Estimates for order statistics in terms of quantiles

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Abstract

Let $X_1, \ldots, X_n$ be independent non-negative random variables with cumulative distribution functions $F_1, F_2, \ldots, F_n$, each satisfying certain (rather mild) conditions. We show that the median of k-th smallest order statistic of the vector $(X_1, \ldots, X_n)$ is equivalent to the quantile of order $(k-1/2)/n$ with respect to the averaged distribution $F = \frac{1}{n}\sum_{i=1}^{n} F_i$.

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1 Introduction

The goal of this note is to provide sharp estimates for order statistics of independent, not necessarily identically distributed random variables, whose distributions satisfy certain (rather mild) conditions. Order statistics are among very important objects in probability and statistics with many applications. We refer to [AN, BC, DN] and references therein for information on the subject, especially in the case of i.i.d. random variables. The case of independent but not identically distributed random variables is less studied, we refer to [DN, Chapter 5] for some results in this direction. Understanding this setting is important in some applications, for example in connection with the Mallat–Zeitouni problem [MZ, LT], the study of asymptotic behaviour of some classes normed spaces [GLSW1], some problems in reconstruction [GLMP], to name a few.

Given $1 \leq k \leq n$ and a sequence of real numbers $a_1, a_2, \ldots, a_n$, let $k\text{-}\min_{i \leq n} a_i$ and $k\text{-}\max_{i \leq n} a_i$ denote its k-th smallest and k-th largest elements, in particular,

$$k\text{-}\min_{i \leq n} a_i = (n-k+1)\text{-}\max_{i \leq n} a_i.$$

Let F be cdf (cumulative distribution function) of a non-negative random variable. We employ the following condition:

$$\textit{there exists } K > 1 \textit{ such that } \frac{F(Kt)}{1-F(Kt)} \geq \frac{2F(t)}{1-F(t)} \textit{ for all } t > 0 \tag{1}$$

(see the next section for discussion and examples).

The main result of this note, Theorem 3.1, states that given $K > 1$, $1 \leq k \leq n$, and independent non-negative random variables $X_1, \ldots, X_n$ with cdf's $F_1, F_2, \ldots, F_n$, each satisfying condition (1) with parameter K, one has

$$K^{-10}\, q_F\left(\frac{k-1/2}{n}\right) \leq \mathrm{Med}\Big(k\text{-}\min_{1 \leq i \leq n} X_i\Big) \leq K^{13}\, q_F\left(\frac{k-1/2}{n}\right),$$

where $q_F(t)$ is the quantile of order t with respect to the averaged distribution $F = \frac{1}{n}\sum_{i=1}^n F_i$.

This result improves and complements the results from [GLSW2, GLSW3, GLSW4], where, under somewhat stronger conditions on distributions, the authors proved estimates for the corresponding expectations up to a factor logarithmic in k. More precisely, in [GLSW2, GLSW3] it was shown that given $\alpha, \beta, p > 0$, $1 \le k \le n$, real numbers $0 < x_1 \le x_2 \le \ldots \le x_n$, and independent random variables $\xi_1, \ldots, \xi_n$ satisfying

$$\forall t > 0 \quad \mathbb{P}\left(|\xi| \le t\right) \le \alpha t \qquad \text{and} \qquad \mathbb{P}\left(|\xi| > t\right) \le e^{-\beta t}$$

one has

$$\frac{1}{2^{1/p}\, 4\, \alpha} \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \le \left(\mathbb{E}\, k\text{-}\min_{1\le i\le n} |x_i \xi_i|^p\right)^{1/p} \le C(p,k)\, \beta^{-1} \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where $C(p,k) := C\max\{p, \ln(k+1)\}$, and C is an absolute positive constant. In [GLSW4] this was extended further to a larger class of distributions, namely it was shown that the expectation above is equivalent to some Orlicz norm of the sequence $(1/x_i)_i$, again up to a factor logarithmic in k.

We would also like to mention that order statistics of random vectors with independent but not identically distributed coordinates were studied in [Sen], where a result of Hoeffding [Ho] was used, in particular, to estimate the difference between the median of $k\text{-}\min_{1\le i\le n}(X_i)$ and the median of the k-th order statistic of a random vector with i.i.d. coordinates distributed according to the law $F = \frac{1}{n}\sum_{i=1}^n F_i$ (see also [DN, pp 96–97]). However, the results of [Sen] do not seem to directly imply the relations which we prove in Theorem 3.1.

2 Notation and preliminaries

Given a subset $A \subset \mathbb{N}$, we denote its cardinality by $|A|$. Next, for a natural number n and a set $E \subset \{1, 2, \ldots, n\}$, we denote by E^c the complement of E inside $\{1, 2, \ldots, n\}$. Similarly, for an event $\mathcal{E}$ we denote by $\mathcal{E}^c$ the complement of the event. Further, we say that a collection of sets $(A_j)_{j\le k}$ is a partition of $\{1, 2, \ldots, n\}$ if each A_j is non-empty, the sets are pairwise disjoint and their union is $\{1, 2, \ldots, n\}$. The canonical Euclidean norm and the canonical inner product in $\mathbb{R}^n$ will be denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively. We adopt the conventions $1/0 = \infty$ and $1/\infty = 0$ throughout the text.

Let ξ be a real-valued random variable. As usual, we use the abbreviation cdf for the cumulative distribution function (that is, the cdf of ξ is $F_\xi(t) = \mathbb{P}(\xi \le t)$). Given $r \in [0,1]$, by $q(r) = q_F(r) = q_\xi(r)$ we denote a quantile of order r, that is a number satisfying

$$\mathbb{P}\left\{\xi < q(r)\right\} \le r \qquad \text{and} \qquad \mathbb{P}\left\{\xi \le q(r)\right\} \ge r$$

(note that in general $q(r)$ is not uniquely defined).

Now we discuss our main condition on the distributions, the condition (1). Clearly, if the cdf of a non-negative random variable ξ satisfies condition (1) with some K then for every $x > 0$ the cdf of $x\xi$ satisfies (1) with the same K. Note that (1) is equivalent to

$$\mu\big((t, Kt]\big) \ge \mu\big([0,t]\big)\, \mu\big((Kt, \infty)\big), \quad t > 0, \tag{2}$$

where μ is the probability measure on $\mathbb{R}$ (actually, on $\mathbb{R}_+$) induced by F. It is not difficult to see that the uniform distribution on $[0,1]$ satisfies the condition (1) with $K = 2$. Another example of a random variable satisfying (1) (with $K = 2^{1/p}$) is a random variable ξ taking values in $[1,\infty)$ with $\mathbb{P}(\xi \geq 1) = 1/t^p$, $t \geq 1$, where $p > 0$ is a fixed parameter. Next we show that the absolute value of any log-concave random variable satisfies (1). In particular, this includes Gaussian and exponential distributions.

Lemma 2.1. *Let η be a log-concave variable. Then the cdf of $|\eta|$ satisfies* (1) *with $K = 3$.*

The lemma is an immediate consequence of the following statement and the fact that conditions (1) and (2) are equivalent.

Lemma 2.2. *Let μ_0 be a non-degenerate log-concave probability measure on $\mathbb{R}$ and let $t > 0$. Then*

$$\mu_0\big((t,\infty)\big)\,\mu_0\big((t,3t]\big) \geq \mu_0\big([-t,t]\big)\,\mu_0\big((3t,\infty)\big)$$

and

$$\mu_0\big((-\infty,-t)\big)\,\mu_0\big([-3t,-t)\big) \geq \mu_0\big([-t,t]\big)\,\mu_0\big((-\infty,-3t)\big).$$

In particular, we have

$$\begin{aligned}\mu\big((t,3t]\big) = \mu_0\big([-3t,-t)\cup(t,3t]\big) &\geq \mu_0\big([-t,t]\big)\,\mu_0\big((-\infty,-3t)\cup(3t,\infty)\big)\\ &= \mu\big([0,t]\big)\,\mu\big((3t,\infty)\big),\end{aligned}$$

where μ is defined by $\mu(S) := \mu_0(-S \cup S)$, $S \subset \mathbb{R}_+$.

Proof. We prove the first inequality only, the second one is similar. Note that

$$(t,\infty) = \tfrac{1}{2}\,[-t,\infty) + \tfrac{1}{2}\,(3t,\infty).$$

By log-concavity of μ_0 this implies

$${\mu_0}^2\big((t,\infty)\big) \geq \mu_0\big([-t,\infty)\big)\,\mu_0\big((3t,\infty)\big) = \big(\mu_0\big([-t,t]\big) + \mu_0\big((t,\infty)\big)\big)\,\mu_0\big((3t,\infty)\big).$$

Thus

$$\mu_0\big((t,\infty)\big)\ \big(\mu_0\big((t,\infty)\big) - \mu_0\big((3t,\infty)\big)\big) \geq \mu_0\big([-t,t]\big)\,\mu_0\big((3t,\infty)\big),$$

which implies the result. □

Remark 2.3. We would also like to notice that (1) implies that

$$F(t) \geq 2F(t/K^2), \qquad \text{whenever} \qquad F(t) \leq 1/2.$$

This (weaker) assumption on F was employed in [LT].

3 Main result

In this section we prove our main result, stating that medians of order statistics in case of independent components are equivalent to corresponding quantiles of an averaged distribution.

Theorem 3.1. *Let $K > 1$ and $k \le n$. Let $X_1, \dots, X_n$ be independent non-negative random variables with cdf's $F_1, F_2, \dots, F_n$, each satisfying condition* (1) *with parameter K. Set $F = \frac{1}{n}\sum_{i=1}^n F_i$. Then for $0 < t < K^{-5}$ one has*

$$\mathbb{P}\Big\{k\text{-}\min_{1\le i\le n} X_i < t\, q_F\Big(\frac{k-1/2}{n}\Big)\Big\} \le 4\, t^{1/(4\ln K)},$$

and for $t > K^5$ one has

$$\mathbb{P}\Big\{k\text{-}\min_{1\le i\le n} X_i > t\, q_F\Big(\frac{k-1/2}{n}\Big)\Big\} \le 4\, t^{-1/(6\ln K)}.$$

In particular,

$$K^{-10}\, q_F\left(\frac{k-1/2}{n}\right) \le \mathrm{Med}\Big(k\text{-}\min_{1\le i\le n} X_i\Big) \le K^{13}\, q_F\left(\frac{k-1/2}{n}\right).$$

In the proof of the theorem, we will use two following auxiliary statements.

Lemma 3.2. *Let $F : (0,\infty) \to [0,1]$ be a non-decreasing function satisfying* (1). *Let $\ell \ge 1$, $\gamma \in (0,1)$, and $t > 0$. Then*

$$F(t) \ge 2^\ell (1 - F(t))\, F\left(t/K^\ell\right) \tag{3}$$

and, assuming that $F(t) \ge 1 - \gamma$,

$$1 - F\left(t/K^\ell\right) \ge \frac{2^\ell}{2^\ell \gamma + 1}\,(1 - F(t)). \tag{4}$$

Proof. Applying (1) ℓ times we obtain

$$\frac{F(t)}{1-F(t)} \ge \frac{2^\ell\, F(t/K^\ell)}{1 - F(t/K^\ell)},$$

which implies (3). Fix a parameter $\beta \in (0,1)$, which will be specified later. If $F\left(t/K^\ell\right) \ge \beta$ then the above inequality implies

$$1 - F\left(t/K^\ell\right) \ge 2^\ell\, \beta\, (1 - F(t)).$$

Otherwise, if $F\left(t/K^\ell\right) < \beta$, we get

$$1 - F\left(t/K^\ell\right) \ge 1 - \beta \ge \frac{1-\beta}{\gamma}(1 - F(t)).$$

Choosing $\beta := 1/(2^\ell\gamma + 1)$, we get (4) and complete the proof. □

The next simple lemma can be verified by considering the expectation and the variance of the sum of random Bernoulli variables and using the Chebyshev inequality.

Lemma 3.3. *Let $\eta_1, \dots, \eta_n$ be independent Bernoulli $0/1$ random variables with probabilities of success $p_1, p_2, \dots, p_n$. Then for every $t > 0$ we have*

$$\mathbb{P}\Big\{\Big|\sum_{i=1}^n \eta_i - \sum_{i=1}^n p_i\Big| \geq t\Big\} \leq \frac{1}{t^2} \sum_{i=1}^n p_i.$$

Proof of Theorem 3.1. We start with the first bound. Take any positive $q < q_F\big(\frac{k-1/2}{n}\big)$. By definition of the quantile, we have $\sum_{i=1}^n F_i(q) \leq k - 1/2$. To estimate $k\text{-}\min_{i\leq n} X_i$ from below it is enough to show that the set of indices i corresponding to "small" X_i's has cardinality at most $k-1$.

Fix $\ell \geq 5$ such that $K^{-\ell-1} \leq t < K^{-\ell}$ and put $\gamma := 1/2^{\ell/2}$, $t_0 := q/K^\ell$. Further, set

$$I := \{i \leq n \,:\, F_i(q) < 1-\gamma\} \quad \text{and} \quad I^c := \{i \leq n \,:\, F_i(q) \geq 1-\gamma\}.$$

We want to estimate the number of indices $i \in I$ corresponding to "small" X_i. Denote

$$A := \sum_{i\in I} F_i(q) \quad \text{and} \quad B := \sum_{i\in I} F_i(t_0).$$

Applying (3) to F_i, $i \in I$, we get that $A \geq 2^\ell \gamma B$. Therefore, if $A \leq 2^\ell\gamma/(2^\ell\gamma - 1)$ then

$$\mathbb{P}\{\exists i \in I \,:\, X_i \leq t_0\} \leq \sum_{i\in I} \mathbb{P}\{X_i \leq t_0\} = B \leq \frac{A}{2^\ell\gamma} \leq \frac{1}{2^\ell\gamma - 1}.$$

If $A > 2^\ell\gamma/(2^\ell\gamma - 1)$ then, applying Lemma 3.3, we get

$$\begin{aligned}
\mathbb{P}\Big\{|\{i \in I : X_i \leq t_0\}| < A\Big\} &= \mathbb{P}\Big\{\sum_{i\in I} \chi_{\{X_i \leq t_0\}} < A\Big\} \\
&\geq 1 - \frac{B}{(A-B)^2} \\
&\geq 1 - \frac{2^\ell\gamma}{A(2^\ell\gamma - 1)^2} \\
&\geq 1 - \frac{1}{2^\ell\gamma - 1}.
\end{aligned}$$

Thus, in both cases we have

$$\mathbb{P}\Big\{|\{i \in I : X_i \leq t_0\}| < \sum_{i\in I} F_i(q)\Big\} \geq 1 - \frac{1}{2^\ell\gamma - 1}. \tag{5}$$

Next, we estimate the number of indices $i \in I^c$ corresponding to "small" X_i's. If $a := \sum_{i\in I^c}(1 - F_i(q)) < 1/2$, then we have

$$|I^c| < \frac{1}{2} + \sum_{i\in I^c} F_i(q) \leq k - \sum_{i\in I} F_i(q),$$

and from (5) we obtain

$$\mathbb{P}\Big\{|\{i \le n : \ X_i \le t_0\}| < k\Big\} \ge 1 - \frac{1}{2^\ell \gamma - 1}.$$

Now, assume that $a \ge 1/2$. Set $b := \sum_{i \in I^c}(1 - F_i(t_0))$. Applying (4) to F_i, $i \in I^c$, we get $b \ge \frac{2^\ell}{2^\ell \gamma + 1} a$. Note that $\sum_{i \in I^c} F_i(q) = |I^c| - a$. Therefore, by Lemma 3.3 we obtain

$$\begin{aligned}
\mathbb{P}\Big\{|\{i \in I^c : \ X_i \le t_0\}| < \sum_{i \in I^c} F_i(q)\Big\} &= \mathbb{P}\Big\{\sum_{i \in I^c} \chi_{\{X_i \le t_0\}} < |I^c| - a\Big\} \\
&= \mathbb{P}\Big\{b - \sum_{i \in I^c} \chi_{\{X_i > t_0\}} < b - a\Big\} \\
&\ge 1 - \frac{b}{(b-a)^2} \\
&\ge 1 - \frac{2 \cdot 2^\ell (2^\ell \gamma + 1)}{(2^\ell - 2^\ell \gamma - 1)^2}.
\end{aligned}$$

Combining the last relation with (5) and using that $\sum_{i \le n} F_i(q) \le k - 1/2 < k$, we obtain

$$\mathbb{P}\Big\{|\{i \le n : \ X_i \le t_0\}| < k\Big\} \ge 1 - \frac{1}{2^\ell \gamma - 1} - \frac{2 \cdot 2^\ell (2^\ell \gamma + 1)}{(2^\ell - 2^\ell \gamma - 1)^2} \ge 1 - \frac{4}{2^{\ell/2}},$$

where in the last inequality we used the assumption $\ell \ge 5$ and the identity $\gamma = 1/2^{\ell/2}$. This proves

$$\mathbb{P}\Big\{k\text{-}\min_{1 \le i \le n} X_i \le q/K^\ell\Big\} \le \frac{4}{2^{\ell/2}}.$$

Finally, by the choice of ℓ we have $\ell \ge (4\ln(1/t))/(5\ln K)$, which implies the first part of the theorem.

The second part is somewhat similar. To make comparison with the first part of the proof straightforward, we will use the same letters for corresponding sets or numbers, just adding a bar. Let $\bar{q} := q_F\left(\frac{k-1/2}{n}\right)$. By definition, we have $\sum_{i=1}^n F_i(\bar{q}) \ge k - 1/2$. To estimate $k\text{-}\min_{i \le n} X_i$ from above we will show that the set of indices i corresponding to "small" X_i typically has cardinality at least k. Fix $\bar{\ell} \ge 5$ such that such that $K^{\bar{\ell}} \le t < K^{\bar{\ell}+1}$, and set $\bar{\gamma} := 2/4^{\bar{\ell}/3}$, $\bar{t}_0 := K^{\bar{\ell}} \bar{q}$. Further, let

$$\bar{I} := \{i \le n \ : \ F_i(\bar{t}_0) < 1 - \bar{\gamma}\} \quad \text{and} \quad \bar{I}^c := \{i \le n \ : \ F_i(\bar{t}_0) \ge 1 - \bar{\gamma}\}.$$

Let us bound the number of indices $i \in \bar{I}$ corresponding to "small" X_i. Denote

$$\bar{A} := \sum_{i \in \bar{I}} F_i(\bar{q}) \quad \text{and} \quad \bar{B} := \sum_{i \in \bar{I}} F_i(\bar{t}_0).$$

Assume that $\bar{A} \geq 1/2$. Applying Lemma 3.3, we get

$$\begin{aligned}
\mathbb{P}\big\{|\{i \in \bar{I} : X_i \leq \bar{t}_0\}| > \bar{A}\big\} &= \mathbb{P}\Big\{\sum_{i \in \bar{I}} \chi_{\{X_i \leq \bar{t}_0\}} > \bar{A}\Big\} \\
&= \mathbb{P}\Big\{\bar{B} - \sum_{i \in \bar{I}} \chi_{\{X_i \leq \bar{t}_0\}} < \bar{B} - \bar{A}\Big\} \\
&\geq 1 - \frac{\bar{B}}{(\bar{B} - \bar{A})^2} \\
&\geq 1 - \frac{2^{\bar{\ell}+1}\,\bar{\gamma}}{(2^{\bar{\ell}}\,\bar{\gamma} - 1)^2},
\end{aligned}$$

where we used the estimate $\bar{B} \geq 2^{\bar{\ell}}\bar{\gamma}\bar{A}$, which follows from (3). Thus, in both cases $\bar{A} < 1/2$ and $\bar{A} \geq 1/2$ we have

$$\mathbb{P}\Big\{|\{i \in \bar{I} : X_i \leq \bar{t}_0\}| > \sum_{i \in \bar{I}} F_i(\bar{q}) - 1/2\Big\} \geq 1 - \frac{2^{\bar{\ell}+1}\,\bar{\gamma}}{(2^{\bar{\ell}}\,\bar{\gamma} - 1)^2}. \tag{6}$$

Next, we estimate the number of indices $i \in \bar{I}^c$ corresponding to "small" X_i's. Fix

$$\lambda := \sqrt{2^{\bar{\ell}}(2^{\bar{\ell}}\bar{\gamma} + 1)}/(2^{\bar{\ell}} - 2^{\bar{\ell}}\bar{\gamma} - 1) < 1.$$

If $\bar{a} := \sum_{i \in \bar{I}^c}(1 - F_i(\bar{q})) < \lambda$, then $|\bar{I}^c| < \lambda + \sum_{i \in \bar{I}^c} F_i(\bar{q})$. In this situation we have

$$|\{i \in \bar{I}^c : X_i \leq \bar{t}_0\}| \geq \sum_{i \in \bar{I}^c} F_i(\bar{q})$$

if and only if $X_i \leq \bar{t}_0$ for all $i \in \bar{I}^c$. Note also that for every $a_1, \ldots, a_m \in (0, 1]$ one has

$$\text{if} \quad \sum_{i=1}^m a_i \geq m - \lambda \quad \text{then} \quad \prod_{i=1}^m a_i \geq 1 - \lambda.$$

This and independence of X_i's imply

$$\mathbb{P}\Big\{|\{i \in \bar{I}^c : X_i \leq \bar{t}_0\}| \geq \sum_{i \in \bar{I}^c} F_i(\bar{q})\Big\} = \prod_{i \in \bar{I}^c} \mathbb{P}\big\{X_i \leq \bar{t}_0\big\} \geq \prod_{i \in \bar{I}^c} \mathbb{P}\big\{X_i \leq \bar{q}\big\} \geq 1 - \lambda.$$

Together with (6), it gives

$$\mathbb{P}\Big\{|\{i \leq n : X_i \leq \bar{t}_0\}| > \sum_{i=1}^n F_i(\bar{q}) - 1/2\Big\} \geq 1 - \frac{2^{\bar{\ell}+1}\,\bar{\gamma}}{(2^{\bar{\ell}}\,\bar{\gamma} - 1)^2} - \lambda.$$

It remains to consider the case $\bar{a} \geq \lambda$. Set

$$\bar{b} := \sum_{i \in \bar{I}^c}(1 - F_i(\bar{t}_0)).$$

Applying (4) to F_i, $i \in \bar{I}^c$, we get that $\bar{a} \geq \frac{2^{\bar{\ell}}}{2^{\bar{\ell}}\bar{\gamma}+1}\bar{b}$. Note that $\sum_{i\in\bar{I}^c} F_i(\bar{q}) = |\bar{I}^c| - \bar{a}$. Therefore, by Lemma 3.3, we obtain

$$\begin{aligned}\mathbb{P}\Big\{|\{i \in \bar{I}^c:\ X_i \leq \bar{t}_0\}| > \sum_{i\in\bar{I}^c} F_i(\bar{q})\Big\} &= \mathbb{P}\Big\{\sum_{i\in\bar{I}^c} \chi_{\{X_i\leq\bar{t}_0\}} > |\bar{I}^c| - \bar{a}\Big\}\\ &= \mathbb{P}\Big\{\sum_{i\in\bar{I}^c} \chi_{\{X_i>\bar{t}_0\}} - \bar{b} < \bar{a} - \bar{b}\Big\}\\ &\geq 1 - \frac{\bar{b}}{(\bar{a}-\bar{b})^2}\\ &\geq 1 - \frac{2^{\bar{\ell}}(2^{\bar{\ell}}\bar{\gamma}+1)}{(2^{\bar{\ell}} - 2^{\bar{\ell}}\bar{\gamma} - 1)^2\,\lambda}.\end{aligned}$$

Combining this with (6), we obtain

$$\mathbb{P}\Big\{|\{i \leq n:\ X_i \leq \bar{t}_0\}| > \sum_{i=1}^n F_i(\bar{q}) - 1/2\Big\} \geq 1 - \frac{2^{\bar{\ell}+1}\,\bar{\gamma}}{(2^{\bar{\ell}}\,\bar{\gamma}-1)^2} - \frac{2^{\bar{\ell}}(2^{\bar{\ell}}\bar{\gamma}+1)}{(2^{\bar{\ell}} - 2^{\bar{\ell}}\bar{\gamma} - 1)^2\,\lambda}.$$

Since $\bar{\ell} \geq 5$ and in view of the definitions of $\bar{\gamma}$ and λ, in both cases $\bar{a} < \lambda$ and $\bar{a} \geq \lambda$ one has

$$\mathbb{P}\Big\{|\{i \leq n:\ X_i \leq \bar{t}_0\}| > \sum_{i=1}^n F_i(\bar{q}) - 1/2\Big\} \geq 1 - \frac{4}{2^{\bar{\ell}/3}}.$$

Note that $\sum_{i=1}^n F_i(\bar{q}) - 1/2 \geq k - 1$, thus the last estimate implies

$$\mathbb{P}\Big\{k\text{-}\min_{1\leq i\leq n} X_i > K^{\bar{\ell}}\,\bar{q}\Big\} \leq \frac{4}{2^{\bar{\ell}/3}}.$$

Finally, observe that by the choice of $\bar{\ell}$ we have $\bar{\ell} \geq (4\ln t)/(5\ln K)$, which implies the second estimate in the theorem. □

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