# Anti-concentration property for random digraphs and invertibility of their adjacency matrices

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**Abstract**

Let $\mathcal{D}_{n,d}$ be the set of all directed $d$-regular graphs on $n$ vertices. Let $G$ be a graph chosen uniformly at random from $\mathcal{D}_{n,d}$ and $M$ be its adjacency matrix. We show that $M$ is invertible with probability at least $1 - C\ln^3 d/\sqrt{d}$ for $C \leq d \leq cn/\ln^2 n$, where $c, C$ are positive absolute constants. To this end, we establish a few properties of directed $d$-regular graphs. One of them, a Littlewood–Offord type anti-concentration property, is of independent interest: Let $J$ be a subset of vertices of $G$ with $|J| \leq cn/d$. Let $\delta_i$ be the indicator of the event that the vertex $i$ is connected to $J$ and $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \{0,1\}^n$. Then $\delta$ is not concentrated around any vertex of the cube. This property holds even if a part of the graph is fixed.

**Résumé**

**Propriété d'anti-concentration pour les digraphes aléatoires et invertibilité de leur matrice d'adjacence.** Soit $\mathcal{D}_{n,d}$ l'ensemble des graphes orientés $d$-réguliers à $n$ sommets. Soit $G$ un élément choisi uniformément au hasard dans $\mathcal{D}_{n,d}$ et $M$ sa matrice d'adjacente. On montre que $M$ est inversible avec probabilité supérieure à $1 - C\ln^3 d/\sqrt{d}$ pour $C \leq d \leq cn/\ln^2 n$, où $c, C$ sont des constantes universelles positives. Afin d'établir ce résultat, nous montrons certaines propriétés des graphes orientés $d$-réguliers. Parmi celles-ci, une propriété d'anti-concentration de type Littlewood–Offord. Soit $J$ un sous-ensemble de sommets de $G$ de taille $|J| \leq cn/d$. Soit $\delta_i$ l'indicateur de l'évènement que le sommet $i$ est connecté à $J$ et on note $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \{0,1\}^n$. On montre alors que $\delta$ n'est concentré autour d'aucun sommet du cube. Cette propriété reste vraie si une partie du graphe est fixée.

**1. Introduction.** An undirected (resp., directed) graph $G$ with $n$ vertices is *$d$-regular* if every vertex has exactly $d$ neighbors (resp., $d$ in-neighbors and $d$ out-neighbors). In this definition we allow graphs to have

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loops and, for directed graphs, opposite (anti-parallel) edges, but no multiple edges. Thus directed graphs (*digraphs*) can be viewed as bipartite graphs with both parts of size $n$. We denote sets of all such graphs by $\mathcal{G}_{n,d}$ and $\mathcal{D}_{n,d}$, respectively, and the corresponding sets of adjacency matrices by $\mathcal{S}_{n,d}$ and $\mathcal{M}_{n,d}$. Note that $\mathcal{M}_{n,d}$ coincides with the set of $n \times n$ matrices with 0/1-entries and such that every row and every column has exactly $d$ ones. Clearly, $\mathcal{S}_{n,d}$ consists of symmetric matrices from $\mathcal{M}_{n,d}$. Probability is always given by the normalized counting measure on the corresponding set.

Spectral properties of adjacency matrices of random $d$-regular graphs attracted considerable attention of researchers in the recent years. Many works were devoted to the eigenvalue distribution. At the same time, much less is known about the singular values of the matrices.

Our work is motivated by related questions on singular probability. One conjecture was mentioned by Vu in his survey [12, Problem 8.4] (see also 2014 ICM talks by Frieze and Vu [5, Problem 7], [13, Conjecture 5.8]). It asserts that for $3 \le d \le n-3$ the probability that a random matrix uniformly distributed on $\mathcal{S}_{n,d}$ is singular goes to zero as $n$ grows to infinity. We formulate here the corresponding question for non-symmetric adjacency matrices (cf., [3, Conjecture 1.5]):

*Is it true that for every $3 \le d \le n-3$, one has*

$$p_{n,d} := \mathbb{P}\{M \in \mathcal{M}_{n,d} : M \text{ is singular}\} \longrightarrow 0 \text{ as } n \to \infty? \qquad (*)$$

Singularity of random square matrices is a subject with a long history and many results. A fundamental role in this topic is played by what is nowadays called *the Littlewood-Offord theory*. In its classical form, established by Erdös [4], the Littlewood–Offord inequality states that for every fixed $z \in \mathbb{R}$, a vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ with non-zero coordinates and independent random signs $r_k$ $(k \le n)$, the probability $\mathbb{P}\{\sum_{k=1}^n r_k a_k = z\}$ is bounded from above by $n^{-1/2}$. This combinatorial result has been substantially strengthened and generalized in subsequent years, leading to a much better understanding of interrelationship between the law of the sum $\sum_{k=1}^n r_k a_k$ and the arithmetic structure of the vector $a$. For more information and further references, we refer the reader to [11, Section 3] and [9, Section 4].

The use of the Littlewood-Offord theory in context of random matrices can be illustrated as follows: given an $n \times n$ matrix $A$ with i.i.d. elements, $A$ is non-singular if and only if the inner product of a normal vector to the span of any subset of $n-1$ columns of $A$ with the remaining column is non-zero. Thus, knowing the typical arithmetic structure of the random normal vectors and conditioning on their realization, one can estimate the probability that $A$ is singular.

The main difficulty in singularity questions such as $(*)$ stems from the restrictions on row/column-sums, and from possible symmetry constraints for the entries. Note that for a random matrix uniformly distributed on $\mathcal{M}_{n,d}$ every two entries/rows/columns are probabilistically dependent; moreover, a realization of the first $n-1$ columns uniquely defines the last column. This makes a straightforward application of the Littlewood–Offord theory (as illustrated in the previous paragraph) impossible. Thus, an extension of the theory covering this probabilistic model is needed.

In this note we address the question $(*)$ and provide a Littlewood–Offord type anti-concentration property of random graphs. For the complete proofs see [7].

**2. Main results.** The question $(*)$ has been recently studied in [3] by Cook who obtained the bound $p_{n,d} \le d^{-c}$ for a small constant $c > 0$ and $d$ satisfying $\omega(\ln^2 n) \le d \le n - \omega(\ln^2 n)$, where $f \ge \omega(a_n)$ means $f/a_n \to \infty$ as $n \to \infty$. Our main result shows that one can drop the condition $d \ge \omega(\ln^2 n)$.

**Theorem 1** *There are positive constants $c, C$ such that for $3 \le d \le cn/\ln^2 n$ one has $p_{n,d} \le C \ln^3 d/\sqrt{d}$.*

Thus we proved that $p_{n,d} \to 0$ as $d \to \infty$, which in particular verifies $(*)$ whenever $d = \omega(1)$, without any restrictions on the rate of convergence. We would like to notice that even in the range $\omega(\ln^2 n) \le d \le cn/\ln^2 n$ covered in [3], our bound in Theorem 1 is better.

The following *anti-concentration* property plays a crucial role in our proof. Let a random matrix $M$

be uniformly distributed on $\mathcal{M}_{n,d}$. Denote its rows by $R_i(M)$ and its columns by $X_i(M)$, $i \le n$. Given a subset $J$ of $[n] := \{1, 2, \dots, n\}$, consider a random 0/1-vector $\delta^J(M)$ such that its $i$-th coordinate $\delta_i^J(M)$ equals one if and only if the support of $i$-th row of $M$ intersects $J$. In other words, $\delta_i^J(M)$ is the indicator of the event $J \cap \operatorname{supp} R_i(M) \ne \emptyset$. We prove that for $J$ whose cardinality $|J|$ is not too close to $n$ the vector $\delta^J(M)$ cannot concentrate around any vertex of the cube, i.e., that $\delta^J(M)$ behaves similarly to a uniformly distributed random vector on $\{0,1\}^n$ (in the abstract we described this property in terms of graphs). In a sense, this shows that 1's are located rather uniformly across $M$. This property can be seen as an anti-concentration result for random graphs, matching anti-concentration properties of a weighted sum of independent random variables (or random vectors) studied in the Littlewood–Offord theory. In order to combine this property with an $\varepsilon$-net approximation (discussed later on), we will need to determine the action of $M$ on a part of a given vector corresponding to "small" coordinates. To achieve this the columns of the matrix corresponding to the remaining part of the vector should be fixed and the anti-concentration is proved under this conditioning. Theorem 2 makes this precise.

**Theorem 2** *There are two positive absolute constants $c$ and $c_1$ such that the following holds. Let $32 \le d \le cn$ and $I, J$ be disjoint subsets of $[n]$ such that $|I| \le d|J|/32$ and $8 \le |J| \le 8cn/d$. Let vectors $a^i \in \{0,1\}^n$, $i \in I$, be such that the event $\mathcal{E} := \{X_i(M) = a^i \text{ for all } i \in I\}$ has non-zero probability (if $I = \emptyset$ we set $\mathcal{E} = \mathcal{M}_{n,d}$). Then for every $v \in \{0,1\}^n$ one has*

$$\mathbb{P}\{\delta^J(M) = v \mid \mathcal{E}\} \le 2 \exp\left(-c_1 d|J| \ln\left(\frac{n}{d|J|}\right)\right).$$

**3. Methods of proof.** In this section, we discuss the scheme and the methods of the proof. We also explain several novel ideas allowing to drop the restriction $d \ge \omega(\ln^2 n)$ and to treat very sparse matrices.

The proof is naturally split into two distinct parts. First we establish certain properties of random $d$-regular directed graphs and their adjacency matrices. Then we use these results to deal with the singularity. Below, considering a random matrix $M$, we always mean a random matrix uniformly distributed on $\mathcal{M}_{n,d}$. Saying that it has some property means that this property holds with probability going to one.

To work with the "shuffling" procedure described below, we show that supports of any two rows of a random matrix have small intersection. Moreover, the proof of Theorem 2 requires a stronger property:

**Lemma 3** *There exists an absolute constant $c > 0$ such that for every $\varepsilon \in (\sqrt{\ln d/d}, 1)$ and $k \le c\varepsilon n/d$, the union of supports of any $k$ rows (or columns) of a random matrix on $\mathcal{M}_{n,d}$ has cardinality exceeding $(1-\varepsilon)dk$ with probability at least $1 - \exp(-c\varepsilon^2 d \ln(c\varepsilon n/d))$.*

Properties of this type are known for random undirected graphs (see [6] and references therein). A key ingredient in the proofs of these results is the *simple switching* (also called *transfusion*), which was introduced for general graphs by Senior [10]. In the context of $d$-regular graphs it was first applied by McKay [8]. We also use this technique to show that a random matrix has no large zero minors, namely:

**Lemma 4** *There are absolute positive constants $c$ and $C$ such that for $Cn \ln d/d \le \ell \le r \le n/4$ a random matrix on $\mathcal{M}_{n,d}$ has no $\ell \times r$ zero minors with probability at least $1 - \exp(-cr\ell d/n)$.*

Both properties ("no large intersections" and "no large zero minors") illustrate a general phenomenon that a random graph has good "regularity" properties. Analogous statements for the Erdös-Rényi graphs (in this random model an edge between every two vertices is included/excluded in a graph independently of other edges) follow from standard Bernstein type inequalities. For related results on $d$-regular random graphs, we refer the reader to [6] where concentration properties of *co-degrees* were established in the undirected setting, and to [2] for concentration of co-degrees and the "edge counts" for directed graphs. In paper [2] which serves as a basis for the main theorem of [3], rather strong concentration properties were established; however, the results provided in that paper are valid only for $d \ge \omega(\ln n)$. The proof in [3] is based on the method of exchangeable pairs introduced by Stein and developed for concentration inequalities by Chatterjee (see survey [1] for more information and references). On the contrary, our proof of the aforementioned statements is simpler, completely self-contained, and works for $d \geq C$.

After establishing properties of random $d$-regular directed graphs and their adjacency matrices, we turn to the proof of Theorem 1. We follow the scheme and expand on some of the techniques developed in [3] adding new crucial ingredients to remove logarithmic lower bound on $d$. In this scheme, at the first step, one shows that a random matrix does not have any (left or right) null vectors with many (more than $Cnd^{-c}$) equal coordinates, provided that $d \geq \omega(\ln^2 n)$. At the second step, one shows that, conditioned on this event, a random matrix is not singular.

The "no large zero-minors property," which we apply on the second step, allows to modify this scheme so that at first step it is enough to consider a much smaller class of *almost constant* vectors. We show that for every $C \leq d \leq cn$, a random matrix does not have null vectors having $n - n/\ln d$ equal coordinates. This step essentially uses Theorem 2 together with a new delicate approximation argument dealing with tails of appropriately rescaled vectors in $\mathbb{R}^n$. Note that a logarithmic lower bound on $d$ is not required.

Then, conditioning on the event that $M$ does not have almost constant null vectors, we show that a random matrix $M$ is non-singular with high probability. In [3], a sophisticated approach based on "shuffling" of two rows was developed to treat this case. The shuffling consists in a random perturbation of two rows of a fixed matrix $M \in \mathcal{M}_{n,d}$ in such a way that the sum of the rows remains unchanged. Then one uses a variant of the classical Erdös anti-concentration inequality to show that the number of "bad" perturbations is small. To apply this we need that the supports of these two rows have a small intersection. As shuffling involves supports of only two rows, at this step we get that probability tends to zero with $d$ and not with $n$ (and this is the only such step – in all our other statements the probability converges with $n$). We developed further the shuffling technique to simplify the proof and to obtain better probability estimates.

Finally, we give more details about the completion of the proof. Using that there are no almost constant vectors and that there are no large zero minors, we show that for singular matrices with high probability the minor $M^{1,2}$ obtained by removing the first two rows has largest possible rank, that is, either $\operatorname{rk} M^{1,2} = \operatorname{rk} M$ when $\operatorname{rk} M \leq n-2$ or $\operatorname{rk} M^{1,2} = n-2$ when $\operatorname{rk} M = n-1$. We consider the equivalence classes of matrices with the same minor $M^{1,2}$. Noticing that fixing such a minor determines the support of the first two rows, we use the shuffling procedure for the first two rows and show that the set of matrices of rank $\leq n-2$ (resp. $= n-1$) is small inside the set of matrices of rank $\leq n-1$ (resp. $= n$). This implies the bound on the probability that $M$ is singular.

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