

# Random section and random simplex inequality.

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## Abstract

Consider some convex body  $K \subset \mathbb{R}^d$ . Let  $X_1, \dots, X_k$ , where  $k \leq d$ , be random points independently and uniformly chosen in  $K$ , and let  $\xi_k$  be a uniformly distributed random linear  $k$ -plane. We show that for  $p \geq -d+k+1$ ,

$$\mathbb{E} |K \cap \xi_k|^{d+p} \leq c_{d,k,p} \cdot |K|^k \mathbb{E} |\text{conv}(0, X_1, \dots, X_k)|^p,$$

where  $|\cdot|$  and  $\text{conv}$  denote the volume of correspondent dimension and the convex hull. The constant  $c_{d,k,p}$  is such that for  $k > 1$  the equality holds if and only if  $K$  is an ellipsoid centered at the origin, and for  $k = 1$  the inequality turns to equality.

If  $p = 0$ , then the inequality reduces to the Busemann intersection inequality, and if  $k = d$  – to the Busemann random simplex inequality.

We also present an affine version of this inequality which similarly generalizes the Schneider inequality and the Blaschke-Grömer inequality.

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# 1 Introduction

## 1.1 Busemann intersection inequality

For  $d \in \mathbb{N}$  and  $k \in \{1, \dots, d\}$ , the linear Grassmannian of  $k$ -dimensional linear subspaces of  $\mathbb{R}^d$  is denoted by  $G_{d,k}$  and is equipped with a unique rotation invariant probabilistic Haar measure  $\nu_{d,k}$ . By  $|\cdot|$  we denote the  $d$ -dimensional volume. Given  $k \leq d$ , slightly abusing notation, considering the sets intersected with  $k$ -dimensional affine subspaces or the convex hulls of  $k + 1$  points, we denote the  $k$ -dimensional volume by  $|\cdot|$  as well.

The seminal Busemann intersection inequality states that for any convex compact set  $K \subset \mathbb{R}^d$  with non-empty interior (i.e., a convex body),

$$\int_{G_{d,k}} |K \cap L|^d \nu_{d,k}(dL) \leq \frac{\kappa_k^d}{\kappa_d^k} |K|^k, \quad (1)$$

where  $\kappa_k$  denotes the volume of the  $k$ -dimensional unit ball.

Originally Busemann [1] proved this inequality for  $k = d - 1$  and later it was generalized in [3, 10] for all  $k = 1, \dots, d - 1$ . If  $k > 1$ , then the equality holds if and only if  $K$  is an ellipsoid centered at the origin, and in this case (1) turns to the classical Fustenberg–Tzkonon formula [7].

Using the polar coordinates, it is easy to see that for  $k = 1$  the inequality turns to the equality:

$$\int_{G_{d,1}} |K \cap L|^d \nu_{d,1}(dL) = \frac{2^d}{\kappa_d} |K|. \quad (2)$$

Moreover, this equation can be generalized to other moments as follows:

$$\int_{G_{d,1}} |K \cap L|^{d+p} \nu_{d,1}(dL) = \frac{(d+p)2^{d+p}}{d\kappa_d} \int_{K^k} |\mathbf{x}|^p d\mathbf{x}, \quad p \geq -d + k + 1. \quad (3)$$

**Problem 1.1** *Is it possible for all  $k = 1, \dots, d - 1$  to obtain a generalization of (1) of the form*

$$\int_{G_{d,k}} |K \cap L|^{d+p} \nu_{d,k}(dL) \leq \dots$$

which turns to (3) for  $k = 1$ ?

Now let us consider an affine version of (1). To this end, denote by  $A_{d,k}$  the affine Grassmannian of  $k$ -dimensional affine subspaces of  $\mathbb{R}^d$  equipped with a unique measure  $\nu_{d,k}$  invariant with respect to the rigid motions in  $\mathbb{R}^d$  and normalized by

$$\mu_{d,k}(\{E \in A_{d,k} : E \cap \mathbb{B}^d \neq \emptyset\}) = \kappa_{d-k}.$$

Schneider [14] showed that

$$\int_{A_{d,k}} |K \cap E|^{d+1} \mu_{d,k}(dE) \leq \frac{\kappa_k^{d+1} \kappa_{d(k+1)}}{\kappa_d^{k+1} \kappa_{k(d+1)}} |K|^{k+1}, \quad (4)$$

and for  $k > 1$  the equality holds if and only if  $K$  is an ellipsoid.

As above, for  $k = 1$  the inequality turns to equality, although it is not as trivial as in the linear case, see [5] for  $d = 2$  and [12] for any  $d$ . As in the linear case, this equality can be generalized to other moments. It was done independently in [4, Eq. (21)] and [13, Eq. (34)]: for  $p \geq -d + k + 1$ ,

$$\int_{A_{d,1}} |K \cap E|^{p+d+1} \mu_{d,1}(dE) = \frac{(d+p)(d+p+1)}{2d\kappa_d} \int_{K^2} |\mathbf{x}_0 - \mathbf{x}_1|^p d\mathbf{x}_0 d\mathbf{x}_1. \quad (5)$$

**Problem 1.2** *As in the linear case, it is natural to ask: is it possible for all  $k = 1, \dots, d-1$  to obtain a generalization of (4) of the form*

$$\int_{A_{d,k}} |K \cap E|^{d+p+1} \mu_{d,k}(dE) \leq \dots,$$

which turns to (5) for  $k = 1$ ?

To conclude this section, let us note that Gardner [8] generalized (1) and (4) to bounded Borel sets and characterized the equality cases. Recently Dann, Paouris, and Pivovarov [6] extended (1), (4) to bounded integrable functions.

## 1.2 Busemann random simplex inequality

Another group of inequalities deals with the volume of the random simplex in a body. The classical Busemann random simplex inequality states that

$$|K|^{d+1} \leq (d+1)! \frac{\kappa_d^{d+1}}{2\kappa_{d+1}^{d-1}} \int_{K^d} |\text{conv}(0, \mathbf{x}_1, \dots, \mathbf{x}_d)| d\mathbf{x}_1 \dots d\mathbf{x}_d.$$

This inequality can be generalized (see, e.g., [15, Theorem 8.6.1.]) as follows: for every  $p \geq 1$ ,

$$|K|^{p+d} \leq (d!)^p \frac{\kappa_d^{p+d}}{\kappa_{d+p}^d} b_{d+p,d} \int_{K^d} |\text{conv}(0, \mathbf{x}_1, \dots, \mathbf{x}_d)|^p d\mathbf{x}_1 \dots d\mathbf{x}_d, \quad (6)$$

where for a non-integer  $p > 0$  we denote

$$\kappa_p := \frac{\pi^{p/2}}{\Gamma\left(\frac{p}{2} + 1\right)} \quad \text{and} \quad b_{q,k} := \binom{q}{k} \frac{\kappa_{q-k+1} \cdots \kappa_q}{\kappa_1 \cdots \kappa_k}. \quad (7)$$

The equality holds if and only if  $K$  is a centered ellipsoid.

**Problem 1.3** *Is it possible to obtain a generalization of (6) for a random simplex of arbitrary dimension  $k$ ?*

The affine counterpart of (6) is known as the Blaschke-Grömer inequality [11]: for every  $p \geq 1$ ,

$$|K|^{p+d+1} \leq (d!)^p b_{d+p,d} \frac{\kappa_d^{p+d+1}}{\kappa_{d+p}^{d+1}} \frac{\kappa_{(d+1)(d+p)}}{\kappa_{d(d+p+1)}} \int_{K^{d+1}} |\text{conv}(\mathbf{x}_0, \dots, \mathbf{x}_d)|^p d\mathbf{x}_0 \dots d\mathbf{x}_d. \quad (8)$$

The equality holds if and only if  $K$  is an ellipsoid.

**Problem 1.4** *As in the linear case, it is natural to ask: Is it possible to obtain a generalization of (8) for a random simplex of arbitrary dimension  $k$ ?*

The aim of this note is to derive a general inequality which simultaneously solves Problems 1.1 and 1.3 thus generalizing both Busemann inequalities in one form.

positively solve Problems 1.1 and 1.3 thus presenting a general form for both the Busemann intersection inequality and the Busemann random simplex inequality. We also present an affine version of the inequality which solves Problems 1.2 and 1.4 and thus implies the Schneider inequality and the Blaschke-Grömer inequality.

## 2 Main results

Our first theorem generalizes (1) and (6).

**Theorem 2.1.** *For any convex body  $K \subset \mathbb{R}^d$ ,  $k \in \{0, 1, \dots, d\}$ , and any real number  $p \geq -d + k + 1$ ,*

$$\int_{G_{d,k}} |K \cap L|^{p+d} \nu_{d,k}(dL) \leq (k!)^p \frac{\kappa_k^{d+p}}{\kappa_{d+p}^k} \frac{b_{d+p,k}}{b_{d,k}} \int_{K^k} |\text{conv}(0, \mathbf{x}_1, \dots, \mathbf{x}_k)|^p d\mathbf{x}_1 \dots d\mathbf{x}_k. \quad (9)$$

For  $k > 1$  the equality holds if and only if  $K$  is a non-degenerate ellipsoid centered at the origin.

**Remarks.**

1. Applying (9) with  $p = 0$  we obtain (1), while applying it with  $k = d$  we obtain (6).
2. It was shown in [9, Theorem 1.6] that if  $K$  is a non-degenerate ellipsoid centered at the origin, then one has the equality in (9).
3. In the probabilistic language it may be formulated as

$$\mathbb{E} |K \cap \xi_k|^{p+d} \leq (k!)^p \frac{\kappa_k^{d+p}}{\kappa_{d+p}^k} \frac{b_{d+p,k}}{b_{d,k}} |K|^k \mathbb{E} |\text{conv}(0, X_1, \dots, X_k)|^p,$$

where  $X_1, \dots, X_k$  are independently and uniformly distributed points in  $K$  and  $\xi_k$  is a random linear  $k$ -plane uniformly distributed in  $G_{d,k}$ .

Our second theorem generalizes (4) and (8).

**Theorem 2.2.** For any convex body  $K \subset \mathbb{R}^d$ ,  $k \in \{0, 1, \dots, d\}$ , and any real number  $p \geq -d + k + 1$ ,

$$\int_{A_{d,k}} |K \cap E|^{p+d+1} \mu_{d,k}(dE) \leq C(k, p, d) \int_{K^{k+1}} |\text{conv}(\mathbf{x}_0, \dots, \mathbf{x}_k)|^p d\mathbf{x}_0 \dots d\mathbf{x}_k, \quad (10)$$

where

$$C(k, p, d) = (k!)^p \frac{\kappa_k^{p+d+1}}{\kappa_{d+p}^{k+1}} \frac{\kappa_{(k+1)(d+p)}}{\kappa_{k(d+p)+k}} \frac{b_{d+p,k}}{b_{d,k}}.$$

For  $k > 1$  the equality holds if and only if  $K$  is a non-degenerate ellipsoid.

**Remarks.**

1. Applying (10) with  $p = 0$  we obtain (4), while applying it with  $k = d$  we obtain (8).
2. It was shown in [9, Theorem 1.4] that if  $K$  is a non-degenerate ellipsoid, then one has the equality in (10).

3. In probabilistic language (10) may be formulated as

$$\mathbb{E} |K \cap \eta_k|^{p+d+1} \leq C'(k, p, d) \frac{|K|^{k+1}}{V_{d-k}(K)} \mathbb{E} |\text{conv}(X_0, X_1, \dots, X_k)|^p,$$

where

$$C'(k, p, d) = \frac{d! (k!)^{p-1}}{(d-k)!} \frac{\kappa_d}{\kappa_{d-k}} \frac{\kappa_k^{p+d}}{\kappa_{d+p}^{k+1}} \frac{\kappa_{(k+1)(d+p)}}{\kappa_{k(d+p)+k}} \frac{b_{d+p,k}}{b_{d,k}},$$

$X_0, X_1, \dots, X_k$  are independently and uniformly distributed points in  $K$ ,  $\eta_k$  is uniformly distributed among all affine  $k$ -planes intersected  $K$ , and  $V_{d-k}$  is the  $(d-k)$ -th intrinsic volume of  $K$  defined by the Crofton formula [15, Theorem 5.1.1] as the normalized measure of all affine  $k$ -planes intersected  $K$ :

$$V_{d-k}(K) := \binom{d}{k} \frac{\kappa_d}{\kappa_k \kappa_{d-k}} \mu_{d,k}(\{E \in A_{d,k} : E \cap K \neq \emptyset\}).$$

## 3 Proofs

### 3.1 Blaschke–Petkantschin formula

Recall that  $b_{d,k}$  is defined by (7). Given points  $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^d$  we denote

$$V_k = V(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k) := |\text{conv}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k)|$$

and

$$V_{0,k} = V(\mathbf{x}_1, \dots, \mathbf{x}_k) := |\text{conv}(0, \mathbf{x}_1, \dots, \mathbf{x}_k)|.$$

In our further calculations we will need to integrate some non-negative measurable function  $h$  of  $k$ -tuples of points in  $\mathbb{R}^d$ . To this end, we first integrate over the  $k$ -tuples of points in a fixed  $k$ -dimensional linear subspace  $L$  and then we integrate over  $G_{d,k}$ . The corresponding transformation formula is known as the linear Blaschke–Petkantschin formula (see [15, Theorem 7.2.1]):

$$\int_{(\mathbb{R}^d)^k} h \, d\mathbf{x}_1 \dots d\mathbf{x}_k = (k!)^{d-k} b_{d,k} \int_{G_{d,k}} \int_{L^k} h V_{0,k}^{d-k} \lambda_L(d\mathbf{x}_1) \dots \lambda_L(d\mathbf{x}_k) \nu_{d,k}(dL), \quad (11)$$

where  $h = h(\mathbf{x}_1, \dots, \mathbf{x}_k)$ . The following is an affine counterpart of (11):

$$\int_{(\mathbb{R}^d)^{k+1}} h \, d\mathbf{x}_0 \dots d\mathbf{x}_k = (k!)^{d-k} b_{d,k} \int_{A_{d,k}} \int_{E^{k+1}} h V_k^{d-k} \lambda_E(d\mathbf{x}_0) \dots \lambda_E(d\mathbf{x}_k) \mu_{d,k}(dE), \quad (12)$$

where  $h = h(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k)$  (see [15, Theorem 7.2.7]).

### 3.2 Proof of Theorem 2.1

Let

$$J := \int_{K^k} V_{0,k}^p \, d\mathbf{x}_1 \dots d\mathbf{x}_k = \int_{(\mathbb{R}^d)^k} V_{0,k}^p \prod_{i=1}^k \mathbb{1}_K(\mathbf{x}_i) \, d\mathbf{x}_1 \dots d\mathbf{x}_k.$$

Applying the linear Blaschke–Petkantschin formula (11) with the function

$$h(\mathbf{x}_1, \dots, \mathbf{x}_k) := V_{0,k}^p \prod_{i=1}^k \mathbb{1}_K(\mathbf{x}_i),$$

we get

$$\begin{aligned} J &= (k!)^{d-k} b_{d,k} \int_{G_{d,k}} \int_{L^k} V_{0,k}^{p+d-k} \prod_{i=1}^k \mathbb{1}_K(\mathbf{x}_i) \lambda_L(d\mathbf{x}_1) \dots \lambda_L(d\mathbf{x}_k) \nu_{d,k}(dL) \\ &= (k!)^{d-k} b_{d,k} \int_{G_{d,k}} \int_{(K \cap L)^k} V_{0,k}^{p+d-k} \lambda_L(d\mathbf{x}_1) \dots \lambda_L(d\mathbf{x}_k) \nu_{d,k}(dL). \end{aligned} \quad (13)$$

Fix  $L \in G_{d,k}$ . Applying (6) with  $p + d - k$  and  $k$  instead of  $p$  and  $d$ , we obtain

$$(k!)^{p+d-k} \frac{\kappa_k^{d+p}}{\kappa_{d+p}^k} b_{d+p,k} \int_{(K \cap L)^k} V_{0,k}^{p+d-k} \lambda_L(d\mathbf{x}_1) \dots \lambda_L(d\mathbf{x}_k) \geq |K \cap L|^{p+d}, \quad (14)$$

which together with (13) implies (9).

Finally we consider the equality case. As was mentioned above, the equality holds for ellipsoids, see [9, Theorem 1.6]. Conversely, suppose that (9) turns to equality. Then it follows from (13) that (14) turns to equality for almost all  $L \in G_{d,k}$  which, in fact, means that it is true for *all*  $L \in G_{d,k}$ . Indeed, if for some  $L \in G_{d,k}$  we had a strict inequality in (14), then the same would be true for some neighborhood of  $L$  which would contradict to the fact that (14) turns to equality for almost all  $L \in G_{d,k}$ . Thus, according to the equality case in (6),  $K \cap L$  is a centered ellipsoid for all  $L \in G_{d,k}$ . Now it remains to apply the following lemma from [2, (16.12)]: if for some fixed  $k > 1$  for any  $E \in A_{d,k}$  passing through some fixed point from the interior of  $K$  the intersection  $K \cap E$  happens to be a  $k$ -dimensional ellipsoid, then  $K$  is an ellipsoid itself.  $\square$

### 3.3 Proof of Theorem 2.2

The proof is similar to the previous one. Let

$$J := \int_{K^{k+1}} V_k^p \, d\mathbf{x}_0 \dots d\mathbf{x}_k = \int_{(\mathbb{R}^d)^{k+1}} V_k^p \prod_{i=0}^k \mathbb{1}_{\mathcal{E}}(\mathbf{x}_i) \, d\mathbf{x}_0 \dots d\mathbf{x}_k.$$

Applying the affine Blaschke–Petkantschin formula (12) with the function

$$h(\mathbf{x}_0, \dots, \mathbf{x}_k) := |\operatorname{conv}(\mathbf{x}_0, \dots, \mathbf{x}_k)|^p \prod_{i=0}^k \mathbb{1}_{\mathcal{E}}(\mathbf{x}_i),$$

we get

$$\begin{aligned} J &= (k!)^{d-k} b_{d,k} \int_{A_{d,k}} \int_{E^{k+1}} V_k^{p+d-k} \prod_{i=0}^k \mathbb{1}_K(\mathbf{x}_i) \lambda_E(d\mathbf{x}_0) \dots \lambda_E(d\mathbf{x}_k) \mu_{d,k}(dE) \\ &= (k!)^{d-k} b_{d,k} \int_{A_{d,k}} \int_{(K \cap E)^{k+1}} V_k^{p+d-k} \lambda_E(d\mathbf{x}_0) \dots \lambda_E(d\mathbf{x}_k) \mu_{d,k}(dE). \end{aligned} \quad (15)$$

Fix  $E \in A_{d,k}$ . Applying (8) with  $p + d - k$  and  $k$  instead of  $p$  and  $d$ , we obtain

$$\begin{aligned} (k!)^{d-k+p} b_{d+p,k} \frac{\kappa_k^{p+d+1}}{\kappa_{d+p}^{k+1}} \frac{\kappa_{k(k+1)(d+p)}}{\kappa_{k(d+p+1)}} \int_{(K \cap E)^{k+1}} V_k^{p+d-k} \lambda_E(d\mathbf{x}_0) \dots \lambda_E(d\mathbf{x}_k) \\ \geq |K \cap E|^{d+p+1} \end{aligned}$$

which together with (15) implies (10). The equality case is treated the same way as in the linear case.  $\square$

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