A remark on vertex index of the convex bodies

E. D. Gluskin A. E. Litvak

Abstract

The vertex index of a symmetric convex body $\mathbf{K} \subset \mathbb{R}^n$, $\mathrm{vein}(\mathbf{K})$, was introduced in [BL]. Bounds on the vertex index were given in the general case as well as for some basic examples. In this note we improve these bounds and discuss their sharpness. We show that

$$\mathrm{vein}(\mathbf{K}) \leq 24 n^{3/2},$$

which is asymptotically sharp. We also show that the estimate

$$\frac{n^{3/2}}{\sqrt{2\pi e}\ \mathrm{ovr}(\mathbf{K})} \leq \mathrm{vein}(\mathbf{K}),$$

obtained in [BL] (here $\mathrm{ovr}(\mathbf{K})$ denotes the outer volume ratio of $\mathbf{K}$), is not always sharp. Namely, we construct an example showing that there exists a symmetric convex body $\mathbf{K}$ which simultaneously has large outer volume ratio and large vertex index. Finally, we improve the constant in the latter bound for the case of the Euclidean ball from $\sqrt{2\pi e}$ to $\sqrt{3}$, providing a completely new approach to the problem.

1 Introduction

Let $\mathbf{K}$ be a convex body symmetric about the origin 0 in $\mathbb{R}^n$ (such bodies below we call 0-symmetric convex bodies). The vertex index of $\mathbf{K}$, $\mathrm{vein}(\mathbf{K})$, was introduced in [BL] as

$$\mathrm{vein}(\mathbf{K}) = \inf \left\{ \sum_i \|x_i\|_{\mathbf{K}} \mid \mathbf{K} \subset \mathrm{conv}\, \{x_i\} \right\},$$

where $\|x\|_{\mathbf{K}} = \inf\{\lambda > 0 \mid x \in \lambda \mathbf{K}\}$ denotes the Minkowski functional of $\mathbf{K}$. In other words, given $\mathbf{K}$ one looks for the convex polytope that contains

$\mathbf{K}$ and whose vertex set has the smallest possible closeness to 0 in metric generated by $\mathbf{K}$. Let us note that $\mathrm{vein}(\mathbf{K})$ is an affine invariant of $\mathbf{K}$, i.e. if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map, then $\mathrm{vein}(\mathbf{K}) = \mathrm{vein}(T(\mathbf{K}))$.

The vertex index is closely connected to some important quantities in analysis and geometry including the illumination parameter of convex bodies, introduced by Bezdek; the Boltyanski-Hadwiger illumination conjecture, which says that every convex body in $\mathbb{R}$ can be illuminated by 2^n sources; the Gohberg-Marcus conjecture, which avers that a convex body can be covered by 2^n smaller positive homothetic copies of itself). We refer to [B1, B2, BL, MS] for the related discussions, history, and references.

Denote the volume by $|\cdot|$, the canonical Euclidean ball in $\mathbb{R}^n$ by $\mathbf{B}_2^n$, and as usual define the outer volume ratio of $\mathbf{K}$ by $\mathrm{ovr}(\mathbf{K}) = \inf \left(|\mathcal{E}|/|\mathbf{K}|\right)^{1/n}$, where the infimum is taken over all ellipsoids $\mathcal{E} \supset \mathbf{K}$. In [BL] the following theorem has been proved.

Theorem 1.1 *There exists a positive absolute constant C such that for every $n \geq 1$ and every 0-symmetric convex body $\mathbf{K}$ in $\mathbb{R}^n$ one has*

$$\frac{n^{3/2}}{\sqrt{2\pi e}\ \mathrm{ovr}(\mathbf{K})} \leq \mathrm{vein}(\mathbf{K}) \tag{1}$$

and

$$\mathrm{vein}(\mathbf{K}) \leq C\ n^{3/2}\ \ln(2n). \tag{2}$$

Moreover, in [GL] it was shown that $\mathrm{vein}(\mathbf{K}) \geq 2n$ for every n-dimensional 0-symmetric convex body $\mathbf{K}$.

The purpose of this note is to discuss sharpness of estimates 1 and 2. We start our discussion with the first estimate. Note that it is sharp (especially in view of estimate (3) below) for the class of bodies with finite outer volume ratio, that is bodies such that $\mathrm{ovr}(\mathbf{K}) \leq C$, where C is a positive absolute constant (fixed in advance). This class is very large, it includes in particular the unit balls of ℓ_p-spaces for $p \geq 2$ as well as 0-symmetric convex polytopes having at most $C_1 n$ facets (here C_1 is another absolute constant). In Section 3 we show that in fact (1) is not sharp, i.e. that in general $\mathrm{vein}(\mathbf{K})$ is not equivalent to $n^{3/2}/\mathrm{ovr}(\mathbf{K})$. Namely, we construct a 0-symmetric convex body $\mathbf{K}$ which has simultaneously large outer volume ratio and large vertex index (in fact both are largest possible up to a logarithmic factor): $\mathrm{vein}(\mathbf{K}) \approx n^{3/2}$ and $\mathrm{ovr}(\mathbf{K}) \approx \sqrt{n}/\sqrt{\ln(2n)}$. It shows that for some bodies the gap in (1) can be of the order $\sqrt{n}/\sqrt{\ln(2n)}$. Note that despite of our example, there are bodies with large outer volume ratio for which (1) is sharp, e.g. for the n-dimensional octahedron $\mathbf{B}_1^n$ we have $\text{vein}(\mathbf{B}_1^n) = 2n$ ([BL]) and

$$\text{ovr}(\mathbf{B}_1^n) = \frac{\sqrt{\pi}}{2} \left(\frac{n}{\Gamma(1+n/2)} \right)^{1/n} \approx \frac{\sqrt{\pi}}{\sqrt{2e}} \sqrt{n}.$$

The construction of our example is of the random nature, essentially we take the absolute convex hull of n^2 random points on the sphere and show that it works with high probability.

Next, in Section 4, we remove the logarithmic factor in the estimate (2), improving it to the asymptotically best possible one. The main new ingredient in our improvement is a recent result of Batson, Spielman, and Srivastava ([BSS]) on the decomposition of a linear operator acting on $\mathbb{R}^n$ (see Theorem 4.1 below). The application of their theorem instead of corresponding Rudelson's Theorem used in [BL] allows us to remove the unnecessary logarithm.

In Section 5 we turn to the vertex index of the Euclidean ball. In [BL] it was conjectured that

$$\text{vein}(\mathbf{B}_2^n) = 2n^{3/2},$$

i.e., the best configuration for the Euclidean ball is provided by the vertices of the n-dimensional octahedron. The conjecture was verified for $n = 2$ and $n = 3$. Note that by (1)

$$\frac{n^{3/2}}{\sqrt{2\pi e}} \leq \text{vein}(\mathbf{B}_2^n).$$

We improve this bound to $n^{3/2}/\sqrt{3}$. Our proof uses completely different approach via operator theory (recall that in [BL] the approach via volumes was used). We think that this new approach is interesting by itself and could lead to more results. Thus the results of Sections 4 and 5 can be summarized in the following theorem.

Theorem 1.2 *For every $n \geq 1$ and every 0-symmetric convex body $\mathbf{K}$ in $\mathbb{R}^n$ one has*

$$\text{vein}(\mathbf{K}) \leq 24\, n^{3/2}. \tag{3}$$

Moreover

$$\text{vein}(\mathbf{B}_2^n) \geq n^{3/2}/\sqrt{3}. \tag{4}$$

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2 Preliminaries and Notation

By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^n$. The canonical basis of $\mathbb{R}^n$ we denote by $e_1, \ldots, e_n$. By $\|\cdot\|_p$, $1 \leq p \leq \infty$, we denote the ℓ_p-norm, i.e.

$$\|x\|_p = \left(\sum_{i \geq 1} |x_i|^p \right)^{1/p} \quad \text{for} \quad p < \infty \quad \text{and} \quad \|x\|_\infty = \sup_{i \geq 1} |x_i|.$$

In particular, $\|\cdot\|_2 = |\cdot|$. As usual, $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$, and the unit ball of ℓ_p^n is denoted by $\mathbf{B}_p^n$.

Given points $x_1, \ldots, x_k$ in $\mathbb{R}^n$ we denote their convex hull by $\operatorname{conv}\{x_i\}_{i \leq k}$ and their absolute convex hull by $\operatorname{abs\,conv}\{x_i\}_{i \leq k} = \operatorname{conv}\{\pm x_i\}_{i \leq k}$. Similarly, the convex hull of a set $A \subset \mathbb{R}^n$ is denoted by $\operatorname{conv} A$ and absolute convex hull of A is denoted by $\operatorname{abs\,conv} A$ $(= \operatorname{conv}\{A \cup -A\})$.

Given convex compact body $\mathbf{K} \subset \mathbb{R}^n$ with 0 in its interior by $|\mathbf{K}|$ we denote its volume and by $\|\cdot\|_{\mathbf{K}}$ its Minkowski functional. $\mathbf{K}^\circ$ denotes the polar of $\mathbf{K}$, i.e.

$$\mathbf{K}^\circ = \{x \mid \langle x, y \rangle \leq 1 \quad \text{for every} \quad y \in \mathbf{K}\}.$$

The outer volume ratio of $\mathbf{K}$ is

$$\operatorname{ovr}(\mathbf{K}) = \inf \left(\frac{|\mathcal{E}|}{|\mathbf{K}|} \right)^{1/n},$$

where infimum is taken over all 0-symmetric ellipsoids in $\mathbb{R}^n$ containing $\mathbf{K}$. It is well-known that

$$\operatorname{ovr}(\mathbf{K}) \leq \sqrt{n}$$

for every convex symmetric about the origin body $\mathbf{K}$.

Finally we recall some notations from the Operator Theory. Given $u, v \in \mathbb{R}^n$, $u \otimes v$ denotes the operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $(u \otimes v)(x) = \langle u, x \rangle\, v$ for every $x \in \mathbb{R}^n$. The identity operator on $\mathbb{R}^n$ is denoted by Id. Given two operators $T, S : \mathbb{R}^n \to \mathbb{R}^n$ we write $T \leq S$ if $S - T$ is positive semidefinite, i.e., $\langle (S - T)x, x \rangle \geq 0$ for every $x \in \mathbb{R}^n$.

3 Example.

Theorem 3.1 *There exists an absolute positive constant c such that for every $n \geq 1$ there exists a convex symmetric body $\mathbf{K}$ satisfying*

$$\text{ovr }(\mathbf{K}) \geq c\sqrt{\frac{n}{\ln(2n)}} \qquad \textit{and} \qquad \text{vein }\mathbf{K} \geq cn^{3/2}.$$

Proof: Let $m = n^2$ and $u_1, u_2, \ldots, u_m$ be independent random vectors uniformly distributed on S^{n-1}. Let

$$\mathbf{K} := \text{abs conv }\{u_i, e_j\}_{i \leq m, j \leq n}.$$

Clearly

$$\frac{1}{\sqrt{n}}\mathbf{B}_2^n \subset \mathbf{K} \subset \mathbf{B}_2^n.$$

Moreover, it is well-known [G, CP, BF] that there exists an absolute positive constant C_0 such that for every linear transformation T satisfying $T\mathbf{K} \subset \mathbf{B}_2^n$ one has

$$|T\mathbf{K}| \leq C_0 \frac{\sqrt{\ln(2(m+n)/n)}}{n},$$

which immediately implies that

$$\text{ovr }(\mathbf{K}) \geq c_0\sqrt{\frac{n}{\ln(2n)}}$$

for an absolute positive constant c_0.

Now we prove the lower bound on vein $(\mathbf{K})$. First note that if $\mathbf{T}$ is an absolute convex hull of vectors $x_1, x_2, \ldots, x_M$ satisfying

$$a := \sum_{i=1}^{M} |x_i| \leq \frac{n^{3/2}}{4\sqrt{2\pi e}}$$

then by Santaló inequality and a result of Ball and Pajor (Theorem 2 in [BP]) we have

$$\frac{|\mathbf{T}|}{|\mathbf{B}_2^n|} \leq \frac{|\mathbf{B}_2^n|}{|\mathbf{T}^0|} \leq \left(\frac{\sqrt{2\pi e}}{\sqrt{n}}\right)^n \left(\frac{a}{n}\right)^n \leq 4^{-n}.$$

It implies that the probability

$$\mathbb{P}\left(\{\mathbf{K} \subset 2\mathbf{T}\}\right) \le \mathbb{P}\left(\{\forall i \le m : u_i \in 2\mathbf{T}\}\right) = \left(\mathbb{P}\left(\{\forall i : u_i \in 2\mathbf{T}\}\right)\right)^m$$
$$= \left(|2\mathbf{T} \cap S^{n-1}|\right)^m \le \left(\frac{|2\mathbf{T} \cap \mathbf{B}_2^n|}{|\mathbf{B}_2^n|}\right)^m \le 2^{-n^3}.$$

Now we consider a $\frac{1}{2\sqrt{n}}$-net (in the Euclidean metric) $\mathcal{N}$ in $n^{3/2}\mathbf{B}_2^n$ of cardinality less than $A = (6n^2)^n$ (it is well known that such a net exists). We fix $M = [n^{3/2}/8\sqrt{2\pi e}]$ (assuming without loss of generality $M \ge 3$) and consider

$$C_M = \left\{ \mathbf{T} \;\mid\; \mathbf{T} = \operatorname{abs\,conv}\{x_i\}_{i\le N}, N \le M, x_i \in \mathcal{N}, \sum_{i=1}^{N} |x_i| \le \frac{n^{3/2}}{4\sqrt{2\pi e}} \right\}.$$

Then the cardinality of C_M is

$$|C_M| \le \sum_{i=1}^{M} \binom{A}{i} \le \left(\frac{eA}{M}\right)^M \le (6n^2)^{nM}.$$

It implies that

$$\mathbb{P}\left(\{\exists \mathbf{T} \text{ such that } \mathbf{K} \subset 2\mathbf{T}\}\right) \le (6n^2)^{nM} 2^{-n^3} < 1.$$

This proves that there exists $\mathbf{K}$ such that

$$\forall \mathbf{T} \in C_M \;:\; \mathbf{K} \not\subset 2\mathbf{T}. \tag{5}$$

Finally fix $\mathbf{K}$ satisfying (5) and assume

$$\operatorname{vein} \mathbf{K} < \frac{n^{3/2}}{8\sqrt{2\pi e}},$$

i.e., that there exists $\mathbf{L} = \operatorname{conv}\{x_i\}_{i\le k}$ with $\mathbf{K} \subset \mathbf{L}$ and

$$k \le \sum_{i=1}^{k} \|x_i\|_{\mathbf{K}} < \frac{n^{3/2}}{8\sqrt{2\pi e}}.$$

Since $\mathbf{K} \subset \mathbf{B}_2^n$, we observe that

$$\sum_{i=1}^{k} |x_i| < \frac{n^{3/2}}{8\sqrt{2\pi e}},$$

in particular $x_i \in n^{3/2}\mathbf{B}_2^n$, $i \le k$. Then for every i there exist $y_i \in \mathcal{N}$ such that
$$|x_i - y_i| \le \frac{1}{2\sqrt{n}}.$$
Therefore
$$\sum_{i=1}^k |y_i| \le \sum_{i=1}^k |x_i| + \sum_{i=1}^k |x_i - y_i| \le \frac{n^{3/2}}{8\sqrt{2\pi e}} + \frac{k}{2\sqrt{n}} \le \frac{n^{3/2}}{4\sqrt{2\pi e}}.$$
Thus $\mathbf{P} = \operatorname{abs\,conv}\{y_i\}_{i\le N} \in C_M$, so, by (5) one has $\mathbf{K} \not\subset 2\mathbf{P}$. On the other hand we have for every x
$$\|x\|_{\mathbf{L}^0} = \max_{i\le k} \langle x, x_i\rangle \le \max_{i\le k} \langle x, y_i\rangle + \max_{i\le k} \langle x, x_i - y_i\rangle$$
$$\le \|x\|_{\mathbf{P}^0} + \frac{1}{2\sqrt{n}}|x| \le \|x\|_{\mathbf{P}^0} + \frac{1}{2}\|x\|_{\mathbf{L}^0},$$
where the latter inequality holds because $\frac{1}{\sqrt{n}}\mathbf{B}_2^n \subset \mathbf{K} \subset \mathbf{L}$. The above inequality means that $\mathbf{L} \subset 2\mathbf{P}$, which contradicts the fact that $\mathbf{K} \not\subset 2\mathbf{P}$. Hence
$$\operatorname{vein} \mathbf{K} \ge \frac{n^{3/2}}{8\sqrt{2\pi e}},$$
which proves the theorem. □

4 An upper bound for the vertex index

In this section we prove the inequality (3), i.e. we prove the sharp (up to an absolute constant) upper estimate for the vein of a convex symmetric body in the general case, removing the unnecessary logarithmic term from (1). Recall that such bound is attained for any body with a bounded volume ratio as well as for the body from Theorem 3.1.

In [BL] the Rudelson theorem on decomposition of identity was essentially used. It contains a logarithmic term which appeared in the upper bound on the vertex index. Here we use a recent result of Batson, Spielman, and Srivastava instead of Rudelson's theorem. In [BSS], they proved the following theorem.

Theorem 4.1 *Let $m \geq n \geq 1$, $\lambda > 1$, and $u_i \in \mathbb{R}^n$, $i \leq m$ be such that*

$$Id = \sum_{i=1}^{m} u_i \otimes u_i.$$

Then there exist non-negative numbers $c_1, c_2, \ldots, c_m$ such that at most λn of them non-zero and

$$Id \leq \sum_{i=1}^{m} c_i u_i \otimes u_i \leq \left(\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \right)^2 Id.$$

To obtain the upper bound it is enough to apply this theorem combined with the standard John decomposition instead of Rudelson theorem in the proof given in Section 5 of [BL]. For the sake of completeness we provide the details. The following standard lemma proves (3).

Lemma 4.2 *Let $\lambda > 1$, $n \geq 1$, and $\mathbf{K}$ be a 0-symmetric convex body in $\mathbb{R}^n$ such that its minimal volume ellipsoid is $\mathbf{B}_2^n$. Then there exists a 0-symmetric convex polytope $\mathbf{P}$ in $\mathbb{R}^n$ with at most λn vertices such that*

$$\mathbf{P} \subset \mathbf{K} \subset \mathbf{B}_2^n \subset \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \sqrt{n}\ \mathbf{P}.$$

In particular

$$\mathrm{vein}(\mathbf{K}) \leq 24 n^{3/2}.$$

Proof: The John decomposition ([J]) states that there exist points v_i, $i \leq m$, with $\|v_i\|_{\mathbf{K}} = |v_i| = 1$ and scalars $\lambda_i > 0$ such that

$$Id = \sum_{i=1}^{m} \lambda_i v_i \otimes v_i.$$

Then Theorem 4.1 applied to $u_i = \sqrt{\lambda_i} v_i$ implies that there exist non-negative numbers $c_1, c_2, \ldots, c_m$ such that at most λn of them non-zero and

$$Id \leq \sum_{i=1}^{m} c_i \lambda_i v_i \otimes v_i \leq \left(\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \right)^2 Id. \tag{6}$$

Let I denotes the set of indices i such that $c_i \neq 0$. Consider $\mathbf{P} = \text{abs conv}\,\{v_i\}_{i\in I}$. Since $v_i \in \mathbf{K} = -\mathbf{K}$, $i \leq m$, we observe

$$\mathbf{P} \subset \mathbf{K} \subset \mathbf{B}_2^n.$$

By (6) we also have for every $x \in \mathbb{R}^n$

$$|x|^2 = \langle Id\ x, x\rangle \leq \left\langle \sum_{i=1}^m c_i\lambda_i \langle v_i, x\rangle\, v_i, x \right\rangle = \sum_{i=1}^m c_i\lambda_i \langle v_i, x\rangle^2$$

$$\leq \max_{i\leq m} \langle v_i, x\rangle^2 \sum_{i=1}^m c_i\lambda_i = \|x\|^2_{\mathbf{P}^\circ} \sum_{i=1}^m c_i\lambda_i$$

and

$$\sum_{i=1}^m c_i\lambda_i = \sum_{i=1}^m c_i\lambda_i \langle v_i, v_i\rangle = \text{trace} \sum_{i=1}^m c_i\lambda_i v_i \otimes v_i$$

$$\leq \left(\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}\right)^2 \text{trace } Id = \left(\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}\right)^2 n.$$

It implies that $|x| \leq \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}\ \sqrt{n}\ \|x\|_{\mathbf{P}^\circ}$, which means $\mathbf{B}_2^n \subset \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}\ \sqrt{n}\ \mathbf{P}$. This proves

$$\mathbf{P} \subset \mathbf{K} \subset \mathbf{B}_2^n \subset \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}\ \sqrt{n}\ \mathbf{P}$$

and in particular implies

$$\text{vein}(\mathbf{K}) \leq 2\ \sqrt{n}\ \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \sum_{i\in I} \|v_i\|_{\mathbf{K}} \leq 2\lambda\ \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} n^{3/2}.$$

Choosing $\lambda = 4$ we obtain the result. $\square$

5 A lower bound for the vertex index of $\mathbf{B}_2^n$

In this section we prove estimate (4), i.e. we improve the constant in the estimate

$$cn^{3/2} \leq \text{vein}(\mathbf{B}_2^n) \leq 2n^{3/2}$$

from $c = 1/\sqrt{2\pi e}$ proved obtained [BL] to $c = 1/\sqrt{3}$. Recall that the proof in [BL] was based on volume estimates. We use here completely different approach.

Proof: Assume that $\mathbf{B}_2^n \subset \mathbf{L} = \mathrm{conv}\,\{x_i\}_{i\le N}$ for some non zero x_i's and denote

$$a = \sum_{i=1}^{n} |x_i|.$$

Our goal is to show that $a^2 \ge n^3/3$.

Define the operator $T : \mathbb{R}^N \to \mathbb{R}^n$ by $Te_i = x_i$, $i \le N$. Then the rank of T is n (since $\mathbf{B}_2^n \subset \mathbf{L}$), $a = \sum_{i=1}^n |Te_i|$ and for every $x \in \mathbb{R}^n$

$$|x| \le \|x\|_{\mathbf{L}^0} = \max_{i\le N} \langle x, x_i\rangle = \max_{i\le N} \langle T^* x, e_i\rangle . \tag{7}$$

For $i \le N$ denote

$$\lambda_i = \sqrt{|Te_i|/a} \qquad \text{and} \qquad v_i = \frac{Te_i}{a\lambda_i}.$$

Then

$$\sum_{i=1}^{n} \lambda_i^2 = 1 \qquad \text{and} \qquad \sum_{i=1}^{n} |v_i|^2 = 1.$$

We also observe that T^* can be presented as $T^* = a\Lambda S$, where Λ is the diagonal matrix with λ_i's on the diagonal and

$$S = \sum_{i=1}^{N} v_i \otimes e_i.$$

Note that the rank of S equals n. Let $s_1 \ge s_2 \ge \ldots \ge s_n > 0$ be the singular values of S and let $\{w_i\}_{i\le n}$, $\{z_i\}_{i\le n}$ be orthonormal systems such that

$$S = \sum_{i=1}^{n} s_n w_i \otimes z_i.$$

Then

$$\sum_{i=1}^{n} s_i^2 = \|S\|_{HS}^2 = \sum_{i=1}^{n} |v_i|^2 = 1,$$

where $\|S\|_{HS}$ is the Hilbert-Schmidt norm of S. Now for $m \le n$ denote

$$S_m = \sum_{i=m}^{n} s_n w_i \otimes z_i$$

and consider the $(n+1-m)$-dimensional subspace

$$E_m = \text{Im } (\Lambda S_m) \subset \text{Im } T^*.$$

Considering the extreme points of the section of the cube $\mathbf{B}_\infty^N \cap E_m$ we observe that there exists a vector $y = \{y_i\}_{i \le N} \in \mathbf{B}_\infty^N \cap E_m$ such that the set $A = \{i \mid |y_i| = 1\}$ has cardinality at least $n+1-m$. Without loss of generality we assume that $|A| = n+1-m$ (otherwise we choose an arbitrary subset of A with such cardinality). We observe

$$|(a\Lambda)^{-1}y| = \frac{1}{a}\sqrt{\sum_{i=1}^{N} \frac{y_i^2}{\lambda_i^2}} \ge \frac{1}{a}\sqrt{\sum_{i \in A} \frac{1}{\lambda_i^2}}$$

$$\ge \frac{n+1-m}{a\sqrt{\sum_{i\in A} \lambda_i^2}} \ge \frac{n+1-m}{a\sqrt{\sum_{i=1}^{N} \lambda_i^2}} = \frac{n+1-m}{a}.$$

Note that by construction $y \in E_m \subset \text{Im } T^*$, so denoting the inverse of T^* from the image by $(T^*)^{-1}$ we have

$$|(T^*)^{-1}y| = |S^{-1}(a\Lambda)^{-1}y| = |S_m^{-1}(a\Lambda)^{-1}y| \ge \frac{|(a\Lambda)^{-1}y|}{\|S_m\|} \ge \frac{n+1-m}{as_m}.$$

Using (7) we obtain

$$\frac{n+1-m}{as_m} \le |(T^*)^{-1}y| \le \max_{i \le N} \left\langle T^*(T^*)^{-1}y, e_i \right\rangle = \|y\|_\infty = 1.$$

This shows $s_m \ge (n+1-m)/a$ and implies

$$\frac{n^3}{3a^2} \le \frac{1}{a^2}\sum_{m=1}^{n}(n+1-m)^2 \le \sum_{m=1}^{n} s_m^2 = 1,$$

which proves the desired result. $\square$

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A.E. Litvak, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, Canada, T6G 2G1.
e-mail: alexandr@math.ualberta.ca

E.D. Gluskin, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel.
e-mail: gluskin@post.tau.ac.il