On approximation by projections of polytopes with few facets

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Abstract

We provide an affirmative answer to a problem posed by Barvinok and Veomett in [4], showing that in general an n-dimensional convex body cannot be approximated by a projection of a section of a simplex of sub-exponential dimension. Moreover, we prove that for all $1 \leq n \leq N$ there exists an n-dimensional convex body B such that for every n-dimensional convex body K obtained as a projection of a section of an N-dimensional simplex one has

$$d(B, K) \geq c \sqrt{\frac{n}{\ln \frac{2N \ln(2N)}{n}}},$$

where $d(\cdot, \cdot)$ denotes the Banach-Mazur distance and c is an absolute positive constant. The result is sharp up to a logarithmic factor.

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1 Introduction

One of the standard ways to describe a convex body in computational geometry is the *membership oracle*. The membership oracle of a body $K \subset \mathbb{R}^n$

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is an algorithm, which, given a point $x \in \mathbb{R}^n$, outputs whether $x \in K$, or $x \notin K$. If such oracle is constructed, and if the body K has a relatively well-conditioned position, meaning that $rB_2^n \subset K \subset RB_2^n$ with $R/r \leq n^C$, then one can construct efficient probabilistic algorithms for estimating the volume of K, its inertia ellipsoid, and other geometric characteristics (see e.g. [10] and [22]). Yet, constructing an efficient membership oracle for a given convex body may be a hard problem [4]. Because of this, it is important to know whether a convex body can be approximated by another body, for which the membership oracle can be efficiently constructed. One natural class of convex bodies for which the construction of the membership oracle is efficient is the projections of a polytope with a few faces. Such polytopes can be realized as projections of sections of a simplex in a dimension comparable to n. This construction is discussed in details in [4]. In particular, the following problem was posed (Problem 4.7.2 in [4]).

Problem. *Let $K \subset \mathbb{R}^n$ be a symmetric convex body and let $P \subset \mathbb{R}^n$ be a projection of a polytope with N facets, which approximates K within a factor of 2. Is it true that in the worst case the number N should be at least exponential in d: $N \geq e^{cn}$ for some absolute constant $c > 1$?*

Note that if $K = B_p^n$ is the unit ball of ℓ_p^n, then this approximation requires only proportional dimension. To see it recall that a $(2n)$-dimensional simplex possesses a cubic section of dimension n. Since a random projection of such a section is isomorphic to an ellipsoid, we obtain an approximation of the Euclidean ball by a projection of a section of a simplex in a dimension proportional to n. Another deterministic construction of such an approximation was found by Ben-Tal and Nemirovski [5]. A similar construction can be used to approximate all balls B_p^n for $2 \leq p < \infty$. Since the polar of a simplex is a simplex, one can also approximate the balls B_p^n for $1 \leq p \leq 2$. (Also, modifications of these constructions give explicit symmetric "conical subsets" of the proportionally dimensional cube, whose linear projections can arbitrarily close approximate the balls B_p^n for $1 \leq p < \infty$, see [11] for the details.) Moreover, even the existence of an n-dimensional convex body, which cannot be approximated by a projection of a section of a simplex Δ_N with N proportional to n has been an open problem.

The main result of this paper provides an affirmative solution to the Barvinok problem above. Furthermore, we prove a lower estimate for the minimal Banach–Mazur distance between a certain convex symmetric body and a projection of a polytope with N facets. This estimate is optimal for all $N > n$ up to logarithmic terms.

Theorem 1.1. *Let $n \le N$. There exists an n-dimensional convex symmetric body B, such that for every n-dimensional convex body K obtained as a projection of a section of an N-dimensional simplex one has*

$$d(B, K) \ge c\sqrt{\frac{n}{\ln \frac{2N\ln(2N)}{n}}},$$

where c is an absolute positive constant.

Let us note here that any projection of a section of a simplex can be realized as a section of a projection of a simplex and vice versa (see the next section). Thus, Theorem 1.1 holds for bodies K obtained as a section of a projection of a simplex as well.

To see that the estimate of Theorem 1.1 is close to optimal, recall that Barvinok proved in [3] that for every $N \ge 8n$ and every symmetric convex body B in $\mathbb{R}^n$ there exists a section K of an N-dimensional simplex such that

$$d(B, K) \le C \max\left\{1, \sqrt{\frac{n}{\ln N} \cdot \ln \frac{n}{\ln N}}\right\}.$$

Comparison of these two bounds shows that working with projections of sections of a simplex, as opposed to using sections alone, does not significantly improve the approximation. This is in stark contrast with the situation described in the Quotient of a Subspace Theorem. Recall that the Quotient of a Subspace Theorem of Milman ([16], see also [17] and [18] for the non-symmetric case) states that *given $\theta \in (0, 1)$ and an n-dimensional convex body K there exists a projection of a section of K whose dimension is greater than θn and whose Banach-Mazur distance to the Euclidean ball of the corresponding dimension does not exceed $C(\theta)$ (moreover, $C(\theta)$ can be chosen such that $C(\theta) \to 1$ as $\theta \to 0^+$).* On the other hand, it is well-known by a volumetric argument (see Fact 2.2 below) that any n-dimensional section of the N-dimensional cube (or simplex) is at the distance at least $c\sqrt{n/\ln(2N/n)}$ from the n-dimensional Euclidean ball. Thus, in the case of the cube (or simplex) and proportional subspaces/projections, taking just sections leads to $c\sqrt{n}$ distance to the Euclidean ball, while adding one more operation – taking a projection – yields the distance bounded by an absolute constant.

Our result also shows that Quotient of a Subspace Theorem cannot be extended much beyond the Euclidean setting. Even if we start with the simplest (in terms of complexity) convex body – simplex – we cannot obtain an arbitrary convex set by taking a projection of a section. Similar phenomena – that many results of Asymptotic Geometric Analysis cannot be extended much beyond the Euclidean setting were discussed in [12].

It would be interesting to characterize the class of all n-dimensional convex bodies, that can be realized (up to a Banach-Mazur distance less than or equal to 2, say) as a projection of a section of an N-dimensional simplex for $N = O(n)$. As we mentioned above any B_p^n is in this class, clearly any polytope with $O(n)$ vertices or faces is in this class as well. In a related direction we conjecture that there is no convex body K such that an arbitrary body can be obtained (up to Banach-Mazur distance bounded by a constant) from K by taking a projection of a section.

Finally we would like to mention that many aspects of computational complexity of convex bodies were discussed in [21].

The paper is organized as follows. In the next section we introduce notation and auxiliary results, that will be used latter. We also describe a class of random polytopes crucial for our construction in which we will find our example. We model these polytopes on random polytopes introduced by Gluskin in [8]. In Section 3 we prove the main theorem, Theorem 1.1. The proof of this theorem uses Theorem 2.3, which states that with high probability two Gluskin's polytopes are on large Banach-Mazur distance to each other. The last section is devoted to the proof of Theorem 2.3.

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2 Notation and Preliminaries

By $|\cdot|$ and $\langle\cdot,\cdot\rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^d$. B_2^d and S^{d-1} stand for the Euclidean unit ball and the unit sphere, respectively; the standard basis of $\mathbb{R}^d$ is denoted by $e_1, \dots, e_d$.

As usual, $\|\cdot\|_p$, $1 \le p \le \infty$, denotes the ℓ_p-norm, i.e. for every $x =$

$(x_i)_{i=1}^d \in \mathbb{R}^d$

$$\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p} \quad \text{for} \quad p < \infty, \qquad \|x\|_\infty = \sup_{i \le d} |x_i|,$$

and $\ell_p^d = (\mathbb{R}^d, \|\cdot\|_p)$. The unit ball of ℓ_p^d is denoted by B_p^d.

Recall that $\lceil x \rceil$ denotes the smallest integer which is not less than x.

By a convex body we mean a compact set with a non-empty interior. For a convex body $K \subset \mathbb{R}^d$ with 0 in its interior, the Minkowski functional of K is

$$\|x\|_K = \inf\{\lambda > 0 \mid x \in \lambda K\},$$

i.e. it is the homogeneous convex functional, whose unit ball is K. The polar of K is

$$K^\circ = \{x \mid \langle x, y\rangle \le 1 \quad \text{for all} \quad y \in K\}.$$

Note that if K is symmetric, then K° is the unit ball of the space dual to $(\mathbb{R}^d, \|\cdot\|_K)$.

It is well known that for any convex body $K \subset \mathbb{R}^d$ there exists a point $a \in K$ such that

$$-(K - a) \subset d(K - a). \tag{1}$$

For example the center of the maximal volume ellipsoid contained in K satisfies this ([9], see also [1]).

Given a subset $K \subset \mathbb{R}^d$ the convex hull and the absolute convex hull of K are denoted by $\mathrm{conv}(K)$ and $\mathrm{absconv}(K) = \mathrm{conv}(K \cup -K)$ respectively. The volume of K is denoted by $\mathrm{vol}\,(K)$.

A position of a convex body K is a non-degenerate affine image of K.

For two convex bodies K_1 and K_2 in $\mathbb{R}^d$ the Banach-Mazur distance between them is defined as

$$d(K_1, K_2) = \inf\{\lambda > 0 \mid K_1 - a \subset T(K_2 - b) \subset \lambda(K_1 - a)\},$$

where infimum is taken over all non-degenerate linear operators $T : \mathbb{R}^d \to \mathbb{R}^d$ and all $a, b \in \mathbb{R}^d$. Note that if $K_1 = -K_1$ and $K_2 = -K_2$ then a, b can be taken equal to 0. The distance $d(\cdot,\cdot)$ satisfies the multiplicative triangle inequality, i.e. $d(K_1, K_2) \le d(K_1, K_3) d(K_3, K_2)$.

We fix the following notation.

$$S := S(N) = \left\{x = \{x_i\}_{i=1}^{N+1} \in \mathbb{R}^{N+1} \mid x_i \ge 0, i \le N+1\right\},$$

$$H := H(N) = \left\{ x = \{x_i\}_{i=1}^{N+1} \in \mathbb{R}^{N+1} \;\mid\; \sum_{i=1}^{N+1} x_i = 1 \right\},$$

and

$$\Delta = \Delta_N := S \cap H.$$

Note that $\Delta = \mathrm{conv}\{e_i\}_{i=1}^{N+1}$ is an N-dimensional regular simplex.

As we mentioned in the introduction, any projection of a section of a simplex can be realized as a section of a projection of a simplex and vice versa. Indeed, let E and F be linear subspaces, denote $L = E \cap F$. If $K = (P_E \Delta_{N+1}) \cap F$ then $K = (P_E \Delta_{N+1}) \cap L$ and it is easy to check that $K = P_L(\Delta_{N+1} \cap \tilde{L})$, where $\tilde{L} = L \oplus E^{\perp}$. If $K = P_E(\Delta_{N+1} \cap F)$ then $K = (P_{\bar{L}} \Delta_{N+1}) \cap L$, where $\bar{L} = L \oplus F^{\perp}$. The case of affine subspaces E and F is similar.

Recall that a set $F \subset \mathbb{R}^{N+1}$ is an affine subspace if there exists $b \in \mathbb{R}^{N+1}$ such that $F - b$ is a linear subspace of $\mathbb{R}^{N+1}$. Given a set $K \subset \mathbb{R}^{N+1}$ and an affine subspace $F \subset \mathbb{R}^{N+1}$ the section of K by F is denoted by

$$K^F = K \cap F.$$

In particular,

$$\Delta^F = \Delta_N^F = \Delta_N \cap F \quad \text{ and } B_2^F = B_2^{N+1} \cap F.$$

For a metric space (X, ρ) and $\varepsilon > 0$ an ε-net $\mathcal{N}$ is a subset of X such that for every x in X there exists $x_0 \in \mathcal{N}$ satisfying $\rho(x, x_0) \le \varepsilon$.

Let $k \le d$. By $O(d)$ we denote the group of orthogonal operators on $\mathbb{R}^d$ and by $G_{d,k}$ we denote the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^d$ endowed with the distance

$$\rho(E, F) = \inf\{\|U - I\| \mid U \in O(d), UE = F\},$$

where $\|\cdot\|$ denotes the operator norm $\ell_2^d \to \ell_2^d$.

We will use the following result of Szarek ([19, 20]) on the size of ε-nets on $G_{d,k}$.

Theorem 2.1. *Let $k \le d$ and $\varepsilon \in (0,1)$. There exists an ε-net on $G_{d,k}$ with respect to $\rho(\cdot)$ of cardinality not exceeding $(C/\varepsilon)^{Cdk}$, where C is an absolute positive constant.*

Volume estimates play an important role in the theory. Let us recall the following fundamental result ([2, 6, 7]).

Fact 2.2. *Let $M \geq 2d$ be integers. For arbitrary vectors $x_1, \ldots, x_M \in S^{d-1}$ the volume of the absolute convex hull satisfies*

$$\mathrm{vol}(\mathrm{absconv}\{x_1, \ldots, x_M\}) \leq \left(C \frac{\sqrt{\ln(M/d)}}{d} \right)^d ,$$

where C is a positive absolute constant.

The proof of existence of convex bodies that are poorly approximated by projections of sections of a simplex uses a modification of bodies introduced by Gluskin in [8]. This probabilistic construction and its further versions became the main source of counterexamples in asymptotic geometric analysis [15]. However, most polytopes described in the literature have the number of random vertices M proportional to d, while we want M to be arbitrary satisfying $2d \leq M \leq e^d$. To keep this paper self-contained we show an existence with a direct argument.

Let $d \geq 1$ and $2d \leq M \leq e^d$ be integers. Set

$$\ell = \lceil \log_5(M/d) \rceil ,$$

and let $\{1, \ldots, d\} = \bigcup_{k=1}^{\lceil d/\ell \rceil} I_k$ be the decomposition of $\{1, \ldots, d\}$ into the disjoint union of consecutive intervals, with each interval, except possibly the last one, consisting of ℓ numbers. For each $1 \leq k \leq \lceil d/\ell \rceil$ choose a $(1/2)$-net $\mathcal{N}_k \subset S^{d-1} \cap \mathbb{R}^{I_k}$ of cardinality at most 5^ℓ. (It is well known that such a net exists, cf. Lemma 4.3 below; moreover, one can show that such a net can be taken symmetric about the origin.)

Recall that $\mathbb{P}$ is the rotation invariant probability measure on the Euclidean unit sphere S^{d-1}. (We may also denote this probability space by $(\Omega, \mathbb{P})$.) Let X be a random vector uniformly distributed on S^{d-1}, and let $X_1, \ldots, X_M$ be independent copies of X. Then we define Gluskin's polytope $V \subset \mathbb{R}^d$ by

$$V = \mathrm{absconv} \left\{ \bigcup_{i=1}^{d} \{e_i\} \cup \bigcup_{k=1}^{\lceil d/\ell \rceil} \mathcal{N}_k \cup \bigcup_{j=1}^{M} \{X_j\} \right\} . \tag{2}$$

To emphasize the number of random vertices we will denote V by V_M. Since $\mathcal{N}_k$ is symmetric, $2d \leq M$, and by the choice of ℓ, we observe that V_M has less than or equal to $4M$ vertices. Therefore, by Fact 2.2,

$$\mathrm{vol}(V_M) \leq \left(C \frac{\sqrt{\ln(M/d)}}{d} \right)^d. \tag{3}$$

This definition of Gluskin's polytopes differs from the original one in [8] by the inclusion of the nets $\mathcal{N}_k$. This guarantees that the polytope V_M contains a ball of an appropriate radius, which is necessary for the construction below. Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Since $\mathcal{N}_k$ is a $(1/2)$-net in $S^{d-1} \cap \mathbb{R}^{I_k}$, we have $(1/2)B_2^{I_k} \subset \mathrm{conv}(\mathcal{N}_k) \subset V_M$. Therefore,

$$\|x\|_{V_M} = \left\| \sum_{k=1}^{\lceil d/\ell \rceil} \sum_{j \in I_k} x_j e_j \right\|_{V_M} \leq \sum_{k=1}^{\lceil d/\ell \rceil} \left\| \sum_{j \in I_k} x_j e_j \right\|_{V_M}$$

$$\leq 2 \sum_{k=1}^{\lceil d/\ell \rceil} \left| \sum_{j \in I_k} x_j e_j \right| \leq 2 \sqrt{\lceil d/\ell \rceil} \left(\sum_{k=1}^{\lceil d/\ell \rceil} \left| \sum_{j \in I_k} x_j e_j \right|^2 \right)^{1/2} \leq 4 \sqrt{\frac{d}{\ln(M/d)}} \ |x|,$$

which means that

$$B_2^d \subset 4 \sqrt{\frac{d}{\ln(M/d)}} \ V_M. \tag{4}$$

Having two independent Gluskin's polytopes V_M' and V_M'' in $\mathbb{R}^d$ we will represent them on the product space $S^{d-1} \times S^{d-1}$ with the product probability $\mathbb{P} \otimes \mathbb{P}$. The next theorem shows that with high probability two Gluskin polytopes are far apart in the Banach-Mazur distance. The proof of this Theorem will be presented in Section 4.

Theorem 2.3. *There exists a (small) constant $a > 0$ such that for all integers $2d \leq M \leq e^d$ the subset of pairs (V_M', V_M'') of two independent Gluskin's polytopes in $\mathbb{R}^d$ satisfies*

$$\mathbb{P} \otimes \mathbb{P} \left(\left\{ (V_M', V_M'') \mid d(V_M', V_M'') \leq \frac{a\, d}{\ln(M/d)} \right\} \right) \leq 2e^{-dM}. \tag{5}$$

Corollary 2.4. *Let $2d \le M \le e^d$. Let $K \subset \mathbb{R}^d$ be a convex body. Then Gluskin's polytopes V_M in $\mathbb{R}^d$ with M random vertices satisfy*

$$\mathbb{P}\left(\left\{V_M \mid d(V_M, K) \le C\sqrt{\frac{d}{\ln(\frac{M}{d})}}\right\}\right) \le \sqrt{2}e^{-dM/2},$$

where $C > 0$ is an absolute constant.

Proof: Let V_M, V'_M, and V''_M be independent Gluskin's polytopes in $\mathbb{R}^d$ with M random vertices. By Theorem 2.3 and submultiplicativity of the Banach-Mazur distance, for every convex body K we have

$$\begin{aligned}
2e^{-dM} &\ge \mathbb{P}\otimes\mathbb{P}\left(\left\{(V'_M, V''_M) \mid d(V'_M, V''_M) \le \frac{a\,d}{\ln(\frac{M}{d})}\right\}\right) \\
&\ge \mathbb{P}\otimes\mathbb{P}\left(\left\{(V'_M, V''_M) \mid d(V'_M, K)d(K, V''_M) \le \frac{a\,d}{\ln(\frac{M}{d})}\right\}\right) \\
&\ge \mathbb{P}\otimes\mathbb{P}\left(\left\{(V'_M, V''_M) \mid \max\{d(V'_M, K), d(K, V''_M)\} \le \sqrt{\frac{a\,d}{\ln(\frac{M}{d})}}\right\}\right) \\
&= \left(\mathbb{P}\left(\left\{V_M \mid d(V_M, K) \le \sqrt{\frac{ad}{\ln(\frac{M}{d})}}\right\}\right)\right)^2,
\end{aligned}$$

which implies the result. □

3 Proof of the main result

We start with the following lemma, which shows that it is enough to consider only special sections of the cone S.

Lemma 3.1. *Let $m \le N$ and let $F \subset \mathbb{R}^{N+1}$ be an affine subspace such that Δ_N^F is an m-dimensional body. Then there exists a linear subspace $L \subset \mathbb{R}^{N+1}$ such that Δ_N^F has a position K inside L of the form*

$$K = \{x \in \mathbb{R}^{N+1} \mid x \in L \quad \text{and} \quad -1 \le x_i \le m \text{ for all } i \le N+1\}.$$

In particular,

$$B_2^L \subset K \subset m\sqrt{N+1}\,B_2^L.$$

Proof: By (1) there exists $a = \{a_i\}_{i=1}^{N+1} \in \Delta_N^F \subset S$ such that

$$-(\Delta_N^F - a) \subset m(\Delta_N^F - a). \tag{6}$$

Clearly $a_i \geq 0$ for all $i \leq N+1$. Without loss of generality we can assume that $a_i > 0$ for all i. Indeed, note that a is in the relative interior of Δ_N^F. Thus, if for some $j > 0$, $a_j = 0$ then

$$\Delta_N^F \subset H_j := \{x \in \mathbb{R}^{N+1} \mid x_j = 0\}.$$

Therefore Δ_N^F is in fact a corresponding m-dimensional section of the $(N-1)$-dimensional simplex

$$\Delta_{N-1} = S \cap H \cap H_j$$

and we can apply the proof below for this section (or just to take the operator D below with zero j-th row).

Consider the diagonal operator D with $1/a_i$'s on the main diagonal. Denote

$$b = Da = \sum_{i=1}^{N+1} e_i \quad \text{and} \quad K := D(\Delta_N^F - a) = D\Delta_N^F - b.$$

Then

$$D\Delta_N^F = D(S \cap H \cap F) = S \cap D(H \cap F).$$

Therefore, denoting $L := D(H \cap F) - b$, we obtain

$$K = \{x \in \mathbb{R}^{N+1} \mid -1 \leq x_i \text{ and } x \in L\}.$$

By (6) we observe that $-K \subset mK$, hence

$$K = \{x \in \mathbb{R}^{N+1} \mid x \in L \quad \text{and} \quad -1 \leq x_i \leq m \text{ for all } i \leq N+1\}.$$

This implies

$$B_2^{N+1} \cap L \subset K \subset m\sqrt{N+1}\, B_2^{N+1} \cap L.$$

$\square$

Lemma 3.2. *Let $\varepsilon \in (0,1)$ and $m \leq N$. For $j = 1, 2$ let L_j be an m-dimensional linear subspace of $\mathbb{R}^{N+1}$ and put*

$$K_j := \{x \in \mathbb{R}^{N+1} \mid x \in L_j \quad \textit{and} \quad -1 \leq x_i \leq m \textit{ for all } i \leq N+1\}.$$

Assume $\rho(L_1, L_2) \leq \varepsilon$. Then

$$d(K_1, K_2) \leq (1 + \varepsilon m\sqrt{N+1})^2.$$

Proof: By the definition there exists an orthogonal operator U such that $UL_1 = L_2$ and $\|U - I\| \leq \varepsilon$. Therefore for every $x = \{x_i\}_i \in K_1$ we have $|Ux - x| \leq \varepsilon|x| \leq \varepsilon m\sqrt{N+1}$, hence $|(Ux - x)_i| \leq \varepsilon m\sqrt{N+1}$ for every $i \leq N+1$. Thus, for every i we have

$$(Ux)_i = x_i + (Ux - x)_i \geq -(1 + \varepsilon m\sqrt{N+1})$$

and

$$(Ux)_i = x_i + (Ux - x)_i \leq m + \varepsilon m\sqrt{N+1}.$$

Therefore, $UK_1 \subset (1+\varepsilon m\sqrt{N+1})K_2$. Similarly, $U^{-1}K_2 \subset (1+\varepsilon m\sqrt{N+1})K_1$, which implies the result. □

Lemma 3.3. *Let $\varepsilon \in (0,1)$, $n \leq m \leq N$, L be an m-dimensional linear subspace of $\mathbb{R}^{N+1}$ and*

$$K = \{x \in \mathbb{R}^{N+1} \mid x \in L \quad \textit{and} \quad -1 \leq x_i \leq m \textit{ for all } i \leq N+1\}.$$

Let F_1 and F_2 be n-dimensional linear subspaces of $\mathbb{R}^{N+1}$ and P_1 and P_2 be the orthogonal projections on F_1 and F_2, respectively. Assume $\rho(F_1, F_2) \leq \varepsilon$. Then

$$d(P_1K, P_2K) \leq (1 + \varepsilon m\sqrt{N+1})^2.$$

Proof: By the definition there exists an orthogonal operator U such that $UF_1 = F_2$ and $\|U - I\| \leq \varepsilon$. Then $UP_1 = P_2U$ and therefore for every $x \in K$ we have

$$UP_1x = P_2Ux = P_2x + P_2(U - I)x \in P_2K + P_2(U - I)m\sqrt{N+1}\, B_2^{N+1} \cap L.$$

Since $B_2^{N+1} \cap L \subset K$, we obtain

$$UP_1x \in (1 + \varepsilon m\sqrt{N+1})P_2K.$$

Similarly,

$$U^{-1}P_2x \subset (1 + \varepsilon m\sqrt{N+1})P_1K,$$

which implies the result. □

We are now ready to prove our main theorem.

Proof of Theorem 1.1: In this proof C_1, C_2, C_3 are absolute constants greater then one. Without loss of generality we assume that $2 \leq n \leq N \leq$

e^{cn}, where c is an absolute positive constant, which will be specified later (if $n = 1$ or $N \geq e^{cn}$ the conclusion of the theorem is immediate).

For any $k \leq N$ and $\varepsilon \in (0, 1)$, by $\mathcal{A}_k$ we denote an ε-net on the Grassmanian $G_{N+1,k}$ of cardinality

$$|\mathcal{A}_k| \leq (C_1/\varepsilon)^{C_1 N k}.$$

(The existence of such a net follows from Lemma 2.1. Note that we suppress the dependence of the net on ε.)

In the first part of the argument fix an integer m such that $n \leq m \leq N$ and fix $\varepsilon \in (0, 1)$. Put

$$K_m = \{x \in \mathbb{R}^{N+1} \mid -1 \leq x_i \leq m \text{ for all } i \leq N+1\}.$$

Let $2n \leq M \leq e^n$. We apply Corollary 2.4 with $d = n$ and the body $K = P_{E_0}(K_m \cap L_0)$, for arbitrary $L_0 \in \mathcal{A}_m$ and $E_0 \in \mathcal{A}_n$. By the union bound we obtain that for n-dimensional Gluskin's polytopes V_M one has

$$\mathbb{P}\left(\left\{\forall L_0 \in \mathcal{A}_m \, \forall E_0 \in \mathcal{A}_n \quad d(V_M, P_{E_0}(K_m \cap L_0)) \leq C_2 \sqrt{\frac{n}{\ln(\frac{M}{d})}}\right\}\right)$$

$$\leq \sqrt{2}\, (C_1/\varepsilon)^{C_1 N m + C_1 N n} \exp(-Mn/2) \leq \sqrt{2}\, \exp(-Mn/2 + 2C_1 N m \ln(C_1/\varepsilon)).$$

Therefore whenever M satisfies

$$M \geq 8 C_1 N m \ln(C_1/\varepsilon)/n, \tag{7}$$

then

$$\mathbb{P}\left(\left\{\forall L_0 \in \mathcal{A}_m \, \forall E_0 \in \mathcal{A}_n \quad d(V_M, P_{E_0}(K_m \cap L_0)) \leq C_2 \sqrt{\frac{n}{\ln(\frac{M}{n})}}\right\}\right)$$

$$\leq \sqrt{2}\, \exp(-Mn/4) \leq \exp(-Mn/6). \tag{8}$$

Therefore taking M satisfying $2n \leq M \leq e^n$ and (7) (if such an M exists), this implies the result for Gluskin's polytopes V_M and for every n-dimensional projection of an m-dimensional section of an N-dimensional simplex, with high probability. (Note that m is fixed in this argument.) Indeed, let F be any affine subspace of $\mathbb{R}^{N+1}$, such that Δ_N^F is m-dimensional. Let $L = L(F)$ be an m-dimensional linear subspace and $K = K(F) = K_m \cap L$ be the position of Δ_N^F provided by Lemma 3.1. Let P be any orthogonal projection such that PK is n-dimensional and let E be the range of P. Let $L_0 \in \mathcal{A}_m$ and $E_0 \in \mathcal{A}_n$ be such that $\rho(L, L_0) \le \varepsilon$ and $\rho(E, E_0) \le \varepsilon$. Then by Lemmas 3.2 and 3.3 we get

$$d(PK, P_{E_0}(K_m \cap L_0)) \le d(PK, P_{E_0}K)d(P_{E_0}K, P_{E_0}(K_m \cap L_0)) \le (1+\varepsilon m\sqrt{N+1})^4,$$

where in the last estimate we used the obvious inequality $d(P_{E_0}K_1, P_{E_0}K_2) \le d(K_1, K_2)$ valid for all convex bodies $K_1, K_2 \subset \mathbb{R}^N$ of dimension $m \le N$. Therefore, taking $\varepsilon = m^{-1}(N+1)^{-1/2}$ we obtain that

$$d(V_M, P_{E_0}(K_m \cap L_0)) \le 2^4 d(V_M, PK).$$

Combining this with (8), we obtain the probability estimate for

$$\mathbb{P}\left(\left\{\text{for every } \ F, L, K, P \ \text{ as above: } \ d(V_M, PK) \le 2^{-4} C_2 \sqrt{\frac{n}{\ln(\frac{M}{n})}}\right\}\right).$$

More precisely we showed that for any $n \le m \le N$ whenever M satisfies $2n \le M \le e^n$ and (7) with $\varepsilon = m^{-1}(N+1)^{-1/2}$, then the latter probability is less than or equal to $\exp(-Mn/6)$. In particular, let

$$M = \lceil 8C_1 N^2 \ln(C_1 N^{3/2})/n \rceil,$$

so that (7) is satisfied with $\varepsilon = m^{-1}(N+1)^{-1/2}$. Additionally we can find a universal constant $0 < c < 1$ such that the condition $N \le e^{cn}$ implies $M \le e^n$.

Then for some absolute constant C_3,

$$\mathbb{P}\Big(\Big\{\text{for every } \ K, P, \ d(V_M, PK) \le C_3 \sqrt{\frac{n}{\ln \frac{2N\ln(2N)}{n}}}\Big\}\Big) \le \exp(-N^2 \ln(2N)).$$

(Here K and P are as above, in particular, the dimension of a section K is equal to m.)

To obtain the full result for any $n \le N$, for any n-dimensional projection of an arbitrary dimensional section of an N-dimensional simplex we apply the above discussion for an arbitrary m representing the dimension of a section (so $n \le m \le N$). Note that the choice of M does not depend on m, so we are working in the same probability space for all m, leading to the same class of Gluskin's polytopes V_M. Taking the union bound over all integers $n \le m \le N$ concludes the proof. $\square$

Remarks. **1.** In fact, taking $M = \lceil 8C_1 Nm \ln(C_1 m\sqrt{N+1})/n \rceil$ in our proof, we observe that for $n \le m \le N$ there exists an n-dimensional convex body B such that for every convex body K obtained as an n-dimensional projection of an m-dimensional section of an N-dimensional simplex one has

$$d(B, K) \ge c\sqrt{\frac{n}{\ln \frac{2Nm\ln(2N)}{n^2}}}.$$

Moreover, our construction is random – we use Gluskin's polytopes – and we obtain the result with high probability – the estimate above holds with probability larger than $1 - \exp(-Nm\ln(2N))$.

2. If we restrict ourselves to just one operation – projection – then we have almost the same lower bound using the Euclidean ball. Namely, for every n-dimensional projection P one has

$$d(B_2^n, P\Delta_N) \ge c\sqrt{\frac{n}{\ln \frac{2N}{n}}},$$

which follows from volume estimates (see Fact 2.2) as mentioned in the introduction.

3. Also note that, although an N-dimensional simplex clearly has $\lceil N/2 \rceil$-dimensional symmetric projection, a "random" projection is very far from being symmetric. It was shown in Theorem 5.1 of [14] that for a "random" n-dimensional projection P and every centrally symmetric convex body B one has

$$d(B, P\Delta_N) \ge c\sqrt{\frac{n}{\ln N}}.$$

4 Proof of Theorem 2.3

The proof of the theorem is standard and follows the road-map of [8]. The main difference from [8] is the modification of the definition of a Gluskin polytope (2). Adding the nets $\mathcal{N}_k$ to the vertex set of V_M allowed to guarantee the inclusion (4) without significantly increasing the number of vertices. (Of course if the number of vertices is proportional then (4) is automatically satisfied.)

Recall that the underlying probability space is the product space $\Omega' \times \Omega'' = S^{d-1} \times S^{d-1}$ with the product probability $\mathbb{P} \otimes \mathbb{P}$. Our first aim in the proof is to prove two estimates similar to (5): one is for probability on Ω', with $\omega'' \in \Omega''$ fixed, and in the other one the roles of Ω' and Ω'' are interchanged. This is proved in Lemma 4.5 below. Then the full Theorem 2.3 follows by considerations based on Fubini's theorem.

Throughout most of this section, until the final proof of the theorem, we fix an arbitrary $\omega'' \in \Omega''$ and the corresponding Gluskin's polytope $W_M = V''_M(\omega'')$.

For any $\tau > 0$ and any operator $T : \mathbb{R}^d \to \mathbb{R}^d$ with $\det T = 1$ consider the event

$$A(\tau, W_M, T) = \big\{ V_M : \ \|T : V_M \to W_M\| \le \tau \big\} = \big\{ V_M : \ TV_M \subset \tau W_M \big\}. \quad (9)$$

First we estimate the probability of this event.

Lemma 4.1. *One has*

$$\mathbb{P}\left(A(\tau, W_M, T)\right) \le \left(C\tau\sqrt{\ln(M/d)/d} \right)^{dM},$$

where C is a positive absolute constant.

To prove this lemma we need the following well-known simple fact, which can be found in many places, for example in [23], (38.4). We outline the proof for the reader's convenience.

Fact 4.2. *Let $K \subset \mathbb{R}^d$ be a convex body with 0 in its interior. Let X be a random vector uniformly distributed on the sphere S^{d-1}. Then*

$$\mathbb{P}(\{X \in K\}) \le \mathrm{vol}\,(K)/\mathrm{vol}\,(B_2^d).$$

Proof: Obviously we have $\mathbb{P}(\{X \in K\}) = \mathrm{vol}(L)/\mathrm{vol}(B_2^d)$ where $L = \big\{ x \in B_2^d \mid x/|x| \in K \cap S^{d-1} \big\}$. On the other hand, $L \subset \mathrm{conv}(K \cap S^{d-1}) \subset K$, which yields the required estimate for volumes. $\Box$

We use a convenient shortcut for norms of linear operators: for two convex bodies $K_1, K_2 \subset \mathbb{R}^d$ and for $\lambda > 0$ the statement $\|T : K_1 \to K_2\| \le \lambda$ is equivalent to $T(K_1) \subset \lambda K_2$ and is equivalent to $\|T : K_1 \to \lambda K_2\| \le 1$.

Proof of Lemma 4.1: Since V_M contains the vectors X_j, $j \le M$, the condition $T(V_M) \subset \tau W_M$ implies that $TX_j \in \tau W_M$ for all $j \le M$. Therefore

$$\begin{aligned}\mathbb{P}\left(A(\tau, W_M, T)\right) &\le \mathbb{P}\left(\{TX_j \in \tau W_M \quad \text{for } 1 \le j \le M\}\right) \\ &= \Big(\mathbb{P}\left(\left\{X \in \tau\, T^{-1}\, W_M\right\}\right)\Big)^M\end{aligned}$$

(cf. Lemma 38.3 in [23] and Lemma 4 in [15]). By Fact 4.2 and using $\det T^{-1} = 1$ and (3) for W_M, we obtain

$$\mathbb{P}\left(A(\tau, W_M, T)\right) \le \left(\frac{\mathrm{vol}(\tau W_M)}{\mathrm{vol}(B_2^d)}\right)^M = \tau^{dM}\left(\frac{\mathrm{vol}(W_M)}{\mathrm{vol}(B_2^d)}\right)^M \le \left(C\tau\sqrt{\frac{\ln(M/d)}{d}}\right)^{dM},$$

which completes the proof. $\square$

In the next step we discretize certain sets of operators acting on $\mathbb{R}^d$ (see Lemma 38 in [23] and Lemma 7 in [15]). We need more notation. Set

$$B_{op}^d = \{T : \mathbb{R}^d \to \mathbb{R}^d \mid \|T : \ell_2^d \to \ell_2^d\| \le 1\},$$

and for a convex body $K \subset \mathbb{R}^d$,

$$B_{op,K}^d = \{T : \mathbb{R}^d \to \mathbb{R}^d \mid \|T : B_1^d \to K\| \le 1\}.$$

Note that the norm for which $B_{op,K}^d$ is the unit ball is equal to the ℓ_∞-direct-sum of d norms $\|\cdot\|_K$ determined by K.

For the reader's convenience we recall that identifying the set of operators with $\mathbb{R}^{d^2}$ we have

$$\mathrm{vol}(B_{op,K}^d) = (\mathrm{vol}(K))^d \qquad \text{and} \qquad \mathrm{vol}(B_{op}^d) \ge (c/\sqrt{d})^{d^2}, \tag{10}$$

where c is a positive absolute constant.

We also will use the following fact on cardinality of ε-nets. Recall that the smallest cardinality of a 1-net of a set K_1 in the metric of defined by a convex body K_2 is denoted by $N(K_1, K_2)$, hence the smallest cardinality of an ε-net is $N(K_1, \varepsilon K_2)$. The following lemma follows by the standard volumetric argument (in such a formulation it is Lemma 6 from [15]).

Lemma 4.3. *Let $\varepsilon > 0$. Let $K_1, K_2 \subset \mathbb{R}^n$ be two symmetric convex bodies such that $K_1 \subset K_2$. Then every subset $K' \subset K_2$ admits an ε-net $\mathcal{N} \subset K'$ in the metric of K_1 with $|\mathcal{N}| \le (1 + 2/\varepsilon)^n\, (\mathrm{vol}(K_2)/\mathrm{vol}(K_1))$.*

We use this lemma to control the cardinality of an ε-net in $B^d_{op.\eta W}$ in the operator norm.

Lemma 4.4. *Let $\xi > 0$ and let $W \subset \mathbb{R}^d$ be a convex symmetric body such that $B_2^d \subset \xi W$. Let $\eta, \varepsilon > 0$. Every subset K' of $B^d_{op.\eta W}$ admits an ε-net $\mathcal{N}$ in K' in the operator norm on ℓ_2^d with cardinality*

$$|\mathcal{N}| \leq \left(\frac{\xi}{\eta} + \frac{2}{\varepsilon}\right)^{d^2} \left(C\,\eta\sqrt{d}\cdot \mathrm{vol}^{1/d}(W)\right)^{d^2}, \tag{11}$$

where C is an absolute positive constant.

Proof: We will use Lemma 4.3 with $\lambda = \xi/\eta$, $K_1 = (1/\lambda)B^d_{op}$ and $K_2 = B^d_{op,\eta W}$. The assumption $B_2^d \subset \xi W$ yields $(1/\lambda)B^d_{op} \subset B^d_{op,\eta W}$. Thus, by (10),

$$\begin{aligned} N\left(K', \varepsilon B^d_{op}\right) = N\left(K', \lambda\varepsilon\left((1/\lambda)\, B^d_{op}\right)\right) &\leq \left(1 + \frac{2}{\varepsilon\lambda}\right)^{d^2} \frac{\mathrm{vol} K_2}{\mathrm{vol} K_1} \\ &\leq \left(\frac{2}{\varepsilon} + \frac{\xi}{\eta}\right)^{d^2} \left(C\,\eta\sqrt{d}\cdot \mathrm{vol}^{1/d}(W)\right)^{d^2}, \end{aligned}$$

with an absolute positive constant C. $\Box$

We need one more lemma, which estimates the probability of the following event

$$\widetilde{A}(\eta, W_M) = \left\{V_M : \ \exists\, S : \mathbb{R}^d \to \mathbb{R}^d,\ \det S = 1,\ \text{s.t. } \|S : V_M \to W_M\| \leq \eta\right\}, \tag{12}$$

where η is a positive parameter.

Lemma 4.5. *Let $d \leq M \leq e^d$. There exists a positive constant $a_1 > 0$ such that for $\eta = a_1\sqrt{d/\ln(M/d)}$ one has*

$$\mathbb{P}\left(\widetilde{A}(\eta, W_M)\right) \leq e^{-dM}.$$

Proof: Denote for shortness $\xi = 4\sqrt{\frac{d}{\ln(M/d)}}$. Fix an arbitrary $0 < \varepsilon \leq 1$. By K' denote the set of all operators $T \in B^d_{op,\eta W_M}$ with $\det T = 1$. Let $\mathcal{N}$ be an ε-net for K' with respect to the metric given by B^d_{op} and satisfying (11) with $W = W_M$.

We first show that

$$\widetilde{A}(\eta, W_M) \subset \bigcup_{T\in\mathcal{N}} A(\tau, W_M, T), \tag{13}$$

where $\tau = \eta + \varepsilon\xi$.

Pick $\omega \in \widetilde{A}(\eta, W_M)$, and let S be an operator with $\det S = 1$ such that $\|S : V_M(\omega) \to W_M\| \le \eta$. Since $V_M \supset B_1^d$, we have $\|S : B_1^d \to \eta W_M\| \le 1$, which means $S \in B^d_{op,\eta W}$.

Since $\det S = 1$, then S belongs to K'. By the definition of $\mathcal{N}$, we can find $T \in \mathcal{N}$ satisfying $\|T - S : \ell_2^d \to \ell_2^d\| \le \varepsilon$. Since $V_M \subset B_2^d$ and by (4), we get

$$(T - S)(V_M) \subset \varepsilon B_2^d \subset \varepsilon\xi W_M.$$

Equivalently, $\|T - S : V_M \to W_M\| \le \varepsilon\xi$. By the triangle inequality,

$$\|T : V_M \to W_M\| \le \|T - S : V_M \to W_M\| + \|S : V_M \to W_M\| \le \varepsilon\xi + \eta = \tau.$$

This means that $\omega \in A(\tau, W_M, T)$ for every $T \in \mathcal{N}$ and ends the proof of (13).

By the union bound and Lemma 4.1

$$\mathbb{P}\big(\widetilde{A}(\eta, W_M)\big) \le |\mathcal{N}| \left(C\tau \sqrt{\frac{\ln(M/d)}{d}} \right)^{dM}.$$

Combining this with (11), (3) for W_M, and the definitions of ξ and η we observe that

$$\begin{aligned}\mathbb{P}\big(\widetilde{A}(\eta, W_M)\big) &\le \left(\frac{2}{\varepsilon} + \frac{\xi}{\eta}\right)^{d^2} \left(C_1\,\eta\sqrt{d}\cdot \mathrm{vol}^{1/d}(W_M)\right)^{d^2} \cdot \left(C(\eta+\varepsilon\xi)\sqrt{\frac{\ln(M/d)}{d}} \right)^{dM} \\ &\le \left(\frac{2}{\varepsilon} + \frac{4}{a_1}\right)^{d^2} (C_2 a_1)^{d^2} \cdot (C_3(a_1 + 4\varepsilon))^{dM},\end{aligned}$$

where C, C_1, C_2, C_3 are absolute positive constants. To complete the proof it is enough to set $\varepsilon = a_1$ and choose a_1 appropriately small. $\square$

Now we are ready to prove Theorem 2.3.

Proof of Theorem 2.3: Let a_1 and η be as in Lemma 4.4. We consider various subsets of the measure spaces Ω', Ω'', and $\Omega' \times \Omega''$; we will use an expanded notation to avoid confusion.

Denote the set that appears in (5) by D, that is

$$D = \left\{(\omega', \omega'') \mid d(V'_M(\omega'), V''_M(\omega'')) \le \eta^2 \ \right\}.$$

For any $\omega''_0 \in \Omega''$ define the subset $D'_{\omega''_0} \subset \Omega' \times \Omega''$ which depends only on the first variable ω' with the second variable fixed $\omega'' = \omega''_0$ and is given by

$$D'_{\omega''_0} = \left\{(\omega', \omega''_0) \mid \exists\ S \text{ s.t. } \det S = 1 \text{ and } \|S : V'_M(\omega') \to V''_M(\omega''_0)\| \le \eta\right\}.$$

Similarly, for any $\omega'_0 \in \Omega'$ define the subset $D''_{\omega'_0}$ by

$$D''_{\omega'_0} = \left\{(\omega'_0, \omega'') \mid \exists\ R \text{ s.t. } \det R = 1 \text{ and } \|R : V''_M(\omega'') \to V'_M(\omega'_0)\| \le \eta\right\}.$$

Note that both definitions closely follow the model of (12) in that the norm of operators is considered from a random polytope to a fixed polytope.

The following inclusion can be easily checked

$$D \subset \bigcup_{\omega'_0 \in \Omega'} D''_{\omega'_0} \ \cup \bigcup_{\omega''_0 \in \Omega''} D'_{\omega''_0}.$$

Indeed, if $d(V'_M(\omega'_0), V''_M(\omega''_0)) \le \eta^2$ then there exists an invertible operator S such that

$$\|S : V'_M(\omega'_0) \to V''_M(\omega''_0)\| \, \|S^{-1} : V''_M(\omega''_0) \to V'_M(\omega'_0)\| \le \eta^2.$$

Without loss of generality we may assume that $\det S = \det S^{-1} = 1$. Thus one of the norms in the above product is less than or equals to η, which means that either $(\omega'_0, \omega''_0) \in D'_{\omega''_0}$ or $(\omega'_0, \omega''_0) \in D''_{\omega'_0}$.

Finally, using Lemma 4.5 and the Fubini theorem, we obtain

$$\mathbb{P} \otimes \mathbb{P}(D) \le \mathbb{E}_{\omega'_0} \mathbb{P}(D''_{\omega'_0} \mid \ \omega'_0) + \mathbb{E}_{\omega''_0} \mathbb{P}(D'_{\omega''_0} \mid \ \omega''_0) \le 2e^{-dM}.$$

This completes the proof of Theorem 2.3. □

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