Euclidean projections of a p-convex body.

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Abstract

In this paper we study Euclidean projections of a p-convex body in $\mathbb{R}^n$. Precisely, we prove that for any integer k satisfying $\ln n \leq k \leq n/2$, there exists a projection of rank k with the distance to the Euclidean ball not exceeding $C_p(k/\ln(1+\frac{n}{k}))^{1/p-1/2}$, where C_p is an absolute positive constant depending only on p. Moreover, we obtain precise estimates of entropy numbers of identity operator acting between ℓ_p and ℓ_r spaces for the case $0 < p < r \leq \infty$. This allows us to get a good approximation for the volume of p-convex hull of n points in $\mathbb{R}^k$, $p < 1$, which shows the sharpness of the announced result.

0 Introduction.

Our work is motivated by so-called "Isomorphic Dvoretzky Theorem" proved recently by Milman and Schechtman [M-S1], [M-S2] (see also [G1]). The theorem states that given $1 \leq k \leq n/2$ and a centrally symmetric body K in $\mathbb{R}^n$ there exists a k-dimensional subspace $E \subset \mathbb{R}^n$ such that Banach-Mazur distance between $K \cap E$ and the Euclidean ball is bounded by $C \max\{1, \sqrt{k/\ln(1+\frac{n}{k})}\}$, where C is an absolute constant (see below the precise definitions). In the dual setting it means that there exists a projection P of rank k such that Banach-Mazur distance between PK and the Euclidean ball has the same upper bound. Since the condition of the symmetry is natural for functional analysis but not so natural for convex geometry, the work of Milman and Schechtman leads to the investigation of similar question about general convex bodies and even quasi-convex bodies. It was shown by Gordon, Guédon and Meyer [G-G-M] that the same estimate holds for non-symmetric convex bodies as well (for $k \leq cn/(\ln^2 n)$). The authors have used a different approach, based on Rudelson's result ([R1], [R2]) about John

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decomposition. Moreover, very recently Litvak and Tomczak-Jaegermann [L-T] was able to show that the proof of Milman and Schechtman can be extended to the non-symmetric case for all $1 \leq k \leq n/2$. In fact, the authors extended to the non-symmetric case so-called "The proportional Dvoretzky-Rogers factorization" ([B-S], [S-T]), the only place, which needs symmetry in the proof.

In this paper we investigate the behavior of projections of quasi-convex (not necessarily symmetric) bodies. The study quasi-convex bodies has been of interest in the last years, since it turns out that many crucial results of the asymptotic theory hold for non-convex (but quasi-convex) bodies as well. It is rather surprising, because convexity was essentially used in the first proofs of most results. Let us note here that contrary to the convex case the statements about projections and sections of the quasi-convex bodies are completely different and do not follow one from the another, since one cannot use duality in the quasi-convex setting.

We prove that given p-convex body K, there exists a projection P of rank k such that Banach-Mazur distance from PK to the Euclidean space is bounded by $C_p \max\{1, (k/\ln(1+\frac{n}{k}))^{1/p-1/2}\}$ and Banach-Mazur distance from PK to its convex hull is bounded by $C_p^{'} \max\{1, (k/\ln(1+\frac{n}{k}))^{1/p-1}\}$, where $C_p, C_p^{'} > 0$ depend on p only. Of course, we use crucially the corresponding "convex" result. However, the straightforward extension can not be done. Moreover, recall that proofs in the convex case deals with sections and the result for projections follows by duality. We do not know any reasonable estimate for sections of p-convex bodies. The reason is that a projection, as any linear operator, preserves the convex hull of the set, while the convex hull of a section of a set can be very far from the section of the convex hull of the set, as was shown by Kalton [K1]. We expect that estimates for sections are much better for small p.

To show the sharpness of estimates above, we study entropy numbers of identity operators acting between ℓ_p and ℓ_r spaces, when $0 < p < r \leq \infty$. Such investigation was already done by Schütt [Sc] (see also [Pi] and [H]) for $p \geq 1$ and by Edmunds and Triebel [E-T] for $p < 1$. We give here a different proof which leads to a better dependence of the constants on p, when p tends to 0. Our proof also allows to estimate the corresponding Gelfand numbers in the case $r \leq 2$. As a corollary we obtain estimates for the volume of p-convex hull of a set of points in $\mathbb{R}^k$.

1 Definitions and notations.

By a body we always mean a compact set in R^n containing the origin as an interior point and star shaped with respect to the origin. Let K be an arbitrary body in $\mathbb{R}^n$, the gauge functional of K is defined by $\|x\|_K = \inf\{t \geq 0 \mid x \in tK\}$. By ellipsoid we always mean a linear image of the canonical Euclidean ball (thus all ellipsoids below are centered at origin). Given bodies K, B in $\mathbb{R}^n$ we define the Banach–Mazur distance by

$$d(K, B) = \inf\{\lambda > 0 \mid K - z \subset u(B - x) \subset \lambda(K - z)\},$$

where infimum is taken over all linear operators $u : \mathbb{R}^n \to \mathbb{R}^n$, and all $x, z \in \mathbb{R}^n$. We also define the following distance

$$d_0(K, B) = \inf\{\lambda > 0 \mid K \subset uB \subset \lambda K\},$$

where infimum is taken over all linear operators $u : \mathbb{R}^n \to \mathbb{R}^n$. Clearly, if K and B are centrally symmetric bodies, then $d(K, B) = d_0(K, B)$ and it is the standard Banach–Mazur distance. For $q \in (1, 2]$, and a body K, we define the constant $T_q(K)$ as the smallest possible constant C such that for every m, every x_1, ..., $x_m \in K$ the following inequality holds

$$\inf_{\varepsilon_i = \pm 1} \left\{ \left\| \sum_{i=1}^m \varepsilon_i x_i \right\|_K \right\} \leq C m^{1/q}.$$

The constant $T_q(K)$ is closely connected to the equal-norms type constant (see e.g. [G-K]).

Let $p \in (0, 1]$. A body K is called p-convex if for any x, $y \in K$, and any λ, $\mu \in [0, 1]$, $\lambda^p + \mu^p = 1$, the point $\lambda x + \mu y$ belongs to K. Correspondingly, the non-negative homogeneous functional $\| \cdot \|_K$ on $\mathbb{R}^n$ is called p-norm if for every x, $y \in \mathbb{R}^n$ we have $\|x + y\|_K^p \leq \|x\|_K^p + \|y\|_K^p$. Let us note that we do not require the symmetry in our definition. Similarly, a body K is called quasi-convex if there is a constant C such that $K + K \subset CK$ and the non-negative homogeneous functional $\| \cdot \|$ on $\mathbb{R}^n$ is called C-quasi-norm (or just quasi-norm) if for every x, $y \in \mathbb{R}^n$ we have $\|x+y\|_K \leq C \max\{\|x\|_K, \|y\|_K\}$. By Aoki-Rolewicz theorem (see e.g. [K-P-R], or [Kö] p.47), for every C-quasi-norm on $\mathbb{R}^n$ there is a p-norm $\| \cdot \|_0$, where $2^{1/p} = 2C$, such that $\|x\| \leq \|x\|_0 \leq 2C\|x\|$ for every $x \in \mathbb{R}^n$. Because of it we restrict ourselves to the study of p-convex bodies only, however all results can be equally well stated for quasi-convex bodies.

Given $p \in (0, 1]$ and a set A, its p-convex hull is defined as

$$p\text{-conv}\, A = \left\{ \sum_{i=1}^m \lambda_i x_i \mid m \in \mathbb{N}, x_i \in A, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i^p = 1 \right\}.$$

If $p = 1$ we write conv A. The p-absolute convex hull is p-conv $(A \cup -A)$ and we denote it by p-absconv$A$. It was shown in [B-B-P] that for $p \in (0,1)$

$$p\text{-conv } A = \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid m \in \mathbb{N}, x_i \in A, \lambda_i \geq 0, 0 < \sum_{i=1}^{m} \lambda_i^p \leq 1 \right\}.$$

Thus for any $p \in (0,1)$ and for every set A, the origin belongs to closure of p-conv A.
Given p-convex body K, we define

$$\alpha_m = \alpha_m(K) = \sup\{m^{-1} \left\| \sum_{i=1}^{m} x_i \right\|_K \mid x_i \in K, i \leq m\}$$

and if δ_K denotes $d_0(K, \text{conv } K)$ then it is known by a result of Peck [P] that $\delta_K = \inf \{\lambda > 0 \mid \text{conv } K \subset \lambda K\} = \sup \alpha_m$. Note also that by property of p-norm, $\alpha_m \leq m^{-1+1/p}$.

Given set $A \subset \mathbb{R}^n$, by vol(A) we denote the volume of A.

Recall that given bodies K, B in $\mathbb{R}^n$ the covering number $N(K,B)$ is the smallest number of translation of B needed to cover K. For a positive integer k the entropy number $e_k(K,B)$ is the smallest ε such that $N(K, \varepsilon B) \leq 2^{k-1}$. The following properties of covering and entropy numbers are well-known (see e.g. [Pis2], pp.56-63, with obvious modifications in the quasi-convex case). The sequence $\{e_k(K,B)\}_k$ is non-increasing. If B is a body and K is a p-convex body then for every positive ε

$$\varepsilon^{-n}\text{vol}(B)/\text{vol}(K) \leq N(B, \varepsilon K) \leq \text{vol}(B + \varepsilon K/2^{1/p})/\text{vol}(\varepsilon K/2^{1/p}). \qquad (1)$$

For every sets K_1, K_2, K_3 in $\mathbb{R}^n$ and every positive integers k and m we have $e_{k+m-1}(K_1, K_2) \leq e_k(K_1, K_3) e_m(K_3, K_2)$. Given an operator $u : E \to F$ we denote $e_k(u) = e_k(uB_E, B_F)$, where B_E is the unit ball of E and B_F is the unit ball of F and the Gelfand numbers $c_k(u)$ are defined for $k = 1, \dots, n$ by

$$c_k(u) = \inf\{\|u \ : E_k \to F\|, \ E_k \subset E \text{ with codim } E_k < k\}.$$

For $0 < p, q \leq \infty$, $x \in \mathbb{R}^n$ we denote by $\text{id}^n_{p,q}$ the identity operator from ℓ^n_p to ℓ^n_q and $|x|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$. By B^n_p we denote the unit ball of ℓ^n_p, i.e. $\{x \in \mathbb{R}^n, |x|_p \leq 1\}$. As usual $e_1, e_2, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$.

2 Large rank projections of p-convex bodies.

In [G-K] (Lemmas 2, 3 with remark after it) the authors proved that if p-normed space has equal-norms type $q > 1$ then the distance from the space to the corresponding normed space is bounded by a constant depending on p, q, and type constant only. We start with the following non-symmetric analog of their result.

Lemma 1 *Let $p \in (0,1)$, $q \in (1,2]$, K be a p-convex body and B be a symmetric body with respect to the origin. Define ϕ as $\phi = (1/p - 1/q)/(1 - 1/q)$ then*

$$\text{(i)}\ \delta_K \le \left(\frac{c}{p(q-1)}\right)^{1/p\,(1-1/\phi)} (T_q(B)\, d_0(K,B))^{1-1/\phi},$$

$$\text{(ii)}\ \delta_K \le \left(\frac{c}{p(q-1)}\right)^{(\phi-1)/p} (T_q(B)\, d_0(\text{conv } K,B))^{\phi-1},$$

$$\text{(iii)}\ d_0(K,B) \le \left(\frac{c}{p(q-1)}\right)^{(\phi-1)/p} T_q(B)^{\phi-1} d_0(\text{conv } K,B)^{\phi},$$

where $c > 0$ is an absolute constant.

The proof is essentially the same as in the symmetric case but for completeness we outline it here.

Proof. Let $d = d_0(K,B)$ and $T = T_q(B)$. Without loss of generality we can assume that $(1/d)B \subset K \subset B$. Let m be a positive integer and x_i, $i = 1,\ldots,2^m$ be a family of points in K, then $x_i \in B$, $i \le 2^m$ and by definition there is a choice of signs ε_i, $i \le 2^m$, such that $\|\sum \varepsilon_i x_i\|_B \le T2^{m/q}$. Since the body B is symmetric we can assume that $A = \{i \mid \varepsilon_i = 1\}$ has cardinality larger than 2^{m-1}. Thus

$$\begin{aligned}\left\|\sum_{i=1}^{2^m} x_i\right\|_K^p &= \left\|\sum_{i=1}^{2^m} \varepsilon_i x_i + 2\sum_{i\notin A} x_i\right\|_K^p \\ &\le d^p \left\|\sum_{i=1}^{2^m} \varepsilon_i x_i\right\|_B^p + 2^p \left\|\sum_{i\notin A} x_i\right\|_K^p \le d^p T^p 2^{mp/q} + 2^{mp}\alpha_{2^{m-1}}.\end{aligned}$$

Thus for any $k \le m$

$$\alpha_{2^m}^p \le \alpha_{2^k}^p + d^p T^p \sum_{i=k+1}^{\infty} 2^{-ip(1-1/q)} \le 2^{k(1-p)} + d^p T^p \frac{2^{-kp(1-1/q)}}{p(1-1/q)\ln 2}.$$

Choosing k from $2^{k(1-p)}(p(1-1/q)\ln 2) = d^pT^p2^{-kp(1-1/q)}$ we get the first estimate. The second and third estimates follow from the inequality $d_0(K,B) \leq \delta_K d_0(\text{conv } K, B)$. $\square$

This lemma allows us to extend the "Isomorphic Dvoretzky Theorem" to the p-convex setting.

Theorem 2 *There exists an absolute positive constant c such that for every p-convex body K in $\mathbb{R}^n$, $0 < p < 1$, for all integer $1 \leq k \leq n/2$ there exists a projection P of rank k such that*

$$d_0(PK, B_2^k) \leq C_p \max\left\{1, \left(\frac{k}{\ln(1+\frac{n}{k})}\right)^{\frac{1}{p}-\frac{1}{2}}\right\},$$

where $C_p \leq (c/p)^{2/p^2}$.

Proof. It is known [L-T] (see also [M-S1], [M-S2], [G1], [G-G-M]) that for all integer $k = 1, \ldots, [n/2]$ there exists a projection P of rank $k+1$ such that

$$d(P(\text{conv } K), B_2^{k+1}) \leq A := c\max\left\{1, \sqrt{\frac{k}{\ln(1+\frac{n}{k})}}\right\}.$$

In other words there are an ellipsoid centered at the origin $\mathcal{E}$ and a vector $a \in \mathbb{R}^n$ such that $P\mathcal{E} \subset P(\text{conv } K) - a \subset A(P\mathcal{E})$. Let Q be an orthogonal projection of rank k with $\text{Ker } Q \subset \text{span }\{\text{Ker } P, a\}$. Then $QP = Q$ which gives $Q\mathcal{E} \subset QK \subset AQ\mathcal{E}$. The result follows now by Lemma 1, since $T_2(Q\mathcal{E}) = 1$, $\phi = 2(1/p - 1/2)$ and $(\text{conv } QK) = Q(\text{conv } K)$. $\square$

Remark 1. If δ_K is not large then the trivial estimate $c_p\delta_K\sqrt{\frac{k}{\ln(1+\frac{n}{k})}}$ can be better than the one given in the theorem. Thus the theorem is of interest for "essentially" non-convex bodies only.

Remark 2. The theorem with the same proof holds without restriction "origin is an interior point of K" (assuming that interior of K is not empty).

Remark 3. The theorem holds for $k > n/2$ as well. Indeed, let $\varepsilon \in (0, 1/2)$ and $k = [(1-\varepsilon)n]$. Recently, the first name author ([G2], Theorem 2.3) has shown that for any convex body K with baricenter at origin there is a k-dimensional section E such that $d_0(K \cap E, B_2^k) \leq c\sqrt{k}/\varepsilon^{3/2}$, where c is an absolute constant. The proof is based on corresponding result from [L-M-P] and estimate for the parameter $\tilde{M} = \int_K |x|_2 dx/\text{vol}(K)$ ([G2], Theorem 2.2).

In dual formulation it means that for any convex body K there are a projection of rank k and a point $a \in K$ such that

$$d(P(K-a), B_2^k) \le c\sqrt{k}/\varepsilon^{3/2}. \tag{2}$$

Thus repeating the proof above, we get that the theorem holds for any $k = [(1-\varepsilon)n] > n/2$ with an additional factor $\varepsilon^{3(1/2-1/p)}$. Note also that (2) with slightly worse dependence on ε follows immediately from so-called "The proportional Dvoretzky-Rogers factorization" [L-T] (see also [B-S] and [S-T] for the symmetric case).

It is interesting to note that dependence on n and k in the theorem is optimal as simple example of the B_p^n shows. To prove it we need the following volume estimate.

Lemma 3 *There exists an absolute positive constant c such that for every set of points $x_1, \ldots, x_n$ in the k-dimensional space $\mathbb{R}^k$, the following estimate holds*

$$\left(\frac{\operatorname{vol}(p\text{-absconv}\{x_1,\ldots,x_n\})}{\operatorname{vol}(\operatorname{absconv}\{x_1,\ldots,x_n\})}\right)^{1/k} \le C_p \min\left\{1, \left(\frac{\ln(1+\frac{n}{k})}{k}\right)^{\frac{1}{p}-1}\right\},$$

where $C_p = \left(\frac{c}{p}\ln(2/p)\right)^{1/p}$.

We obtain this lemma as a corollary of entropy estimates at the end of our paper.

Remark. In particular the lemma implies that for every set of points x_1, $\ldots$, x_n in the k-dimensional Euclidean ball B_2^k we have

$$\left(\frac{\operatorname{vol}(p\text{-absconv}\{x_1,\ldots,x_n\})}{\operatorname{vol} B_2^k}\right)^{1/k} \le C_p \min\left\{1, \left(\frac{\ln(1+\frac{n}{k})}{k}\right)^{\frac{1}{p}-\frac{1}{2}}\right\},$$

where $C_p = \left(\frac{c}{p}\ln(2/p)\right)^{1/p}$. Indeed, it follows by well-known estimate of $\operatorname{vol}(\operatorname{absconv}\{x_1,\ldots,x_n\})$ obtained independently by Bárány-Füredy [B-F], Carl-Pajor [C-P], and Gluskin [G].

Proposition 4 *For every $p \in (0,1]$, for all integer $k = 1,\ldots,n$, and all projections P of rank k we have*

$$d(PB_p^n, B_2^k) \ge \frac{1}{C_p} \max\left\{1, \left(\frac{k}{\ln(1+\frac{n}{k})}\right)^{\frac{1}{p}-\frac{1}{2}}\right\},$$

where C_p is the same constant as in the lemma above.

Proof. Let $\mathcal{E}$ be an ellipsoid satisfying $PB_p^n \subset \mathcal{E}$ and d be the best constant such that

$$1/d\ \mathcal{E} \subset PB_p^n \subset \mathcal{E}.$$

Denote by v the isomorphism on $\mathbb{R}^k$ such that $v(\mathcal{E}) = B_2^k$ and define for all $i = 1, \ldots, n$, $x_i = v\, P\, e_i$. It is clear that for all $i = 1, \ldots, n$, $|x_i|_2 \le 1$ and that

$$1/d \le \left(\frac{\mathrm{vol}(v(PB_p^n))}{\mathrm{vol} B_2^k}\right)^{1/k} = \left(\frac{\mathrm{vol}(PB_p^n)}{\mathrm{vol}\mathcal{E}}\right)^{1/k}.$$

As $v(PB_p^n) = p\text{-absconv}\{x_1, \ldots, x_n\}$, we conclude applying Lemma 3 and remark after it, that

$$d \ge \frac{1}{C_p} \max\left\{1, \left(\frac{k}{\ln(1+\frac{n}{k})}\right)^{\frac{1}{p}-\frac{1}{2}}\right\}.$$

It is now enough to choose an ellipsoid $\mathcal{E}$ which realizes the distance from PB_p^n to the Euclidean ball. □

We would like to end this section with the version of the theorem dealing with the estimates for distance from projection of p-convex body to its convex hull. Recall that by strong form of Carathéodory theorem if K is a k-dimensional p-convex body then $\delta_K \le k^{1/p-1}$. Repeating the previous argument we obtain the following theorem.

Theorem 5 *There exists an absolute constant c such that for every p-convex body K in $\mathbb{R}^n$, $0 < p < 1$, for all integer $k \le n/2$, there exists a projection P of rank k such that*

$$\delta_{PK} \le c_p \max\left\{1, \left(\frac{k}{\ln(1+\frac{n}{k})}\right)^{\frac{1}{p}-1}\right\},$$

where $c_p \le (c/p)^{2/p^2}$.

Moreover, if $K = B_p^n$ then for all projection P of rank k, we have

$$\delta_{PK} \ge \frac{1}{C_p} \max\left\{1, \left(\frac{k}{\ln(1+\frac{n}{k})}\right)^{\frac{1}{p}-1}\right\},$$

where $C_p \le \left(\frac{c}{p}\ln(2/p)\right)^{1/p}$.

Remark 1. As above we do not need the restriction "origin is an interior point of K". Also, as it follows from the proof, there exists one projection which satisfies both estimates of Theorem 2 and 5.

Remark 2. The last part of Theorem 5 shows that one can not expect in general that there exists a projection of p-convex body of sufficiently large rank which is almost convex. It was known ([K2]) that to get "convex" projection (i.e. projection such that $\delta_{PK} \leq C_p$) the rank of the projection can not exceed $c_p \log n$. Thus the formula above is quantification of this observation. The similar question about sections of p-convex body remains open.

3 Entropy numbers in the p-convex case.

In this section we start by extending a few known convex results to the quasi-convex case. First, we extend result of Schütt [Sc] about entropy numbers of identity operator acting between ℓ_p and ℓ_r spaces (see also [Pi], [H] for the upper estimates), to the case $0 < p < r \leq \infty$. When $p < 1$, the upper estimates are already proved by Edmunds and Triebel [E-T]. The proof we give here is new and gives better dependence in p when p goes to 0.

Theorem 6 *Let $0 < p < r \leq \infty$ and n be an integer. Let $p' = \min\{1, p\}$ and $r' = \min\{1, r\}$. There are absolute positive constants c_0, c_1, c_2 such that for all integer k one has*

(i) if $k < [\log_2 n]$ then $e_k(\mathrm{id}^n_{p,r}) \in [1/2, 1]$.

(ii) if $[\log_2 n] \leq k \leq n$ then

$$c_0^{1/p'} f_1(n,k) \leq e_k(\mathrm{id}^n_{p,r}) \leq 2 \cdot 2^{1/r} C_p^{1/p-1/r} f_1(n,k),$$

where $f_1(n,k) = \left\{ \dfrac{\ln(1+\frac{n}{k})}{k} \right\}^{1/p-1/r}$ *and* $C_p = \frac{c_1}{p'} \ln\left(\frac{2}{p'}\right)$.

(iii) if $k > n$ then

$$c_2 \sqrt{r'/p'} f_2(n,k) \leq e_k(\mathrm{id}^n_{p,r}) \leq 2 \cdot 6^{1/r'} C_p^{1/p-1/r} f_2(n,k)$$

where $f_2(n,k) = 2^{-k/n} n^{1/r-1/p}$ and C_p as above.

The proofs of (i) and (iii) are standard, but we show it for completeness. In the case $k > c_3 n/p'$ one can replace the constant in front of $f_2(n,k)$ with $c_4 8^{1/p'}$. The case $p \geq 1$ holds by Theorem 1 of [Sc], so we restrict ourselves to the case $p < 1$. We will need two auxiliary lemmas which are essentially 12.1.11 and 12.1.12 of [Pi], adapted to the quasi-convex case. The proofs are identical and we only need to substitute the triangle inequality by the quasi-triangle one. All the spaces considered are $\mathbb{R}^n$ equipped with some quasi-norm. When there is some quasi-normed indexed by a parameter i, E_i denotes the space $(\mathbb{R}^n, \|\cdot\|_i)$.

Lemma 7 *For $i = 0, 1$, let $\|\cdot\|_i$ be a symmetric C_i-quasi-norm on $\mathbb{R}^n$ and for $\theta \in [0,1]$ assume that a quasi-norm $\|\cdot\|_\theta$ satisfies $\|x\|_\theta \le \|x\|_0^\theta \|x\|_1^{1-\theta}$ for all $x \in \mathbb{R}^n$. Then for every quasi-normed space F, for every linear operator $T : F \longrightarrow \mathbb{R}^n$, for every integer k, m one has*

$$e_{m+k-1}(T : F \longrightarrow E_\theta) \le (C_0 e_m(T : F \longrightarrow E_0))^\theta (C_1 e_k(T : F \longrightarrow E_1))^{1-\theta}.$$

Remark. This lemma holds true (even without the constants C_0 and C_1 in the last formula) for Gelfand numbers instead of entropy numbers.

Lemma 8 *Let $A > 0$, $\theta \in [0,1]$, and assume that quasi-norms $\|\cdot\|_0$, $\|\cdot\|_1$, $\|\cdot\|_\theta$ satisfy*

$$\inf\{a\|y\|_0 + b\|z\|_1 \,|\, y + z = x\} \le A a^\theta b^{1-\theta} \|x\|_\theta \tag{3}$$

for every $a \ge 0$, $b \ge 0$ and $x \in \mathbb{R}^n$. If F is a C-quasi-normed space then for every linear operator $T : \mathbb{R}^n \longrightarrow F$, for every integer k and m one has

$$e_{m+k-1}(T : E_\theta \longrightarrow F) \le AC \left(e_m\left(T : E_0 \longrightarrow F\right)\right)^\theta \left(e_k\left(T : E_1 \longrightarrow F\right)\right)^{1-\theta}.$$

Remark. We will use this lemma with $\theta = p < 1$, $E_0 = \ell_p^n$, $E_1 = F = \ell_\infty^n$, $E_\theta = \ell_1^n$. In this case, we can take $A = C = 2$. Indeed, let $x \in \mathbb{R}^n$, $a > 0$, $b > 0$ and assume, without loss of generality, that $x_i \ge 0$ for every i and $\sum x_i = 1$. Now, let

$$v = \min\left\{1, \left(\frac{a}{b}\frac{1-p}{p}\right)^p\right\}$$

and define y and z as follows, $z_i = \min\{v, x_i\}$ for every i, $y = x - z$. Then

$$a|y|_p + b|z|_\infty \le aN^{-1+1/p} + bv,$$

where $N = |\{i \,|\, x_i > v\}|$. Since $Nv \le 1$ and $(1-p)^{p-1}p^{-p} \le 2$, inequality (3) follows with $A = 2$.

Proof of Theorem 6.

Case 1. $k < [\log_2 n]$.

One trivially has $e_k\left(\mathrm{id}^n_{p,r}\right) \le 1$. Assume that $B^n_p \subset \cup_1^N x_i + \varepsilon B^n_r$ with $N \le 2^{k-1}$, then there exist $l \in \{1, \dots, N\}$ and two different vectors of the canonical basis e_i and e_j such that $e_i \in x_l + \varepsilon B^n_r$ and $e_j \in x_l + \varepsilon B^n_r$. Thus

$$2^{1/r} = |e_i - e_j|_r \le \varepsilon\, 2^{\max\{1,1/r\}}.$$

That proves the case (i).

Case 2. $[\log_2 n] \le k \le n$.

Upper estimate.

We start to estimate $e_k\left(\mathrm{id}^n_{p,\infty}\right)$. Let us denote by $A = A(n,p)$ the smallest constant which satisfies for all $1 \le k \le n$

$$e_k\left(\mathrm{id}^n_{p,\infty}\right) \le A \left\{\frac{\ln(1+\frac{n}{k})}{k}\right\}^{1/p}.$$

By Lemma 7 used with $F = E_0 = \ell^n_p$, $E_1 = \ell^n_\infty$, $\theta = p$, $E_\theta = \ell^n_1$ and $m = 1$, we get

$$e_k\left(\mathrm{id}^n_{p,1}\right) \le 4 e_k\left(\mathrm{id}^n_{p,\infty}\right)^{1-p}. \tag{4}$$

Factorizing identity from ℓ^n_p to ℓ^n_∞ through ℓ^n_1, we obtain by properties of entropy numbers and by (4)

$$e_k\left(\mathrm{id}^n_{p,\infty}\right) \le e_{[(1-p)k]}\left(\mathrm{id}^n_{p,1}\right) e_{[pk]}\left(\mathrm{id}^n_{1,\infty}\right) \le 4\left(e_{[(1-p)k]}\left(\mathrm{id}^n_{p,\infty}\right)\right)^{1-p} e_{[pk]}\left(\mathrm{id}^n_{1,\infty}\right).$$

It is well known that for all $k = 1, \dots, n$,

$$e_k\left(\mathrm{id}^n_{1,\infty}\right) \le c \min\left\{1, \frac{\ln(1+\frac{n}{k})}{k}\right\}.$$

Since for $a \in (0,1)$

$$\frac{\ln(1+\frac{n}{ak})}{ak} \le \min\left\{\frac{1}{a^2}, \frac{2}{a}\ln\left(\frac{2}{a}\right)\right\} \frac{\ln(1+\frac{n}{k})}{k},$$

we obtain

$$e_k\left(\mathrm{id}^n_{p,\infty}\right) \le 4c\left(\frac{1}{1-p}\right)^{2\,\frac{1-p}{p}} \frac{2}{p}\ln\left(\frac{2}{p}\right) A^{1-p}\left\{\frac{\ln(1+\frac{n}{k})}{k}\right\}^{1/p}$$

for every $[\log_2 n] \le k \le n$. Hence, by definition of A,

$$A \le 4c\left(\frac{1}{1-p}\right)^{2\,\frac{1-p}{p}} \frac{2}{p}\ln\left(\frac{2}{p}\right) A^{1-p} \le \frac{c_1}{p}\ln\left(\frac{2}{p}\right) A^{1-p},$$

which implies the desired estimate for $r=\infty$ and $k\le n$.

The proof of the upper estimate for every r and $k\le n$ is based on the interpolation inequality: if $0<p<q<r\le\infty$ and $\theta\in[0,1]$ be such that $\frac{\theta}{p}+\frac{1-\theta}{r}=\frac{1}{q}$ then for all $x\in\mathbb{R}^n$, $|x|_q\le|x|_p^\theta|x|_r^{1-\theta}$. Using Lemma 7 with $F=E_0=\ell_p^n$, $E_1=\ell_\infty^n$, $\theta=p/r$, $E_\theta=\ell_r^n$ and $m=1$ we obtain

$$e_k(\mathrm{id}_{p,r}^n)\le 2^{1+1/r}(e_k(\mathrm{id}_{p,\infty}^n))^{1-p/r}$$

which is the announced result.

Lower estimate.

It follows from Lemma 8 (and remark after it) that

$$e_{2k}(\mathrm{id}_{1,\infty}^n)\le 4(e_{2k}(\mathrm{id}_{p,\infty}^n))^p(e_1(\mathrm{id}_{\infty,\infty}^n))^{1-p}.$$

Applying properties of entropy numbers we obtain

$$e_{2k}(\mathrm{id}_{1,\infty}^n)^{1/p}\le 4^{1/p}e_{2k}(\mathrm{id}_{p,\infty}^n)\le 4^{1/p}e_k(\mathrm{id}_{p,r}^n)e_k(\mathrm{id}_{r,\infty}^n).$$

We deduce the lower bound for $e_k(\mathrm{id}_{p,r}^n)$ combining the upper bound obtained in the preceding part for $e_k(\mathrm{id}_{r,\infty}^n)$ and the well known estimate: $e_k(\mathrm{id}_{1,\infty}^n)\ge c\sqrt{k/\ln(1+n/k)}$ for these values of k.

Case 3. $k>n$**.**

This case follows from standard volume consideration. By (1) we have $e_{m+1}(\mathrm{id}_{r,r}^n)\le 3^{1/r'}2^{-m/n}$ for every m. Since

$$e_k(\mathrm{id}_{p,r}^n)\le e_n(\mathrm{id}_{p,r}^n)e_{k-n+1}(\mathrm{id}_{r,r}^n)$$

by the preceding result we obtain

$$e_k(\mathrm{id}_{p,r}^n)\le 2\cdot 6^{1/r'}\left\{\frac{c_1}{p}\ln\left(\frac{2}{p}\right)\right\}^{1/p-1/r}2^{-k/n}n^{1/r-1/p}.$$

On the other hand if $N(B_p^n,\varepsilon B_r^n)\le 2^{k-1}$ then by (1)

$$\varepsilon\ge\left(\frac{\mathrm{vol}\left(B_p^n\right)}{N\mathrm{vol}\left(B_r^n\right)}\right)^{1/n}\ge c\sqrt{\frac{\min\{1,r\}}{\min\{1,p\}}}\,2^{-k/n}\,n^{1/r-1/p},$$

where c is an absolute positive constant. This proves the theorem. □

Corollary 9 *Let $X=(\mathbb{R}^n,\|\cdot\|)$ be a quasi normed space and u be a linear operator from $\mathbb{R}^n$ to X. For all integers $k\le n$ and $m\ge 1$ one has*

$$e_{k+m-1}(u\ :\ell_p^n\to X)\le C_p\min\left\{1,\left(\frac{\ln(1+\frac{n}{k})}{k}\right)^{\frac{1}{p}-1}\right\}e_m(u\ :\ell_1^n\to X),$$

where $C_p\le\left(\frac{c}{p}\ln\left(\frac{2}{p}\right)\right)^{1/p}$ for an absolute constant $c>0$.

Remark. In some cases the estimate of $e_m(u : \ell_1 \to X)$ is well known. For example, the case of Banach space X with non-trivial type was studied in [Pis1] and in [C].

Proof. Let $u : \ell_p^n \to X$ and factorize it as $u = v\ \mathrm{id}_{p,1}^n$ where $v : \ell_1^n \to X$. By property of entropy numbers we obtain for all integers k and m

$$e_{k+m-1}(u\ : \ell_p^n \to X) \leq e_k(\mathrm{id}_{p,1}^n) e_m(v).$$

Now Corollary 9 follows from Theorem 6. □

Remark. Repeating the argument of Theorem 6 one can get the same upper estimate of Gelfand numbers $c_k(\mathrm{id}_{p,r}^n)$ for the case $0 < p < r \leq 2$. In particular, Corollary 9 remains true for Gelfand numbers instead of entropy numbers.

From this entropy estimate we shall deduce a good approximation from above for the volume of the p-convex hull of n points $x_1, \ldots, x_n$ in $\mathbb{R}^k$ as was stated in Lemma 3.

Proof of Lemma 3. Consider the operator $u : \mathbb{R}^n \to \mathbb{R}^k$ defined by $u(e_i) = x_i$ for all integer $i = 1, \ldots, n$. Let X be $(\mathbb{R}^k, \|\cdot\|)$, where unit ball of $\|\cdot\|$ is $K = \mathrm{absconv}\{x_1, \ldots, x_n\}$. By the previous corollary applied with $m = 1$, we have

$$\begin{aligned} e_k(u : \ell_p^n \to X) &\leq C_p \min\left\{1, \left(\frac{\ln(1+\frac{n}{k})}{k}\right)^{\frac{1}{p}-1}\right\} \|u : \ell_1^n \to X\| \\ &= C_p \min\left\{1, \left(\frac{\ln(1+\frac{n}{k})}{k}\right)^{\frac{1}{p}-1}\right\}. \end{aligned}$$

Clearly,

$$2\, e_k(u : \ell_p^n \to X) \geq \left(\frac{\mathrm{vol}\, u(B_p^n)}{\mathrm{vol}\, K}\right)^{1/k}.$$

Since $u(B_p^n) = p\text{-absconv}\{x_1, \ldots, x_n\}$, the result follows. □

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