ON THE DUAL FORM OF "LOW M^*-ESTIMATE" IN THE QUASI-CONVEX CASE.

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Abstract. Let B_2^n denote the Euclidean ball in $\mathbb{R}^n$, and, given closed star-shaped body $K \subset \mathbb{R}^n$, M_K denote the average of the gauge of K on the Euclidean sphere. Let $p \in (0,1)$ and let $K \subset \mathbb{R}^n$ be a p-convex body. In [17] we proved that for every $\lambda \in (0,1)$ there exists an orthogonal projection P of rank $(1-\lambda)n$ such that

$$\frac{f(\lambda)}{M_K} PB_2^n \subset PK,$$

where $f(\lambda) = c_p \lambda^{1+1/p}$ for some positive constant c_p depending on p only. In this note we prove that $f(\lambda)$ can be taken equal to $C_p \lambda^{1/p-1/2}$. In terms of Kolmogorov numbers it means that for every $k \le n$

$$d_k(\mathrm{Id} : \ell_2^n \longrightarrow (\mathbb{R}^n, \|\cdot\|_K)) \le C_p \frac{n^{1/p-1}}{k^{1/p-1/2}} \ell(\mathrm{Id} : \ell_2^n \longrightarrow (\mathbb{R}^n, \|\cdot\|_K)),$$

where $\ell(\mathrm{Id}) = \mathbf{E}\|\sum_{i=1}^n g_i e_i\|_K$ for the independent standard Gaussian random variables $\{g_i\}$ and the canonical basis $\{e_i\}$ of $\mathbb{R}^n$. All results do not require the symmetry of K.

1. Introduction and notations

In this note B_2^n denotes the Euclidean ball in $\mathbb{R}^n$, S^{n-1} denotes the Euclidean sphere. Given a closed star-shaped body K we denote by $\|\cdot\|_K$ its Minkowski functional (the gauge), i.e. $\|x\|_K = \inf\{t > 0 \mid x \in tK\}$. Note that K is the unit ball of K. By M_K we denote

$$\int_{S^{n-1}} \|x\|_K d\nu(x),$$

where $d\nu$ is the normalized Lebesgue measure on S^{n-1}.

Let $p \in (0,1)$. A closed body K is called p-convex body if for every positive λ, μ satisfying $\lambda^p + \mu^p \le 1$ and every x, $y \in K$ one has $\lambda x + \mu y \in K$. If K is a p-convex body then $\|\cdot\|_K$ is a homogeneous non-degenerate functional satisfying p-triangle inequality $\|x+y\|^p \le \|x\|^p + \|y\|^p$ for every x, $y \in \mathbb{R}^n$.

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Vice versa, if $\| \cdot \|$ is a homogeneous non-degenerate functional satisfying the p-triangle inequality then the unit ball of it is a p-convex body. A closed star-shaped body is called quasi-convex with quasi-convexity constant $C > 0$ if its Minkowski functional satisfies a quasi-triangle inequality $\|x + y\| \leq C(\|x\| + \|y\|)$ for every $x, y \in I\!R^n$. Clearly, any p-convex body is quasi-convex with the quasi-convexity constant $2^{-1+1/p}$. Moreover, any quasi-convex body can be approximated by a p-convex body. Aoki-Rolewicz theorem says that for every quasi-convex body K with quasi-convexity constant C there exists a p-convex body K_0, where $2^{-1+1/p} = C$, such that $K \subset K_0 \subset 2^{1/p}K$ ([12, 14, 28]). In view of this theorem we work with p-convex bodies only, but all results can be reformulated in terms of quasi-convex bodies.

It turns out that many crucial theorems of the Asymptotic Theory of Banach Spaces hold in the quasi-convex case. This is somehow surprising since the first proofs used convexity essentially. We refer to [1, 2, 3, 5, 6, 8, 9, 10, 11, 13, 22, 26] for different results in this direction. Of course, to extend the theory to the class of quasi-convex bodies we need to develop new methods and tools. In many cases such development led to the new direct proofs, which simplify also the convex case and thus gave the new understanding of the behavior of convex bodies (see e.g. [17, 18, 22]).

The purpose of this note is to investigate the dependence on p of function appearing in following theorem proved in [17].

Theorem 1.1. *Let $p \in (0, 1)$ and K be a p-convex body. Then for every $\lambda \in (0, 1)$ there exists an orthogonal projection P of rank $[(1 - \lambda)n]$ such that*

$$\frac{f(\lambda)}{C_p M_K} P B_2^n \subset PK,$$

where $f(\lambda) = \lambda^{1+1/p}$, $C_p = (c/p)^{c/p}$ for some absolute positive constant c.

Let us note that in the convex case this theorem is well-known. It is the dual form of the so-called "Low M^*-estimate" by Milman and plays an essential role in the Asymptotic Theory. For example it was a crucial step for obtaining such results as the Quotient of Subspace Theorem and the Reverse Brunn-Minkowski inequality by Milman ([23, 27]). In the convex case ($p = 1$) Theorem 1.1 was proved in [19] with some function $f(\lambda)$. In [20, 21] Milman improved the function $f(\lambda)$ to the polynomial one. Then Pajor and Tomczak-Jaegermann ([25]) were able to show that $f(\lambda)$ can be taken equal to $c\sqrt{\lambda}$, where c is an absolute positive constant. Although it was formulated for convex centrally-symmetric bodies (or for norms), all proofs hold in the non-symmetric case as well. Finally, Gordon ([4]) found a direct proof showing that the absolute constant c in $f(\lambda) = c\sqrt{\lambda}$ can be taken as $1 - \alpha_n$, where $\alpha_n \to 0$, as n grows to ∞. He also gave a precise measure estimate of the set of such projections. In particular, he showed that for $f(\lambda) = \sqrt{\lambda}/2$ this set has measure larger than $1 - e^{-c\lambda n}$, where c is an absolute positive constant.

In this note we show that in the p-convex case $f(\lambda)$ can be taken as $C_p \lambda^{1/p-1/2}$, where C_p is the positive constant, depending on p only.

Theorem 1.1 can be formulated in terms of the Kolmogorov numbers and the ℓ-functional. Recall, that the k-th Kolmogorov number of an operator $u : X \longrightarrow Y$ is

$$d_k(u) = \inf \{\|Qu\| \ : \ Q \text{ is a quotient map } Y \longrightarrow Y/S, \text{ where } \dim S < k\}.$$

The ℓ-functional of $u : \ell_2^n \longrightarrow X$ is $\mathbf{E}\| \sum_{i=1}^n g_i u e_i\|$, where g_i's are independent standard Gaussian random variables, $\{e_i\}$ is the canonical basis of $I\!R^n$. It is well-known (and can be directly calculated) that

$$\ell(\mathrm{Id} : \ell_2^n \longrightarrow (I\!R^n, \|\cdot\|_K))) = c_n \sqrt{n} M_K,$$

where

$$c_n = \frac{\sqrt{2}\, \Gamma\left((1+n)/2\right)}{\sqrt{n}\, \Gamma(n/2)}.$$

Note that $1 \geq c_n \longrightarrow 1$ as n tends to ∞.

Theorem 1.1 says that

$$d_k(\mathrm{Id} : \ell_2^n \longrightarrow (I\!R^n, \|\cdot\|_K)) \leq \ell(\mathrm{Id} : \ell_2^n \longrightarrow (I\!R^n, \|\cdot\|_K))/\left(\sqrt{n} f(k/n)\right).$$

2. The main result

Theorem 2.1. *Let $p \in (0,1)$ and K be a p-convex body. Then for every $\lambda \in (0,1)$ there exists an orthogonal projection P of rank $[(1-\lambda)n]$ such that*

$$(*) \qquad \frac{\lambda^{1/p-1/2}}{C_p M_K} P B_2^n \subset PK,$$

where $C_p = (c/p)^{(2-p)/p^2}$ for some absolute positive constant c.

Equivalently,

$$d_k(Id : \ell_2^n \longrightarrow (I\!R^n, \|\cdot\|_K)) \leq C_p \, \frac{n^{1/p-1}}{k^{1/p-1/2}} \, \ell(Id : \ell_2^n \longrightarrow (I\!R^n, \|\cdot\|_K)).$$

Remark 1. In fact “randomly” chosen projections satisfy $(*)$ with high probability. More precisely, let $G_{n,k}$ denote the Grassmanian of k-dimensional subspaces of $I\!R^n$. Let μ be the normalized Haar measure on $G_{n,k}$. Define the measure σ on the class of all orthogonal projections of rank k by

$$\sigma(A) = \mu\left(\{E \subset I\!R^n \ : \ \text{there is } P \in A \text{ with } PI\!R^n = E\}\right).$$

Then the measure of the set of projections satisfying $(*)$ is larger than $1-e^{-ck}$, where $k = [\lambda n]$ and c is an absolute positive constant.

Remark 2. Let M denote the median of K, i.e. the number satisfying $\nu(\{x \in S^{n-1} : \|x\| \geq M\}) \geq 1/2$ and $\nu(\{x \in S^{n-1} : \|x\| \leq M\}) \geq 1/2$. Then the theorem holds with M_K replaced by M with the same measure estimates as in Remark 1. Let us note that trivially $M_K \geq M/2$.

Remark 3. Using standard concentration inequalities (see e.g. [24], ch. 2) one can show that $M_K \leq (c/p)^{1/p} M$ for some absolute positive constant c. Moreover, using ideas of Latała ([15], see also [7]) it was shown in [16] that M_K is equivalent to the $M_{q,K}$ for $q \in (-p, \infty)$, where

$$M_{q,K} = \left(\int_{S^{n-1}} \|x\|_K^q d\nu \right)^{1/q}.$$

Namely, for every $s \geq 2$ and every $q \in (-p, -p/2)$ one has

$$\frac{1}{sC_p} M_{s,K} \leq M_{2,K} \leq C_p M_{-p/2,K} \leq C_p (p+q)^{1/q} M_{q,K},$$

where $C_p = (c/p)^{1/p}$ for some absolute positive constant c. Note also that $M_{q,K}$ in an increasing function of q.

We shall use the following result of Gordon and Kalton, which says that if the convex hull of a p-convex body is not far from the Euclidean ball then so is the body itself ([6], see also Lemma 1 of [8] and its proof for the precise estimates and for the non-symmetric case).

Lemma 2.2. *Let $p \in (0,1)$ and K be a p-convex body satisfying*

$$(1/d) B_2^n \subset \text{conv} K \subset B_2^n$$

for some positive number d. Then

$$(c/p)^{-(2-p)/p^2} (1/d)^{2(1/p - 1/2)} B_2^n \subset K \subset B_2^n$$

for an absolute positive constant c.

Proof of the theorem. By Remark 2 it is enough to prove the theorem for M, the median of K. Consider the p-convex body $K_0 = K \cap (1/M) B_2^n$. Clearly $\nu(\{x \in S^{n-1} : \|x\| = M\}) \geq 1/2$. Thus M is the median of K_0 and is the median of $K_1 :=$ convK_0. Then, by the corresponding "convex" result, for every $\lambda \in (0,1)$ with high probability there exists an orthogonal projection P of rank $[(1-\lambda)n]$ such that

$$\frac{\sqrt{\lambda}}{2M} P B_2^n \subset P K_1$$

(see [4] for the best possible constant and for the measure estimates). But $PK_1 = P$ conv$K_0 =$ convPK_0. Applying Lemma 2.2 to the body MPK_0, we get

$$(c/p)^{-(2-p)/p^2}\left(\sqrt{\lambda}/2\right)^{2(1/p-1/2)} PB_2^n \subset MPK_0,$$

where c is an absolute positive constant. Since $PK_0 \subset PK$, we obtain the desired result. □

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