K \cap -K$ such that

$$\mathrm{Id} = \sum_{j=1}^m c_j u_j \otimes u_j, \tag{3}$$

where Id denotes the identity operator on $\mathbb{R}^n$. Replacing if necessary $u_j$ by $-u_j$, we can assume that points $u_1, \ldots, u_m$ belong to $\mathbb{S}^{n-1}$ and the boundary of $K$, i.e. $|u_j|_2 = \|u_j\|_K = \|u_j\|_{K^o} = 1$.

We deduce from this decomposition that

$$\int_K |x|_2^2 \, d\lambda_K = \sum_{j=1}^m c_j \int_K \langle x, u_j \rangle^2 \, d\lambda_K(x).$$

Since the centroid of $K$ is at the origin, we can apply the lemma of Kannan, Lovász and Simonovits [KLS] quoted above (see inequality (2)). We obtain

$$\int_K |x|_2^2 \, d\lambda_K \leq \frac{n}{n+2} \sum_{j=1}^m c_j \|u_j\|_{K^0}^2.$$

Since for every $j \leq m$, $\|u_j\|_{K^0} = 1$ and $\sum_{j=1}^m c_j = n$ (by taking the trace in the identity decomposition (3)), we have

$$\int_K |x|_2^2 \, d\lambda_K \leq \frac{n}{n+2} \sum_{j=1}^m c_j \|u_j\|_{K^0}^2 = \frac{n^2}{n+2}.$$

Moreover, if $\Delta_n$ is the regular simplex circumscribed to the Euclidean unit ball, then $\Delta_n$ has the centroid at 0 and there exist $u_1, \ldots, u_m \in \mathbb{S}^{n-1}$ (in fact, $m = n+1$), such that $\sum_{j=1}^m u_j = 0$ and

$$\mathrm{Id} = \frac{n}{n+1} \sum_{j=1}^m u_j \otimes u_j.$$

Thus we have equality in inequality (2), which means that

$$\int_{\Delta_n} \langle x, u_j \rangle^2 \, d\lambda_{\Delta_n} = \frac{n}{n+2}.$$

This proves that

$$\int_{\Delta_n} |x|_2^2 \, d\lambda_{\Delta_n} = \frac{n^2}{n+2}$$

and concludes the proof of Theorem 4.1. □

We conclude this section providing the sharp lower bound for $sav(K)$. As one can intuitively guess, $sav(K)$ is the smallest when $K$ is centrally symmetric and the infumum in the definition of $sav(K)$ is attained when $a$ is chosen to be the center of symmetry of $K$.

**Proposition 4.3** *Let $K \subset \mathbb{R}^n$ be a convex body. Then*

$$sav(K) \geq \frac{n}{n+1}$$

*and equality holds if and only if $K$ is centrally symmetric.*

**Proof:** Let $a$ be an interior point of $K$ and denote $L = K_a$. Using polar coordinates we have

$$I := \int_L \|-x\|_L \, dx = \frac{1}{n+1} \int_{\mathbb{S}^{n-1}} \frac{\|-\omega\|_L}{\|\omega\|_L^{n+1}} \, d\omega.$$

By the symmetry of the sphere,

$$\int_{\mathbb{S}^{n-1}} \frac{\|-\omega\|_L}{\|\omega\|_L^{n+1}} \, d\omega = \int_{\mathbb{S}^{n-1}} \frac{\|\omega\|_L}{\|-\omega\|_L^{n+1}} \, d\omega.$$

Thus,

$$I = \frac{1}{2(n+1)} \int_{\mathbb{S}^{n-1}} \left( \frac{\|-\omega\|_L}{\|\omega\|_L^{n+1}} + \frac{\|\omega\|_L}{\|-\omega\|_L^{n+1}} \right) \, d\omega.$$

Now note that for every positive $a$, $b$ one has

$$\frac{a}{b^{n+1}} + \frac{b}{a^{n+1}} \geq \frac{1}{a^n} + \frac{1}{b^n}, \tag{4}$$

with equality if and only if $a = b$. Indeed, the inequality is equivalent to $a^{n+1}(a-b) \geq b^{n+1}(a-b)$, which is obviously true. Applying it to our integral we observe

$$I \geq \frac{1}{2(n+1)} \int_{\mathbb{S}^{n-1}} \left( \frac{1}{\|\omega\|_L^n} + \frac{1}{\|-\omega\|_L^n} \right) \, d\omega = \frac{1}{n+1} \int_{\mathbb{S}^{n-1}} \frac{1}{\|\omega\|_L^n} \, d\omega.$$

Since

$$\lambda(K) = \lambda(L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \frac{1}{\|\omega\|_L^n} \, d\omega,$$

we obtain that $I \geq n/(n+1)$.

The equality case follows from the fact that in (4) we have equality if and only if $a = b$. Thus, if $\|\omega\|_L \neq \|-\omega\|_L$ for some point $\omega \in \mathbb{S}^{n-1}$ (and hence on a set of positive measure) then $I > n/(n+1)$. □

# 5 Applications

In this section we apply our results to obtain covering estimates. Namely, we show upper bounds for the covering number $N(K, t(K \cap -K))$, where $K$ is a convex body with centroid at 0. Recall, that the covering number $N(K, L)$ is the minimal number of translates of $L$ needed to cover $K$. We will use the following estimate, proved in [LMP] (see Lemma 4 there).

**Lemma 5.1** *Let $K$, $L$ be convex bodies. Assume that $L - L \subset sL$ for some positive $s$. Then for every positive $t$*

$$N(K, tL) \leq 2 \exp\left(csnM(K, L)/t\right),$$

*where $c$ is an absolute positive constant and*

$$M(K, L) = \int_K \|x\|_L \, d\lambda_K.$$

Note that

$$M(K, K \cap -K) = \int_K \|x\|_{K \cap -K} \, d\lambda_K \leq \int_K \left(\|x\|_K + \|-x\|_K\right) \, d\lambda_K.$$

Thus Lemma 5.1 together with Theorems 3.1 and 4.2 immediately imply

**Corollary 5.2** *There is an absolute positive constant $c$ such that for every convex body $K \subset \mathbb{R}^n$ with centroid at $0$ and every $t > 0$ one has*

$$N(K, t(K \cap -K)) \leq 2 \exp\left(cn^{3/2}/t\right).$$

*Moreover, if $K$ is non degenerated polytope with $N$ facets*

$$N(K, t(K \cap -K)) \leq 2 \exp\left(cn \ln N/t\right).$$

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