When random proportional subspaces are also random quotients

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Abstract

We discuss when a generic subspace of some fixed proportional dimension of a finite-dimensional normed space can be isomorphic to a generic quotient of some proportional dimension of another space. We show (in Theorem 4.1) that if this happens (for some natural random structures) then for *any* proportion arbitrarily close to 1, the first space has a lot of Euclidean subspaces and the second space has a lot of Euclidean quotients.

0 Introduction

In the paper [BM1], Bourgain and Milman studied Banach–Mazur distances between finite-dimensional normed spaces, their subspaces and quotients. In particular they proved that given any two normed spaces X and Y, for a large set of (proportional dimensional) subspaces of X and a large set of quotients of Y, the distance betwen any two representatives is less than or equal to $c\sqrt{n}(\log n)^2$, where c depends on the proportion only. In fact, these sets of subspaces and quotients have (Haar) measure close to 1, as subsets of Grassman manifolds naturally determined by the spaces X and Y. This result should be compared to the result of Gluskin [Gl] which says that for a large set of (proportional dimensional) subspaces of ℓ_∞^n, the distance between two distinct subspaces is larger than or equal to cn, where $c > 0$ is an absolute constant. It was then observed in [BM1] that "random"

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subspaces and "random" quotients are not of the same nature, and should have some very different properties.

The present paper answers a vaguely put question from [BM1] and opens a new direction in understanding of what "random" subspaces and "random" quotients are. We consider a critical case and we show that if, for some random structure (described below), a generic subspace of *some* fixed proportional dimension is isomorphic (essentially the same) to a generic quotient of some proportional dimension of another space (with a similarly selected random structure) then for *any* proportion arbitrarily close to 1, the first space has a lot of Euclidean subspaces and the second space has a lot of Euclidean quotients. So a complete similarity between a generic subspace and a generic quotient implies that most subspaces (respectively, quotients) are Euclidean.

Of course, the notion of randomness is crucially important and we introduce and discuss the corresponding Euclidean structure in Section 3. Just to describe our general point of view, for an arbitrary n-dimensional normed space X and an arbitrary so-called M-ellipsoid on X (see Section 3 for the definition), we identify X with $\mathbb{R}^n$ in such a way that the ellipsoid becomes the standard Euclidean ball. Then for every $0 < \lambda \leq 1$ we define a certain subset $\mathcal{F}_{\lceil \lambda n \rceil}(B_X)$ of the Grassman manifold $G_{n,\lceil \lambda n \rceil}$ of all $\lceil \lambda n \rceil$-dimensional subspaces of $\mathbb{R}^n$, depending on X, whose (Haar) measure is exponentially close to 1. Our main result (Theorem 4.1) says that if $K, L \subset \mathbb{R}^n$ are the unit balls of two n-dimensional spaces X and Y with the above identification, and for some $0 < \lambda < 1$ and some $d > 1$ there exist $E \in \mathcal{F}_{\lceil \lambda n \rceil}(K)$ and $F \in \mathcal{F}_{\lceil \lambda n \rceil}(L)$ such that the Banach-Mazur distance satisfies

$$d\Big((F, L \cap F), (E, Q_E K)\Big) \leq d,$$

then the volume ratio of Y and the outer volume ratio of X are both bounded by a function depending on λ and d only. Here $(F, L \cap F)$ denotes the space F with the unit ball $L \cap F$ (which makes it into a subspace of Y) and similarly for $(E, P_E K)$, where P_E is the orthogonal projection onto E, which makes it into a quotient of X. Let us also recall for non-specialists, that the condition of bounded volume ratio implies the existence of a large family of Euclidean subspaces of proportional dimension (for any proportion less than 1), and dually, the boundness of the outer volume ratio is similarly related to Euclidean quotients.

The proof of the main theorem is based upon some new properties of the minimal and maximal volume ellipsoids which are described in

Section 2. In our opinion, these properties should play a role in the theory for many other problems as well, and should be independently noted.

It is well known that every "local" fact in the asymptotic theory (which means a fact about subspaces or quotients) corresponds to some global statment, about the body in the whole space, without a reduction of dimension. It also often happens that some of the facts are very non trivial but other are very easy. In our case the global analogies are easy, nevertheless they are presented in the second part of Section 4 (Theorem 4.6 and before) to complete the picture.

In order to keep our arguments relatively transparent we did not make an attempt to get the dependence of constants in our inequalities on appropriate parameters asymptotically sharpest possible. Some strengthenings of our results as well as their versions for non-symmetric and p-convex cases will be presented in the forthcoming paper [LMT].

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1 Basic notations

We consider $\mathbb{R}^n$ with the standard Euclidean structure and the Euclidean unit ball denoted by B_2. The canonical Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$, and the corresponding inner product by $\langle\cdot,\cdot\rangle$. We shall also consider other Euclidean structures on $\mathbb{R}^n$, with the unit balls given by ellipsoids.

By a body we mean a compact set with a non-empty interior. We shall call a convex body symmetric if it is centrally symmetric. For a symmetric convex body K in $\mathbb{R}^n$ the polar body K^0 is defined by

$$K^0 := \{x \in \mathbb{R}^n \mid |\langle x, y\rangle| \leq 1 \text{ for every } y \in K\}.$$

We recall that for every subspace E of $\mathbb{R}^n$ the polar (in E) of $K \cap E$ is $P_E K^0$, where P_E is the orthogonal projection onto E.

The n-dimensional volume of a body K in $\mathbb{R}^n$ is denoted by $|K|$. For a symmetric convex body $K \subset \mathbb{R}^n$ we shall occasionally use the notation $\|\cdot\|_K$ for the Minkowski functional of K. The normed space $(\mathbb{R}^n, \|\cdot\|_K)$ will be also denoted by $(\mathbb{R}^n, K)$. If $L \subset \mathbb{R}^m$ is another symmetric convex body and $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator, by $\|T : K \to L\|$ we shall denote the operator norm of T from $(\mathbb{R}^n, K)$ to

$(\mathbb{R}^m, L)$. If $m = n$, the geometric distance between K and L is defined by

$$d_g(K, L) := \inf \{b/a \mid a > 0, b > 0,\ aK \subset L \subset bK\}.$$

If $d_g(K, L) \leq C$ then we say that K and L are *C-equivalent*. The Banach-Mazur distance between K and L is defined by

$$d(K, L) := \inf \{d_g(K, TL)\},$$

where the infimum is taken over all invertible linear operators T from $\mathbb{R}^n$ to $\mathbb{R}^n$. The Banach-Mazur distance between normed spaces is the Banach-Mazur distance between their unit balls. If the Banach-Mazur distance between a space and the Euclidean space is bounded by C we say that the space is *C-Euclidean*.

For a real number $a > 0$, by $\lceil a \rceil$ we denote the smallest integer larger than or equal to a.

Given an ellipsoid $\mathcal{E}$ on $\mathbb{R}^n$, by $G^{\mathcal{E}}_{n,k}$ for $1 \leq k \leq n$ we shall denote the Grassman manifold of k-dimensional linear subspaces of $\mathbb{R}^n$ equiped with the normalized Haar measure $\mu^{\mathcal{E}}_{n,k}$ determined by the Euclidean structure given by $\mathcal{E}$. If $\mathcal{E} = B_2$ we shall write $G_{n,k}$ and $\mu_{n,k}$ instead of $G^{\mathcal{E}}_{n,k}$, and $\mu^{\mathcal{E}}_{n,k}$. We say that some property holds for a *random orthogonal* (in $\mathcal{E}$) projection of rank k whenever the measure of the set of all subspaces $E \in G^{\mathcal{E}}_{n,k}$ for which P_E has the property, is larger than $1 - \exp{(ck)}$ for some absolute constant $c > 0$.

For a symmetric convex body $K \subset \mathbb{R}^n$, by $\mathcal{E}_K \supset K$ and $\mathcal{E}'_K \subset K$ we denote the ellipsoids of minimal and maximal volume for K respectively.

Recall that the volume ratio of K and the outer volume ratio of K are defined by

$$\mathrm{vr}\,(K) = \left(|K|/|\mathcal{E}'_K|\right)^{1/n} \qquad \text{and} \qquad \mathrm{outvr}\,(K) = \left(|\mathcal{E}_K|/|K|\right)^{1/n}.$$

For a symmetric convex body $K \subset \mathbb{R}^n$, an ellipsoid $\mathcal{E}$ on $\mathbb{R}^n$, and any $0 < \lambda < 1$ we shall consider certain subsets $\mathcal{F}_{\lceil \lambda n \rceil}(K) \subset G^{\mathcal{E}}_{n,\lceil \lambda n \rceil}$ of $\lceil \lambda n \rceil$-dimensional subspaces of $\mathbb{R}^n$. Each element of $\mathcal{F}_{\lceil \lambda n \rceil}(K)$ gives rise to two different normed spaces. Firstly, it can be treated as a *subspace* of the normed space $(\mathbb{R}^n, K)$, in which case we may use a generic notation sK, that is, $sK := (E, K \cap E)$. The set of all these subspaces will be denoted by $\mathcal{F}_{s,\lceil \lambda n \rceil}(K)$. Secondly, every $E \in \mathcal{F}_{\lceil \lambda n \rceil}(K)$ gives rise to a *quotient* space of $(\mathbb{R}^n, K)$, via the orthogonal (in $\mathcal{E}$) projection P_E onto E, and in this case we may use a generic notation of qK; that is, $qK := (E, P_E K)$. The set of all these quotient spaces will be denoted by $\mathcal{F}_{q,\lceil \lambda n\rceil}(K)$. (It should be noted that given a family $\mathcal{F}_k$, the definition of $\mathcal{F}_{s,k}$ does not depend on the ellipsoid $\mathcal{E}$, while the definition of $\mathcal{F}_{q,k}$ depends on this ellipsoid in an essential way.)

2 The minimal and maximal volume ellipsoids

We present in this section some new properties of the minimal (resp., maximal) volume ellipsoid associated to a convex body, which play an essential role in our constructions. They deal with relations to any other ellipsoid containing (resp., contained in) the same body. These new properties depend on an abstract condition of Dvoretzky-Rogers-type. All results can be dualized in a standard way to the corresponding statements for the maximal volume ellipsoids and their sections.

Let B be a symmetric convex body in $\mathbb{R}^m$ and let $\mathcal{E} \subset \mathbb{R}^m$ be an ellipsoid. Let $\phi : (0,1] \to (0,1]$ be a function. We say that $\mathcal{E}$ has property $(*)$ with respect to B with function ϕ, whenever

$(*)$ for any $1 \le k \le m$ and any projection Q of rank k on $\mathbb{R}^m$ orthogonal with respect to $\mathcal{E}$ we have $\|Q : B \to \mathcal{E}\| \ge \phi(k/m)$.

It is well known that the minimal volume ellipsoid satisfies $(*)$ with the function $\phi(t) = \sqrt{t}$. This is connected to, but simpler than, the Dvoretzky–Rogers Lemma. (We shall show in Lemma 2.2 below that proportional-dimensional projections of the minimal volume ellipsoids satisfy $(*)$ as well.)

Theorem 2.1 *Let $\mathcal{E} \subset \mathbb{R}^m$ and $\mathcal{D} \subset \mathbb{R}^m$ be two ellipsoids, let $B := \mathcal{E} \cap \mathcal{D}$. Let $\phi : (0,1] \to (0,1]$ and set $A := \left(\prod_{l=1}^m \phi(l/m)\right)^{-1/m}$. Assume that $\mathcal{E}$ has property $(*)$ with respect to B with function ϕ. Then*

$$|\mathcal{E}|^{1/m} \le A |\mathcal{D}|^{1/m}. \tag{2.1}$$

Furthermore, if $\phi(t) \ge (1/a)t^\alpha$ for some $a \ge 1$ and $\alpha \ge 1/2$, then for every $0 < \xi < 1$ there is $f_{a,\alpha} = f_{a,\alpha}(\xi) \ge 1$ such that a random projection Q of rank $\lceil \xi m\rceil$ orthogonal with respect to $\mathcal{E}$ satisfies

$$Q\mathcal{E} \subset f_{a,\alpha}(\xi)\, Q\mathcal{D}.$$

The difficulty of the second part of the theorem lies in the fact that we prove it for a random projection Q. A deterministic statement of this type is immediate (by dualizing the proof of Proposition 2.4 (a) below).

A typical situation when this theorem may be used is when a symmetric convex body $\tilde{B} \subset \mathbb{R}^m$ is given, $\mathcal{E} \supset \tilde{B}$ is any ellipsoid satisfying property $(*)$ with respect to $\tilde{B}$ (see e.g., Lemma 2.2 below), and $\mathcal{D} \supset \tilde{B}$ is arbitrary.

Proof The first part of the theorem is elementary. Without lost of generality we may assume that $\mathcal{E} = B_2$. Let $\rho_1 \geq ... \geq \rho_m > 0$ and let $\{e_i\}_{i=1}^m$ be an orthonormal basis such that $\mathcal{D}$ is of the form

$$\mathcal{D} = \Big\{x = \sum_{i=1}^m x_i e_i \in \mathbb{R}^m \mid \sum_{i=1}^m x_i^2/\rho_i^2 \leq 1\Big\}.$$

Considering the orthogonal projection Q on the span $\{e_i\}_{i=m-k+1}^m$, we obtain, by property $(*)$,

$$\rho_{m-k+1} = \|Q : \mathcal{D} \to B_2\| \geq \|Q : B \to B_2\| \geq \phi(k/m).$$

Let $\bar{\rho}_i = \min\{1, \rho_i\}$. Clearly we have

$$\mathcal{E}_1 := \Big\{x = \sum_{i=1}^m x_i e_i \in \mathbb{R}^m \mid \sum_{i=1}^m x_i^2/\bar{\rho}_i^2 \leq 1\Big\} \subset \mathcal{E} \cap \mathcal{D}.$$

Thus

$$|\mathcal{E} \cap \mathcal{D}|/|\mathcal{E}| \geq \prod_{l=1}^m \bar{\rho}_l \geq \prod_{l=1}^m \phi(l/m) = A^{-m},$$

which implies the first part of the theorem.

To prove the second part of the theorem, let us note that, by duality, it is enough to prove that

$$\mathcal{D}^0 \cap E \subset f_{a,\alpha}(\xi)\ B_2 \cap E \tag{2.2}$$

for a random (in B_2) subspace E. To show this we shall use the well-known lower M^*-estimate ([M1, PT, Go, M5]) which says that for every convex body $K \subset \mathbb{R}^m$ a random $\lceil \xi m \rceil$-dimensional subspace E satisfies

$$K \cap E \subset \frac{2M^*(K)}{\sqrt{1-\xi}} B_2,$$

where

$$M^*(K) = \left(\frac{1}{m}\ \mathbb{E}\left\|\sum_{i=1}^m g_i e_i\right\|_{K^0}^2\right)^{1/2},$$

for independent identically distributed standard Gaussian random variables $g_1, \dots, g_m$ and the Minkowski functional $\|\cdot\|_{K^0}$.

Note that $\mathcal{D}^0$ is the ellipsoid with the semiaxes $1/\rho_i$, $1 \le i \le m$, and that for every $b > 0$ one has

$$\mathcal{D}^0 \cap bB_2 \subset \sqrt{2}\mathcal{E}_2,$$

where

$$\mathcal{E}_2 := \left\{ x = \sum_{i=1}^m x_i e_i \in \mathbb{R}^m \mid \sum_{i=1}^m x_i^2/\lambda_i^2 \le 1 \right\}$$

for $\lambda_i = \min\{b, 1/\rho_i\}$, $1 \le i \le m$. It is easy to see that

$$M^*\left(\sqrt{2}\mathcal{E}_2\right) = \frac{1}{\sqrt{m}} \left(2 \sum_{i=1}^m \lambda_i^2 \right)^{1/2}$$

Fix $b \ge a$ to be determined later. Since $\rho_{m-k+1} \ge a^{-1}(k/m)^\alpha$, then $\lambda_i \le \min\{b, a(m/i)^\alpha\}$ for $1 \le i \le m$, and we obtain

$$\sum_{i=1}^m \lambda_i^2 \le \sum_{i \le (a/b)^{1/\alpha} m} b^2 + \sum_{m \ge i \ge (a/b)^{1/\alpha} m} a^2 (m/i)^{2\alpha}.$$

In the case $\alpha = 1/2$ the latter expression is less than or equal to

$$a^2 m + a^2 m (1 + \ln(b^2/a^2)) = 2a^2 m (1 + \ln(b/a)).$$

In the case $\alpha > 1/2$, this expression is less than or equal to

$$m a^{1/\alpha} b^{2-1/\alpha} + b^2 + \frac{a^2 m}{2\alpha - 1} (b/a)^{2-1/\alpha} \le \frac{2\alpha + 1}{2\alpha - 1} m a^{1/\alpha} b^{2-1/\alpha},$$

if $m(a/b)^{1/\alpha} \ge 1$. Otherwise, if $m(a/b)^{1/\alpha} \le 1$, the expression is less than or equal to

$$a^2 m^{2\alpha} \frac{2\alpha}{2\alpha - 1} \le \frac{2\alpha}{2\alpha - 1} m a^{1/\alpha} b^{2-1/\alpha}.$$

Thus, by the lower M^*-estimate, we obtain for a random $\lceil \xi m \rceil$-dimensional subspace E

$$\mathcal{D}^0 \cap (bB_2) \cap E \subset \sqrt{2}\mathcal{E}_2 \cap E \subset A_\alpha \; B_2 \cap E, \tag{2.3}$$

where for $\alpha = 1/2$ we set

$$A_{1/2} := \frac{4a}{(1-\xi)^{1/2}} \sqrt{1 + \ln(b/a)},$$

and for $\alpha > 1/2$ we set

$$A_\alpha := 2\sqrt{2}\left(\frac{2\alpha+1}{2\alpha-1}\right)^{1/2}(1-\xi)^{-1/2}a^{1/2\alpha}b^{1-1/2\alpha}.$$

We now treat the two cases separately. Let first $\alpha = 1/2$. Then let

$$b := \frac{4a}{(1-\xi)^{1/2}}\sqrt{\ln(20/(1-\xi))},$$

so that $A_{1/2} < b$. To prove (2.2) fix $x \in \mathcal{D}^0 \cap E$. Let $x' = (b/|x|)x$. If $b < |x|$ then $x' \in \mathcal{D}^0 \cap E$, and hence $x' \in \mathcal{D}^0 \cap (bB_2) \cap E$. By (2.3) we then get $x' \in A_{1/2}B_2 \cap E$. But $|x'| = b > A_{1/2}$, a contradiction. This shows that $b \geq |x|$ and thus (2.2) holds with $f_{a,1/2}(\xi) \leq 4a(1-\xi)^{-1/2}\sqrt{\ln(20/(1-\xi))}$. This concludes the proof in the case $\alpha = 1/2$.

In the case $\alpha > 1/2$ we let

$$b := 8^\alpha\Big(\frac{2\alpha+1}{2\alpha-1}\Big)^\alpha(1-\xi)^{-\alpha}a$$

so that $A_\alpha = b$. A similar argument as before shows (2.2) with $f_{a,\alpha}(\xi) \leq 8^\alpha((2\alpha+1)/(2\alpha-1))^\alpha(1-\xi)^{-\alpha}a$. This concludes the proof of the theorem. □

As the proof above shows the second part of the theorem still holds for certain functions going to 0 faster than a power type functions; however as this case seems less important at the present time we omit the details.

As already mentioned, the minimal volume ellipsoid satisfies $(*)$ with $\phi(t) = \sqrt{t}$. A more general class of examples is provided by proportional-dimensional projections of the minimal volume ellipsoids.

Lemma 2.2 *Let $K \subset \mathbb{R}^n$ be a symmetric convex body and let $\mathcal{E}_K$ be the ellipsoid of minimal volume for K. Let P be an arbitrary projection in $\mathbb{R}^n$ with* rank $P = m = \alpha n$, *for some $0 < \alpha \leq 1$. Then $P\mathcal{E}_K$ has property $(*)$ with respect to PK with function $\phi(t) = \sqrt{\alpha t}$.*

Proof Without loss of generality assume that $\mathcal{E}_K$ is the canonical ball B_2 in $\mathbb{R}^n$. If P is an orthogonal projection on a subspace $E := P(\mathbb{R}^n) \subset \mathbb{R}^n$, then for any orthogonal projection Q in E of rank k, QP can be considered as an orthogonal projection in $\mathbb{R}^n$ of rank k. Thus, by duality,

$$\|Q : PK \to PB_2\| \geq \|QP : K \to B_2\| = \|i_{|H} : B_2 \cap H \to K^0\|,$$

where $i : B_2 \to K^0$ is the formal identity operator and $H = QP(\mathbb{R}^n)$. By the well-known property of the ellipsoid of minimal volume, the last operator has norm larger than or equal to $\sqrt{k/n} = \sqrt{\alpha}\sqrt{k/m}$ (see e.g., [T], §15 or [GM]).

If P is an arbitrary projection then let $F := P(\mathbb{R}^n)$, set $E = (\ker P)^{\perp}$ and let P_E be the orthogonal projection onto E. Then the operator $T := (P_E)_{|F} : F \to E$ is invertible. It is easy to check that $TPx = P_E x$ for all $x \in \mathbb{R}^n$. In particular, $TPK = P_E K$ and $TPB_2 = P_E B_2$, in addition, for any projection $Q : F \to F$ orthogonal in PB_2, the operator TQT^{-1} is an orthogonal projection in E with the same rank as Q. This, and the first part of the proof, clearly imply

$$\begin{aligned} \|Q : PK \to PB_2\| &= \|TQT^{-1} : TPK \to TPB_2\| \\ &= \|TQT^{-1} : P_E K \to P_E B_2\| \geq \sqrt{\alpha}\sqrt{k/m}. \end{aligned}$$

This completes the proof. □

For future reference we formulate an important case.

Corollary 2.3 *Let $m \leq n = \beta m$ for some $\beta \geq 1$. Let $K \subset \mathbb{R}^n$ be a symmetric convex body and let $\mathcal{E}_K$ be the ellipsoid of minimal volume for K. Let P be an arbitrary projection in $\mathbb{R}^n$ with* rank $P = m$. *Set $E = P(\mathbb{R}^n)$ and $\mathcal{E} = P(\mathcal{E}_K)$. Let $\mathcal{D} \subset E$ be an arbitrary ellipsoid such that $\mathcal{D} \supset PK$. Then*

$$|\mathcal{E}|^{1/m} \leq c\sqrt{\beta}|\mathcal{D}|^{1/m}, \tag{2.4}$$

where $c > 0$ is an absolute constant. Furthermore, for every $0 < \xi < 1$, a random projection Q in E of rank $\lceil \xi m \rceil$ orthogonal with respect to $\mathcal{E}$ satisfies

$$Q\mathcal{E} \subset f(\xi)\, Q\mathcal{D},$$

where $f(\xi) = 4\sqrt{\beta}(1-\xi)^{-1/2}\sqrt{\ln(20/(1-\xi))}$.

Proof By Lemma 2.2, $\mathcal{E}$ has property $(*)$ with respect to PK with function $\phi(t) = \sqrt{t/\beta}$. Since $PK \subset \mathcal{E} \cap \mathcal{D}$ implies $\|Q : PK \to \mathcal{E}\| \leq \|Q : \mathcal{E} \cap \mathcal{D} \to \mathcal{E}\|$ for every projection Q, it also has $(*)$ with respect to $\mathcal{E} \cap \mathcal{D}$. The conclusion follows from Theorem 2.1 and the form of function $f_{a,1/2}$. □

Theorem 2.1 suggests a "relaxation" of the relation of containment between two ellipsoids, which seems to be of independent interest.

We formalized it in the following definition. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two ellipsoids on $\mathbb{R}^n$. We say that $\mathcal{E}_1$ is *essentially contained* in $\mathcal{E}_2$ if for every $0 < \lambda < 1$ there is $C(\lambda) \geq 1$, depending on λ only, and a subspace $E \subset \mathbb{R}^n$ with $\dim E \geq \lambda n$ such that

$$\mathcal{E}_1 \cap E \subset C(\lambda)\, \mathcal{E}_2 \cap E. \tag{2.5}$$

In such a case we may also say that $\mathcal{E}_2$ *essentially contains* $\mathcal{E}_1$. We shall say that two ellipsoids are *essentially equivalent* if there is a number $a > 0$ such that $\mathcal{E}_1$ is essentially contained in $a\mathcal{E}_2$ and $a\mathcal{E}_2$ is essentially contained in $\mathcal{E}_1$. (We could also consider the dual notion in terms of projections, but for the time being there does not seem to be much advantage in doing this.)

Since (2.5) deals with sections rather than projections, it is connected with the property $(**)$ dual to property $(*)$, introduced as follows. Let $B \subset \mathbb{R}^m$, $\mathcal{E} \subset \mathbb{R}^m$ and $\phi : (0,1] \to (0,1]$ be the same as in the definition of property $(*)$ above. We say that $\mathcal{E}$ has property $(**)$ with respect to B with function ϕ, whenever

$(**)$ for any $1 \leq k \leq m$ and any subspace $E \subset \mathbb{R}^m$ of dimension k we have $\|i_{|E} : \mathcal{E} \cap E \to B\| \geq \phi(k/n)$.

A prime example of ellipsoids satisfying property $(**)$ are proportional dimensional sections of ellipsoids of maximal volume. If $K \subset \mathbb{R}^n$ is a symmetric convex body and $\mathcal{E}'_K \subset K$ is the ellipsoid of maximal volume for K, and $E \subset \mathbb{R}^n$ with $\dim E = m = \alpha n$, for some $0 < \alpha < 1$, then $\mathcal{E}'_K \cap E$ satsifies $(**)$ with respect to $K \cap E$ with $\phi(t) = \sqrt{\alpha t}$.

An easy straightforward argument shows relations between an ellipsoid satisfying property $(**)$ with respect to a body K and any ellipsoid $\mathcal{D} \subset K$, and in particular a distance ellipsoid for K. Both parts of the proposition below are most interesting for ellipsoids of maximal volume.

Proposition 2.4 *Let $K \subset \mathbb{R}^n$ be a symmetric convex body and let $\mathcal{E}$ be an ellipsoid satisfying property $(**)$ with respect to K with a certain function ϕ.*

(a) *Then $\mathcal{E}$ essentially contains every ellipsoid $\mathcal{D} \subset K$, with $C(\lambda) \leq \frac{1}{\phi(1-\lambda)}$.*

(b) *If additionally $\mathcal{E} \subset K$ then there exists an ellipsoid $\mathcal{D}$ with $\mathcal{D} \subset K \subset \sqrt{2} d_K \mathcal{D}$ which is essentially equivalent to $\mathcal{E}$ (here $d_K = d(K, B_2)$ denotes the Banach–Mazur distance to the Euclidean space).*

Proof (a) With the same notation as at the beginning of the proof of Theorem 2.1, for every $1 \le k \le n$ consider the subspace $F_k := \mathrm{span}\,\{e_i\}_{i=1}^k$. Then our assumptions imply

$$\rho_k^{-1} = \|i_{|E} : \mathcal{E} \cap E \to \mathcal{D}\| \ge \|i_{|E} : \mathcal{E} \cap E \to K\| \ge \phi(k/n).$$

On the other hand, given $0 < \lambda < 1$, let $l = \lceil \lambda n \rceil$ and let $E := \mathrm{span}\,\{e_i\}_{i=n-l+1}^n$. Then $\dim E = l \ge \lambda n$ and we have

$$\mathcal{D} \cap E \subset \rho_{n-l+1} B_2 \cap E \subset \frac{1}{\phi(1-\lambda)} \mathcal{E} \cap E,$$

completing the proof of (a).

(b) Pick any $\mathcal{D}' \subset K \subset d_K \mathcal{D}'$ and consider $B = \mathrm{conv}\,(\mathcal{D}' \cup \mathcal{E})$. Pick an ellipsoid $\mathcal{D}$ such that $\mathcal{D} \subset B \subset \sqrt{2}\mathcal{D}$. Clearly, $\mathcal{D} \subset B \subset K \subset \sqrt{2} d_K \mathcal{D}$. Since $\mathcal{E}$ satisfies $(**)$ then (a) implies that $\mathcal{E}$ essentially contains $\mathcal{D}$ with $C(\lambda) \le 1/\phi(1-\lambda)$. On the other hand, clearly $\mathcal{E} \subset B \subset \sqrt{2}\mathcal{D}$, completing the proof of (b). □

Remark For functions ϕ as in the second part of Theorem 2.1, this theorem provides, by duality, a "randomized" version of Proposition 2.4 in which the existence of a subspace E satisfying (2.5) is replaced by the statement about "random subspaces" E.

3 Bodies in M-position

Let us first recall the definition and a few basic facts about M-ellipsoids and M-positions of symmetric convex bodies.

Let K and L be two sets on $\mathbb{R}^n$. By $N(K, L)$ we denote the covering number, i.e. the minimal number of translations of L needed to cover K.

Let $K \subset \mathbb{R}^n$ be a symmetric convex body and let $C > 0$. We say that B_2 is an M-ellipsoid for K with constant C if we have

$$\max\left\{N(K, B_2), N(B_2, K), N(K^0, B_2), N(B_2, K^0)\right\} \le \exp(Cn). \tag{3.1}$$

In this case we shall often say that K is in M-position with constant C. It is a deep theorem first proved in [M3] that there is an absolute constant $C_0 > 0$ such that for every symmetric convex body K in $\mathbb{R}^n$ there exists a linear transformation taking K into M-position with constant C_0. Throughout the paper we shall use the notation C_0 for such a constant in (3.1). However we shall often omit to mention it explicitely and we may just write, for example, that K is in M-position. Still, the reader should always remember that from now on all our absolute constants later actually depend on this C_0.

It follows from the definition that if K is in M-position then so is K^0 and that

$$|e^{-C_0}B_2| \leq \min(|K|, |K^0|) \leq \max(|K|, |K^0|) \leq |e^{C_0}B_2|.$$

Without loss of generality we assume from now on that whenever K is in M-position then $|K| = |B_2|$.

In fact the estimates (3.1) are consequences of the conditions $|K| = |B_2|$ and one estimate $N(K, B_2) \leq \exp(C_0' n)$ with $C_0' > 0$ (see Lemma 4.2 of [MS2] or Lemma 10 and Remark 1 that follows in [MPa]).

Note, for future reference, that the covering $K \subset \bigcup_{i=1}^N (x_i + B_2)$ (with $N \leq \exp(C_0 n)$) easily implies the volume estimates

$$\left(\frac{|K + B_2|}{|B_2|}\right)^{1/n}, \quad \left(\frac{|K|}{|(B_2 \cap K)|}\right)^{1/n} \leq 2\, e^{C_0}. \tag{3.2}$$

Furthermore, for any two sets in $\mathbb{R}^n$ and every projection P and every subspace E one has

$$N(PK, PL) \leq N(K, L) \quad \text{and} \quad N(K \cap E, (L - L) \cap E) \leq N(K, L). \tag{3.3}$$

The first estimate is trivial. For the second, note that a covering $K \subset \bigcup_{i=1}^N (x_i + L)$ implies the covering $K \cap E \subset \bigcup_{i \in I} (x_i + L) \cap E$, where $I := \{i \mid (x_i + L) \cap E \neq \emptyset\}$. For each $i \in I$ it is easy to see that $(x_i + L) \cap E \subset z_i + (L - L) \cap E$, for any $z_i \in (x_i + L) \cap E$. Also note that if L is symmetric and convex then $L - L = 2L$.

Let us now describe a functorial construction which plays a fundamental role in our results. For each symmetric convex body K in $\mathbb{R}^n$ in M-position and every $0 < \lambda < 1$ we shall define a certain subset $\mathcal{F}_{\lceil \lambda n \rceil}(K) \subset G_{n, \lceil \lambda n \rceil}$ such that

$$\mu_{n, \lceil \lambda n \rceil}\left(\mathcal{F}_{\lceil \lambda n \rceil}(K)\right) \geq 1 - e^{-c_\lambda n}, \tag{3.4}$$

where $c_\lambda > 0$ is a function of λ only. In the future we shall refer to a subset satisfying measure estimates of this type as a *random family*.

Given $K \subset \mathbb{R}^n$ as above, recall that the ellipsoids of minimal and maximal volume for K are denoted by $\mathcal{E}_K \supset K$ and $\mathcal{E}'_K \subset K$, respectively. We shall denote the semi-axes of $\mathcal{E}_K$ by $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$

and a corresponding orthonormal basis by $\{e_i\}_{i=1}^n$. Similar notation is adopted for $\mathcal{E}'_K$ with the semi-axes $\rho'_1 \geq \rho'_2 \geq \ldots \geq \rho'_n$ and a corresponding orthonormal basis $\{e'_i\}_{i=1}^n$.

Define $\mathcal{F}_{\lceil\lambda n\rceil}(K)$ as the set of all $E \in G_{n,\lceil\lambda n\rceil}$ satisfying

(i) $P_E B_2 \subset C_\lambda P_E K$, where P_E is orthogonal in B_2;

(ii) $K \cap E \subset C_\lambda B_2 \cap E$;

(iii) $|P_E x| \geq b_\lambda |x|$ for every $x \in \operatorname{span}\{e_i\}_{i=1}^m \cup \operatorname{span}\{e'_i\}_{i=n-m+1}^n$, where $m = \lceil\lambda n/2\rceil$,

where $C_\lambda > 0$ is an appropriate function on λ, and $b_\lambda = c_0\sqrt{\lambda}$ with an appropriate absolute constant $c_0 > 0$. Below we keep the notation C_λ, b_λ and c_0 for these constants.

Proposition 3.1 *Let K be a symmetric convex body in M-position. Then there exist a choice of C_λ, c_λ and c_0 such that the corresponding family $\mathcal{F}_{\lceil\lambda n\rceil}(K)$ satisfies (3.4).*

Proof The first two conditions in the definition of $\mathcal{F}_{\lceil\lambda n\rceil}(K)$ are closely related to the fact that for a body K in M-position, random proportional-dimensional projections of K have finite volume ratio. This was discovered (even before the existence of an M-ellipsoid) in [M2] (Theorem 4.1, Step d, p. 389), see also [M4], p. 107 for a slightly stronger statement. We shall use the general volume ratio argument that if $K \subset \mathbb{R}^n$ is a symmetric convex body and $(|K + B_2|/|B_2|)^{1/n} \leq a$ then for any $0 < \lambda < 1$, for a set of subspaces $E \in G_{n,\lceil\lambda n\rceil}$ of large measure we have $K \cap E \subset CB_2 \cap E$, where $C \leq (4\pi a)^{1/(1-\lambda)}$ (see e.g. Chapter 6 of [Pi]). By duality, if $(|B_2|/|K \cap B_2|)^{1/n} \leq b$ then for a set of subspaces $E \in G_{n,\lceil\lambda n\rceil}$ of large measure we have $P_E B_2 \subset C_1 P_E K$, where $C_1 = (cb)^{1/(1-\lambda)}$ and $c > 0$ is an absolute constant. Since in our situation K is in M-position, it easily follows from (3.2) that the required upper estimates for a and b are satisfied, with a and b depending on C_0, and hence conditions (i) and (ii) hold with C_λ depending on λ only.

To prove that the third condition is also satisfied for the set of large measure we need the following lemma well known to experts.

Lemma 3.2 *Let $m \leq n$ and $1 \leq k \leq m/2$, and let $H \subset \mathbb{R}^n$ be a k-dimensional subspace. Then*

$$\mu_{n,m}\left(\left\{E \mid |P_E x| \geq c\sqrt{m/n}\,|x| \ \text{ for all } x \in H\right\}\right) \geq 1 - e^{-ck},$$

where $c > 0$ is an absolute constant.

Remark The estimate is also valid for $k \leq \xi m$, for any $0 < \xi < 1$, with the constant c replaced by a function of ξ.

Returning to the proof ot Proposition 3.1, condition (iii) uses the estimate from Lemma 3.2 twice, separately for $H = \operatorname{span}\{e_i\}_{i=1}^{\lceil \lambda n/2 \rceil}$ and for $H' = \operatorname{span}\{e'_i\}_{i=n-\lceil \lambda n/2 \rceil+1}^{n}$. Combining this with the estimates for the sets satisfying (i) and (ii) we finally get

$$\mu_{n,\lambda n}\left(\mathcal{F}_{\lambda n}(K)\right) \geq 1 - \exp\left(-c_\lambda n\right),$$

which is required in (3.4). □

Remark The function C_λ in conditions (i) and (ii) can be improved to a polynomial dependence on $1/(1-\lambda)$ by choosing a stronger definition of an M-ellipsoid. For example, using Theorem 7.13 of [Pi] and Theorem 3.2 of [LT], we immediately get $C_\lambda \leq C\varepsilon^{-3/2}(1-\lambda)^{-(\varepsilon+1/2)}$, for any $\varepsilon > 0$.

Proof of the Lemma 3.2 The lemma can be proved by a reduction to the Gaussian case (as in [MT], Proposition 3.1) and then using a similar fact for $k \times m$ Gaussian matrices (cf. e.g., [Sz], Lemma 2.9). For the reader's convenience we also outline a standard direct argument, which however works for $k \leq c_1 m$ only, where $0 < c_1 < 1$ is a universal constant. First we estimate the measure of the subset of all $E \in G_{n,m}$ satisfying a slightly stronger inequality for a fixed vector x_0 with $|x_0| = 1$; and then we combine this measure estimate with a so-called ε-net argument. To get the first measure estimate we observe that

$$\begin{aligned}
&\mu_{n,m}\left(\left\{E \mid |P_E x_0| \geq 2c\sqrt{m/n}\right\}\right) \\
&= \quad h_n\left(\left\{U \in \mathcal{O}_n \mid |P_m U x_0| \geq 2c\sqrt{m/n}\right\}\right) \\
&= \quad \mu_n\left(\left\{z \in S_{n-1} \mid |P_m z| \geq 2c\sqrt{m/n}\right\}\right),
\end{aligned}$$

where h_n denotes the normalized Haar measure on the orthogonal group $\mathcal{O}_n$, P_m is the orthogonal projection in $\mathbb{R}^n$ on the first m coordinates, and μ_n denotes the normalized measure on the sphere S_{n-1}. The measure of the latter set can be then estimated by noting that $\mathbb{E}|P_m z| \sim \sqrt{m/n}$ and then using the standard concentration inequality for Lipschitz functions on the sphere ([MS1]). □

4 Main results

The main result of this paper is the following theorem.

Theorem 4.1 *Let K and L be two symmetric convex bodies in $\mathbb{R}^n$ in M-position, and assume that for some $0 < \lambda < 1$ and some $d > 1$ there is a quotient space $qK \in \mathcal{F}_{q,\lceil \lambda n \rceil}(K)$ and a subspace $sL \in \mathcal{F}_{s,\lceil \lambda n \rceil}(L)$ such that the Banach-Mazur distance satisfies*

$$d(qK, sL) \leq d.$$

Then

$$\operatorname{outvr}(K) \leq C \quad \textit{and} \quad \operatorname{vr}(L) \leq C,$$

where $C = C(\lambda, d)$ is a function of λ and d only.

The proof of the theorem is based on the following proposition.

Proposition 4.2 *Let $K \subset \mathbb{R}^n$ be a symmetric convex body in M-position. Let $0 < \lambda < 1$. Let $E \in \mathcal{F}_{\lceil \lambda n \rceil}(K)$ and let P_E be the orthogonal projection on E. Then*

$$\operatorname{outvr}(K) \leq C'_\lambda \Big(\operatorname{outvr}(P_E K) \Big)^2, \tag{4.1}$$

where C'_λ depends on λ (and on constant C_0 which defines the M-position we use).

In other words, the proposition says that if K is in M-position then for any quotient space $qK \in \mathcal{F}_{q,\lceil \lambda n \rceil}(K)$ we have $\operatorname{outvr}(K) \leq C'_\lambda (\operatorname{outvr}(qK))^2$.

Remark As it can be seen from the proof below, the power 2 in the estimate (4.1) can be improved to any $\alpha > 1$.

Proof of Proposition 4.2 Recall that $\mathcal{E}_K \supset K$ is the ellipsoid of minimal volume for K, and we denoted its semi-axes by $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$, and the corresponding orthonormal basis by $\{e_i\}_{i=1}^n$. To simplify the notation, set $P := P_E$. Consider the ellipsoid $P\mathcal{E}_K$ in E, and denote its semi-axes by $\rho'_1 \geq \rho'_2 \geq \ldots \geq \rho'_m$ (where $m := \lceil \lambda n \rceil$). (There will be no confusion with the semi-axes of the ellipsoid of maximal volume since we do not consider this ellipsoid in this proof.)

The natural Euclidean structure in E is of course given by $PB_2 = B_2 \cap E$, and by the definition of $\mathcal{F}_m(K)$ we have $PB_2 \subset C_\lambda PK$. On the other hand, clearly, $PK \subset P\mathcal{E}_K$, and hence $\rho'_m \geq C_\lambda^{-1}$.

We first observe that since $E \in \mathcal{F}_m(K)$ then for all $1 \le j \le \lceil m/2 \rceil$ we have

$$\rho'_j \le \rho_j \le (1/b_\lambda)\rho'_j. \tag{4.2}$$

Indeed, given an ellipsoid $\mathcal{D} \subset \mathbb{R}^m$ with semi-axes $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_m$ one has

$$\lambda_j = \inf_L \sup_{x \in L \cap \mathcal{D}} |x|,$$

where infimum is taken over all $(m-j+1)$-dimensional subspaces L. Thus, since $|Px| \le |x|$ for $x \in \mathbb{R}^n$, we have $\rho'_j \le \rho_j$ for every $j \le m$. On the other hand, since $\lceil m/2 \rceil = \lceil \lambda n/2 \rceil$, by the definition of $\mathcal{F}_m(K)$, we have $|Px| \ge b_\lambda |x|$ for every $x \in E_0 := \operatorname{span}\{e_i \mid 1 \le i \le \lceil m/2 \rceil\}$, which means that the operator

$$P|_{E_0} \ : \ (E_0, B_2 \cap E_0) \to (PE_0, PB_2)$$

is invertible with the norm of the inverse bounded by $1/b_\lambda$. That implies $\rho_j \le (1/b_\lambda)\rho'_j$.

Now by (4.2) we get

$$\begin{aligned}
\left(\frac{|\mathcal{E}_K|}{|K|}\right)^{1/n} &= \left(\frac{|\mathcal{E}_K|}{|B_2|}\right)^{1/n} = \left(\prod_{i=1}^{n} \rho_i\right)^{1/n} \le \left(\prod_{i=1}^{\lceil m/2 \rceil} \rho_i\right)^{1/\lceil m/2 \rceil} \\
&\le \frac{C_\lambda}{b_\lambda}\left(\prod_{i=1}^{m} \rho'_i\right)^{2/m} = \frac{C_\lambda}{b_\lambda}\left(\frac{|P\mathcal{E}_K|}{|PB_2|}\right)^{2/m}.
\end{aligned} \tag{4.3}$$

Let $\mathcal{D} \supset PK$ be the ellipsoid of minimal volume for PK, so that

$$|\mathcal{D}|^{1/m} = \operatorname{outvr}(PK)\ |PK|^{1/m}. \tag{4.4}$$

Applying Corollary 2.3 for the ellipsoids $\mathcal{E} = P\mathcal{E}_K \subset E$ and $\mathcal{D} \subset E$ we get, by (2.4),

$$|P\mathcal{E}_K|^{1/m} \le \left(c/\sqrt{\lambda}\right)|\mathcal{D}|^{1/m}.$$

By the definition of M-ellipsoid we have $|PK| \le \exp(C_0 n)|PB_2|$. Thus we get

$$\begin{aligned}
\left(\frac{|P\mathcal{E}_K|}{|PB_2|}\right)^{2/m} &\le (c^2/\lambda)(\operatorname{outvr}(PK))^2 \left(\frac{|PK|}{|PB_2|}\right)^{2/m} \\
&\le \left(c^2 \exp(2C_0/\lambda)/\lambda\right)(\operatorname{outvr}(PK))^2.
\end{aligned}$$

Combining this with (4.3) and the form of b_λ we obtain (4.1) with $C'_\lambda = (c^2/c_0)\; C_\lambda\; \lambda^{-3/2}\; \exp(2C_0/\lambda)$, which completes the proof. $\Box$

Now we are in the position to prove Theorem 4.1.

Proof of Theorem 4.1 By the definition of M-position (3.1) and by (3.3), we have

$$|PK| \le \exp(C_0 n)\; |PB_2| \quad \text{and} \quad |B_2 \cap E| \le \exp(C_0 n)\; |2L \cap E|$$

for every projection P and every subspace E. Thus, by the definitions of $\mathcal{F}_{\lceil \lambda n \rceil}(K)$ we have that every quotient $qK \in \mathcal{F}_{q,\lambda n}(K)$ admits an estimate for the volume ratio,

$$\mathrm{vr}\,(qK) \le \Big(|P_E K| / |C_\lambda^{-1} P_E B_2| \Big)^{1/\lceil \lambda n \rceil} \le C_\lambda \exp(C_0/\lambda).$$

Similarly, every subspace $sL \in \mathcal{F}_{s,\lceil \lambda n \rceil}(L)$ admits an estimate for the outer volume ratio, $\mathrm{outvr}\,(sL) \le a_\lambda := 2C_\lambda \exp(C_0/\lambda)$. Now, let qK and sL satisfy the hypothesis of the theorem, then

$$\mathrm{outvr}\,(qK) \le d\, \mathrm{outvr}\,(sL) \le a_\lambda d.$$

By Proposition 4.2 we obtain

$$\mathrm{outvr}\,(K) \le C'_\lambda\, (a_\lambda d)^2 .$$

The estimate for $\mathrm{vr}\,(L)$ follows by duality. $\Box$

Remark The dependence on d in $C(\lambda, d)$ can be improved by using the remark after Proposition 4.2 and a modification of the family $\mathcal{F}_{\lceil \lambda n \rceil}(K)$. We then obtain that for every $\alpha > 1$, $C(\lambda, d) \le C_{\lambda,\alpha} d^\alpha$, where $C_{\lambda,\alpha}$ depends on λ and α only.

Setting $K = B_2$ in Theorem 4.1 we get an interesting corollary.

Corollary 4.3 *Let L be a symmetric convex body in $\mathbb{R}^n$ in M-position. If for some $0 < \lambda < 1$ and some $d > 1$ a random $\lceil \lambda n \rceil$ section sL of L is d-Euclidean, then $\mathrm{vr}\,(L) \le C$, where $C = C(\lambda, d)$ is a function of λ and d only.*

This corollary was proved in [MS2] in the case when the Euclidean distance was replaced by the geometric distance to the ball B_2. In this case it is shown by combining Theorems 3.1' and 2.2 in [MS2], that for any $0 < \xi < 1$, a random section of L is C-equivalent to B_2.

Theorem 4.1 has the following standard consequence about the existence of a large family of Euclidean quotients and subspaces.

Corollary 4.4 *Let K and L be two symmetric convex bodies in $\mathbb{R}^n$ in M-position, and assume that for some $0 < \lambda < 1$ and some $d > 1$ there is a quotient space $qK \in \mathcal{F}_{q,\lceil \lambda n \rceil}(K)$ and a subspace $sL \in \mathcal{F}_{s,\lceil \lambda n \rceil}(L)$ such that the Banach-Mazur distance satisfies*

$$d(qK, sL) \leq d.$$

Then for every $0 < \xi < 1$ a random orthogonal (in $\mathcal{E}_K$) projection of K is $\bar{C}$-Euclidean and a random (in $\mathcal{E}'_L$) section of L is $\bar{C}$-Euclidean, where $\bar{C} = \bar{C}(\lambda, \xi, d)$ is a function of λ, ξ and d only.

Proof The proof relies on the volume ratio argument. Recall that since $\mathcal{E}_K$ is the ellipsoid of minimal volume for K then a random orthogonal (in $\mathcal{E}_K$) projection satisfies

$$P_E \mathcal{E}_K \subset (4\pi \operatorname{outvr}(K))^{1/(1-\lambda)} P_E K; \tag{4.5}$$

and since $\mathcal{E}'_L$ is the ellipsoid of maximal volume for L then a random (in $\mathcal{E}'_L$) subspace E of $\mathbb{R}^n$ satisfies

$$K \cap E \subset (4\pi \operatorname{vr}(K))^{1/(1-\lambda)} \mathcal{E}'_K \cap E \tag{4.6}$$

(see e.g. Chapter 6 of [Pi]).

The conclusion of the corollary follows directly from Theorem 4.1 and (4.5), (4.6) with $\bar{C} = (4\pi C)^{1/(1-\xi)}$, where C is a function from Theorem 4.1. □

As we have just seen, the closeness of spaces qK and sL in Theorem 4.1 and Corollary 4.4 implies the existence of many Euclidean quotients and subspaces, of an arbitrary proportional dimension, for K and L, respectively. However one may ask whether spaces qK and sL themselves are isomorphic to Euclidean as well? Surprisingly, the answer in general is *no*: an example below shows that for some K and L one may select M-ellipsoids in such a way that random quotients of K and subspaces of L are far from Euclidean, while being close together. At the same time we believe that it might be true that for a judiciously selected M-ellipsoid, the hypothesis of our theorem indeed implies that qK and sL are Euclidean, with a high probability.

Example 4.5 Let $k = \lceil n / \ln n \rceil$ and $m = n - k$. Write $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^k$. Let V be an arbitrary k-dimensional symmetric convex body such that $B_2^k \subset V \subset k B_\infty^k$, where B_p^k is the unit ball of ℓ_p^k. Set $K = B_2^m \oplus_2 V$.

Note that $B_2 = B_2^n$ is contained in K and K has bounded volume ratio with respect to B_2. This immediately implies that a multiple of B_2 by a universal constant is an M-ellipsoid for K. Of course the randomness with respect to this M-ellipsoid is the same as with respect to B_2. Since k is small, random proportional dimensional projections (with respect to B_2) are good isomorphisms, when restricted to $\mathbb{R}^k$ that corresponds to V (see Lemma 3.2). Thus, since B_2^k is contained in V, we have that PK is well isomorphic to $B_2^\ell \oplus_2 V$, where $\ell = \text{rank } P - k$. Actually, PK is isomorphic to the convex hull of V and $B_2^{\text{rank } P}$, but since V contains B_2^k, it is easy to see that this convex hull is isomorphic to the direct sum above.

Now set $L = B_2^m \oplus_2 W$ where W is an arbitary symmetric convex body in $\mathbb{R}^k$ such that $B_2^k \supset W \supset (1/k)B_1^k$. Then applying the above argument for L^0 and then dualizing again we get that (a multiple of) B_2 is an M-ellipsoid for L and L has random sections isomorphic to $B_2^\ell \oplus_2 W$.

Now fix an arbitrary V as above (and set $K = B_2^m \oplus_2 V$). Let W be an affine image of V satisfying $B_2^k \supset W \supset (1/k)B_1^k$ (note that if we make B_2^k to be the ellipsoid of minimal volume for W then the second inclusion is automatically satisfied), and let $L = B_2^m \oplus_2 W$. Then random projections of K (being equivalent to $B_2^\ell \oplus_2 V$) and random sections of L (being equivalent to $B_2^\ell \oplus_2 W$) are well isomorphic to each other, while being very far from Euclidean. $\square$

We now pass to a discussion of the global form of the results of the first part of this section. Although we always have an analogy between local and global results, there is no an abstract argument proving this. In the present context the global result is much easier.

Instead of working with random families of subspaces of $\mathbb{R}^n$ we will work with random families of orthogonal operators. Let $O(n)$ denote the group of orthogonal operators on $\mathbb{R}^n$ and let ν denote the normalized Haar measure on $O(n)$. Given symmetric convex body $K \subset \mathbb{R}^n$ in M-position define $\mathcal{H}(K)$ as the set of all operators $U \in O(n)$ satisfying

(i) $cB_2 \subset K + UK$,

(ii) $c(K \cap UK) \subset B_2$

for some absolute constant $c > 0$. These two conditions are the global form of the conditions (i) and (ii) in the definition of the random family $\mathcal{F}_{[\lambda n]}(K)$. It can be shown ([M5]) that there exists a choice of $c > 0$ such that

$$\nu\left(\mathcal{H}(K)\right) \geq 1 - e^{-c_1 n},$$

where $c_1 > 0$ is an absolute constant.

Note that $(K + UK)^0$ is 2-equivalent to $K^0 \cap (U^*)^{-1}K^0$ and $(K \cap UK)^0$ is 2-equivalent to $K^0 + (U^*)^{-1}K^0$. Thus, since $(U^*)^{-1} = U$ for $U \in O(n)$, we obtain that $\mathcal{H}(K) = \mathcal{H}(K^0)$, possibly replacing the constant c in the definition by $c/2$.

The following theorem is the global version of Theorem 4.1.

Theorem 4.6 *Let K and L be two symmetric convex bodies in $\mathbb{R}^n$ in M-position. Assume that there are operators $U \in \mathcal{H}(K)$, $V \in \mathcal{H}(L)$, and some $d > 1$ such that*

$$d(K_0, L_0) \leq d,$$

where $K_0 = K + UK$ and $L_0 = L \cap VL$. Then

$$\text{outvr}\,(K) \leq (C_1/c)\exp(4C_0)d \quad \textit{and} \quad \text{vr}\,(L) \leq (C_1/c)\exp(4C_0)d,$$

where C_1 is an absolute constant and c is from the definition of the families $\mathcal{H}(K)$ and $\mathcal{H}(L)$.

Proof By the definition of the set $\mathcal{H}(K)$ we have $cB_2 \subset K_0$. On the other hand, by the definition of M-ellipsoid and covering numbers we obtain that $K + UK$ can be covered by $\exp(2C_0 n)$ translations of $2B_2$. That implies

$$\text{vr}\,(K_0) \leq (|K_0|/|cB_2|)^{1/n} \leq (2/c)\exp(2C_0).$$

To find the upper bound for the outer volume ratio of L_0 we could use a similar covering argument (cf. proof of the Claim 6.5 in [LMS]), however it is simplier to use duality. Indeed, $cB_2 \subset L_0^0 = \text{conv}\,(L_0, VL_0) \subset L^0 + VL_0$. Thus repeating the proof above and using Bourgain-Milman's inverse Santaló inequality ([BM2]) we obtain

$$\text{outvr}\,(L_0) \leq \Big(|(1/c)B_2|/|L_0|\Big)^{1/n} \leq C_1\Big(|L_0^0|/|cB_2|\Big)^{1/n} \leq (2C_1/c)\exp(2C_0),$$

where $C_1 > 0$ is an absolute constant. Since $d(K_0, L_0) \leq d$, then K_0 has outer volume ratio bounded by $(2C_1 d/c)\exp(2C_0)$. Let $\mathcal{E}$ be the minimal volume ellipsoid for K_0. Then $K \subset K_0 \subset \mathcal{E}$ and

$$\left(\frac{|\mathcal{E}|}{|K|}\right)^{1/n} = \left(\frac{|\mathcal{E}|}{|K_0|}\right)^{1/n} \left(\frac{|K_0|}{|B_2|}\right)^{1/n} \left(\frac{|B_2|}{|K|}\right)^{1/n} \leq (4C_1/c)\,\exp(2C_0)\,d,$$

which implies boundedness of outvr (K). The result for vr L follows by the similar argument. $\square$

Remark It is clear from the proof that the theorem can be generalized to the case of many orthogonal operators. Namely, let K, L, U, and V be as in the theorem. Assume further that U_1, ..., U_k and V_1, ..., V_m are *arbitrary* orthogonal operators on $\mathbb{R}^n$. Let $K_0 = K + UK + U_1K + ... + U_kK$ and $L_0 = L \cap VL \cap V_1L \cap ... \cap V_mL$. Then if $d(K_0, L_0) \le d$ then

$$\text{outvr}\,(K) \le Cd \quad \text{and} \quad \text{vr}\,(L) \le Cd,$$

where C is a function of k, m, c and C_0 only.

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