Random polytopes obtained by matrices with heavy tailed entries.

O. Guédon A. E. Litvak K. Tatarko

Abstract

Let Γ be an $N \times n$ random matrix with independent entries and such that in each row entries are i.i.d. Assume also that the entries are symmetric, have unit variances, and satisfy a small ball probabilistic estimate uniformly. We investigate properties of the corresponding random polytope $\Gamma^* B_1^N$ in $\mathbb{R}^n$ (the absolute convex hull of rows of Γ). In particular, we show that

$$\Gamma^* B_1^N \supset b^{-1} \left(B_\infty^n \cap \sqrt{\ln(N/n)}\, B_2^n \right),$$

where b depends only on parameters in small ball inequality. This extends results of [18] and recent results of [17]. This inclusion is equivalent to so-called ℓ_1-quotient property and plays an important role in compressed sensing (see [17] and references therein).

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1 Introduction

In this note, we deal with a rectangular $N \times n$ random matrices $\Gamma = \{\xi_{ij}\}_{1 \le i \le N, 1 \le j \le n}$, where ξ_{ij} are independent symmetric random variables with unit variance satisfying uniform small ball probabilistic estimate. More precisely, in the main theorem we assume that there exist $u, v \in (0,1)$ such that

$$\forall i, j \quad \sup_{\lambda \in \mathbb{R}} \mathbb{P}\big\{ |\xi_{ij} - \lambda| \le u \big\} \le v. \tag{1}$$

Of course, if variables have a bounded moment $r > 2$, we will have better estimates. We are interested in geometric parameters of the random polytope generated by Γ, that is, the absolute convex hull of rows of Γ. In other words, the random polytope under consideration is $\Gamma^* B_1^N$, where B_1^N is the N-dimensional octahedron (cross-polytope). Such random polytopes have been extensively studied in the literature, especially in the Gaussian case and in the Bernoulli case. The Gaussian random polytopes in the case when N is proportional to n have many applications in the Asymptotic Geometric Analysis (see e.g., [9] and [30], and the survey [22]). The Bernoulli case corresponds to 0/1 random polytopes. For their combinatorial properties we refer the reader to [7, 3] (see also the survey [32]). Their geometric parameters have been studied in [8, 18]. In the compressed sensing it was shown that the so-called ℓ_1-quotient property is responsible for robustness in certain ℓ_1-minimizations (see [17] and references therein). More precisely, an $n \times N$ (with $N \geq n$) matrix A satisfies the ℓ_1-quotient property with a constant b relative to a norm $\|\cdot\|$ if for every $y \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^N$ such that $Ax = y$ and $\|x\|_1 \leq b\sqrt{n/\ln(eN/n)}\,\|y\|$, where $\|\cdot\|_1$ denotes the ℓ_1-norm. It is easy to see that geometrically this means

$$B_{\|\cdot\|} \subset b\sqrt{n/\ln(eN/n)}\,AB_1^N,$$

where $B_{\|\cdot\|}$ is the unit ball of $\|\cdot\|$. To prove robustness of noise-blind compressed sensing, the authors of [17] dealt with the norm

$$\|\cdot\| = \max\{\|\cdot\|_2, \sqrt{\ln(eN/n)}\|\cdot\|_\infty\},$$

where $\|\cdot\|_2$ is the standard Euclidean norm and $\|\cdot\|_\infty$ is the ℓ_∞-norm. Theorem 5 in [17] states that assuming that entries of A are symmetric i.i.d. random variables with unit variances, and that they have regular (in fact, ψ_α) behaviour of all moments till the moment of order $\ln n$, the matrix $A/\sqrt{n}$ has the ℓ_1-quotient property with high probability. Geometrically this means

$$AB_1^N \supset b^{-1}\left(B_\infty^n \cap \sqrt{\ln(N/n)}\,B_2^n\right). \tag{2}$$

The work [17] complements results of [18], where this inclusion was proved for random matrices with symmetric i.i.d. entries having at least third bounded moment and such that the operator norm of the matrix is bounded with high probability.

The main purpose of this note is to prove such an inclusion with much weaker assumptions on the distribution of the entries. In fact, we require only boundedness of second moments. Thus "robustness" Theorem 8 in [17] holds under much weaker assumptions on the random matrix. Our main result is the following theorem (see Theorem 4.1 for slightly better probability estimates).

Theorem 1.1. *There exist positive constants b, M depending only on u and v and an absolute constant $c > 0$ such that the following holds. Let $N \geq Mn$ and assume that the entries of an $N \times n$ random matrix Γ are independent symmetric random variables with unit variances satisfying condition (1) and such that in each row the entries are i.i.d. Then with probability at least $1 - \exp(-cn)$ the inclusion (2) holds for the matrix $A = \Gamma^*$.*

We use this theorem to study geometric properties of random polytopes $K_N = \Gamma^* B_1^N$ and K_N^0, such as behavior of their volumes and mean widths. Our "volume" theorem states the following (see Theorems 4.10 and 4.11 for more precise statements).

Theorem 1.2. *There exist positive constants C_1, C_2 depending only on u and v and absolute positive constants C, c such that for $C_1 n \leq N \leq e^n$ with probability at least $1 - \exp(-cn)$ one has*

$$|K_N|^{1/n} \geq C_2\sqrt{\frac{\ln(N/n)}{n}} \qquad \textit{and} \qquad |K_N^0|^{1/n} \leq \frac{C}{C_2\sqrt{n\ln(N/n)}},$$

where $K_N = \Gamma^* B_1^N$ and the matrix Γ is as in Theorem 1.1. Moreover, the bounds on the volumes are sharp, provided that the Euclidean lengths of the rows of Γ are of order of $\sqrt{n}$ at most.

Our proof of Theorem 1.1 follows the general scheme of [18] with a very delicate change – in [18] there was an assumption that the operator norm of Γ is bounded by $C\sqrt{N}$ with high probability. However it is known that such a bound does not hold in general unless fourth moments are bounded ([29], see also [20] for quantitative bounds). To avoid using the norm of Γ, we use ideas appearing in [25], where the authors constructed a certain deterministic ε-net (in ℓ_2-metric) $\mathcal{N}$ such that $A\mathcal{N}$ is a good net for AB_2^n for most realizations of a square random matrix A. We extend their construction in three directions. First, we work with rectangular random matrices, not only square matrices. Second, we need a net for the image of a given convex body (not only for the image of the unit Euclidean ball). Finally, instead of approximation in the Euclidean norm only, we use approximation in the following norm

$$\|a\|_{k,2} = \left(\sum_{i=1}^{k} (a_i^*)^2 \right)^{1/2}, \tag{3}$$

where $1 \leq k \leq N$ and $a_1^* \geq a_2^* \geq \ldots \geq a_N^*$ is the decreasing rearrangement of the sequence of numbers $|a_1|, \ldots, |a_N|$. This norm appears naturally and plays a crucial role in our proof of inclusion (2). The generalization of the net from [25] is a new key ingredient, see Theorem 3.1. We would like to emphasize, that norms $\| \cdot \|_{k,2}$ played an important role in proofs of many results of Asymptotic Geometric Analysis, see e.g. [11, 13, 14]. For the systematic studies of norms $\| \cdot \|_{k,2}$ and their unit balls we refer to [12]. We believe that the new approximation in $\| \cdot \|_{k,2}$ norms will find other applications in the theory. In the last section, we present one more application of Theorem 3.1 – we show that it can be used to estimate the smallest singular value of a tall random matrix – see the discussion at the beginning of Section 5.

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2 Notations

By $\langle \cdot, \cdot \rangle$ we denote the canonical inner product on the m-dimensional real space $\mathbb{R}^m$ and for $1 \leq p \leq \infty$, the ℓ_p-norm is defined for any $a \in \mathbb{R}^m$ by

$$\|a\|_p = \left(\sum_{i=1}^{m} |a_i|^p \right)^{1/p} \quad \text{for} \quad p < \infty \qquad \text{and} \qquad \|a\|_\infty = \sup_{i=1,\ldots,m} |a_i|.$$

As usual, $\ell_p^m = (\mathbb{R}^m, \| \cdot \|_p)$, and the unit ball of ℓ_p^m is denoted by B_p^m. The unit sphere of ℓ_2^m is denoted by S^{m-1}, and the canonical basis of ℓ_2^m is denoted by $e_1, \ldots, e_m$.

Given an integer $k \in \{1, \dots, N\}$, we denote by $X_{k,2}$ the normed space $\mathbb{R}^N$ equipped with the norm $\|\cdot\|_{k,2}$ defined by (3). The unit ball of $X_{k,2}$ is denoted by $\mathbf{B}_{k,2}$. Note that for $k = N$ we have $\|a\|_{k,2} = \|a\|_2$ and that for any $k \le N$ and any $a \in \mathbb{R}^N$,

$$\|a\|_{k,2} \le \|a\|_2 \le \sqrt{\frac{N}{k}}\, \|a\|_{k,2} \qquad \text{or, equivalentely,} \qquad B_2^N \subset \mathbf{B}_{k,2} \subset \sqrt{\frac{N}{k}}\, B_2^N.$$

Given integers $\ell \ge k \ge 1$, we denote $[k] = \{1, 2, \dots, k\}$ and $[k, \ell] = \{k, k+1, \dots, \ell\}$. Given a number a we denote the largest integer not exceeding a by $\lfloor a \rfloor$ and the smallest integer larger than or equal to a by $\lceil a \rceil$.

Given points $x_1, \dots, x_k$ in $\mathbb{R}^m$ we denote their convex hull by $\mathrm{conv}\,\{x_i\}_{i \le k}$ and their absolute convex hull by $\mathrm{abs\,conv}\,\{x_i\}_{i \le k} = \mathrm{conv}\,\{\pm x_i\}_{i \le k}$. Given $\sigma \subset [m]$ by P_σ we denote the coordinate projection onto $\mathbb{R}^\sigma = \{x \in \mathbb{R}^m \mid x_i = 0 \text{ for } i \notin \sigma\}$.

Given a finite set E we denote its cardinality by $|E|$. We also use $|K|$ for the volume of a body $K \subset \mathbb{R}^m$ (and, more generally, for the m-dimensional Lebesgue measure of a measurable subset in $\mathbb{R}^m$). Let K be a symmetric convex body with non empty interior. We denote its Minkowski's functional by $\|x\|_K$. The support function of K is $h_K(x) = \sup_{y \in K} \langle x, y \rangle$, the polar of K is

$$K^0 = \{x \in \mathbb{R}^m \mid \langle x, y \rangle \le 1 \text{ for every } y \in K\}.$$

Note that $h_K(\cdot) = \|\cdot\|_{K^0}$.

Given a set $L \subset \mathbb{R}^m$, a convex body $K \subset \mathbb{R}^m$, and $\varepsilon > 0$ we say that a subset $\mathcal{N} \subset \mathbb{R}^m$ is an *ε-net* of L with respect to K if

$$\mathcal{N} \subset L \subset \bigcup_{x \in \mathcal{N}} (x + \varepsilon K).$$

The cardinality of the smallest ε-net of L with respect to K we denote by $N(L, \varepsilon K)$.

For a given probability space, we denote by $\mathbb{P}(\cdot)$ and $\mathbb{E}$ the probability of an event and the expectation respectively. A ± 1 random variable taking values 1 and -1 with probability $1/2$ is called a Rademacher random variable.

In this paper we are interested in rectangular $N \times n$ matrices $\Gamma = \{\xi_{ij}\}_{\substack{1 \le i \le N \\ 1 \le j \le n}}$, with $N \ge n$, where the entries are real-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We will mainly consider the following model of matrix Γ:

$$\begin{cases} \forall i, j \quad \xi_{ij} \text{ are independent, symmetric and } \mathbb{E}\xi_{ij}^2 = 1, \\ \text{in each row the entries are identically distributed.} \end{cases} \tag{4}$$

At the beginning of Section 4, we will also assume that the entries of Γ satisfy a uniform small ball estimate. If $\xi_{ij} \sim \mathcal{N}(0, 1)$ are independent Gaussian random variables we say that Γ is *a Gaussian random matrix*.

3 Construction of a good deterministic net.

In this section we present a key result of this paper. Let T be a subset of $\mathbb{R}^n$, we aim at constructing a deterministic net such that for every general random operator

$\Gamma : \mathbb{R}^n \to \mathbb{R}^N$, with overwhelming probability, the image of the net by the random operator Γ is a good approximation of ΓT. We show that we can quantify this approximation by almost any norm $\|\cdot\|_{k,2}$ defined in (3). For integers $1 \le n \le N$ and for $0 \le \delta \le 1$, set

$$F(\delta, n, N) = \begin{cases} (32\delta N/n)^n & \text{if } \delta \ge n/(2N), \\ (en/(\delta N))^{4\delta N} & \text{if } \delta \le n/(2N). \end{cases} \tag{5}$$

Theorem 3.1. *Let $n \in [N]$, $0 \le \delta \le 1$, $0 < \varepsilon \le 1$. Let $k \in [N]$ such that $k \ln(eN/k) \ge n$. Let T be a non-empty subset of $\mathbb{R}^n$ and denote $M := N(T, \varepsilon B_\infty^n)$. Then there exists a set $\mathcal{N} \subset T$ and a collection of parallelepipeds $\mathcal{P}$ in $\mathbb{R}^n$ such that*

$$\max\{|\mathcal{N}|, |\mathcal{P}|\} \le M\, F(\delta, n, N)\, e^{\delta N}.$$

Moreover, for any random matrix Γ satisfying assumption (4), with probability at least $1 - e^{-k\ln(eN/k)} - e^{-\delta N/4}$, one has

$$\begin{cases} \forall x \in T\ \exists y \in \mathcal{N} \quad \textit{such that} \quad \|\Gamma(x-y)\|_{k,2} \le C\varepsilon \sqrt{\dfrac{kn}{\delta} \ln\left(\dfrac{eN}{k}\right)}, \\ \forall x \in T\ \exists P \in \mathcal{P} \quad \textit{such that} \quad x \in P \quad \textit{and} \quad \Gamma P \subset \Gamma x + C\varepsilon \sqrt{\dfrac{kn}{\delta} \ln\left(\dfrac{eN}{k}\right)}\, \boldsymbol{B}_{k,2}, \end{cases}$$

where $C \ge 1$ is an absolute constant.

Remark 3.2. This result extends Theorem A and Corollary A from [25], where the authors considered the case of square matrices, $T = S^{n-1}$ and $k = N$, which corresponds to the approximation of Γx in the Euclidean norm.

3.1 Basic facts about covering numbers and operator norms of random matrices.

We begin by recalling some classical estimates for covering numbers that will be used later. It is well known that for any two centrally symmetric bodies K and L in $\mathbb{R}^m$ and any $\varepsilon > 0$ there exists an ε-net $\mathcal{N}$ of L with respect to K with cardinality

$$|\mathcal{N}| \le |(2/\varepsilon)L + K|/|K| \tag{6}$$

(see e.g. Lemma 4.16 in [24]). In particular, if $K = L$ are centrally symmetric bodies in $\mathbb{R}^m$ (or if L is the boundary of a centrally symmetric body K) then $|\mathcal{N}| \le (1 + 2/\varepsilon)^m$.

Lemma 3.3. *a) For every $\varepsilon \in (0, 1/\sqrt{m}]$*

$$N(B_2^m, \varepsilon B_\infty^m) \le \left(7/(\varepsilon\sqrt{m})\right)^m$$

and for every $\varepsilon \in (1/\sqrt{m}, 1]$

$$N(B_2^m, \varepsilon B_\infty^m) \le \left(17\varepsilon^2 m\right)^{1/\varepsilon^2}.$$

b) For $J \subset [m]$, let $S^J = \{x \in \mathbb{R}^J \mid \|x\|_2 = 1\}$. For every $\varepsilon \in (0,1)$ and every integer $k \le m$, there exists a finite set $\mathcal{N} \subset \cup_{|J|=k} S^J$ such that

$$\begin{cases} |\mathcal{N}| \le \exp\left(k\ln(3/\varepsilon) + k\ln(em/k)\right), \\ \forall J \subset [m] \textit{ with } |J| = k \ \ \forall y \in S^J \quad \exists z \in \mathcal{N} \cap S^J \textit{ such that } \|y - z\|_2 \le \varepsilon. \end{cases} \tag{7}$$

Proof. a) Note that for every $m \geq 1$ one has $(1/\sqrt{m})B_\infty^m \subset B_2^m$ and $|B_2^m| \leq (2\pi e/m)^{m/2}$. Therefore, by (6), we obtain for every $\varepsilon \leq 1/\sqrt{m}$

$$N(B_2^m, \varepsilon B_\infty^m) \leq \frac{\left|\frac{2}{\varepsilon}B_2^m + B_\infty^m\right|}{|B_\infty^m|} \leq \left(\frac{3}{\varepsilon}\right)^m \frac{|B_2^m|}{|B_\infty^m|} \leq \left(\frac{3\sqrt{\pi e}}{\varepsilon\sqrt{2m}}\right)^m .$$

This implies the first bound. For the second bound note that for every $x \in B_2^n$ the number of coordinates of x larger than ε is at most $1/\varepsilon^2$. Thus every $x \in B_2^n$ can be presented as $x = y + z$, where the cardinality of support of y is at most $1/\varepsilon^2$, $z \in \varepsilon B_\infty^n$, and supports of y and z are mutually disjoint. Therefore, it is enough to cover B_2^σ by $\varepsilon B_\infty^\sigma$ for all $\sigma \subset [n]$ with $|\sigma| = m := \lfloor 1/\varepsilon^2 \rfloor$. Using the above bound we obtain

$$N(B_2^n, \varepsilon B_\infty^n) \leq \binom{n}{m}\left(\frac{3\sqrt{\pi e}}{\varepsilon\sqrt{2m}}\right)^m \leq \left(\frac{3en\sqrt{\pi e}}{\varepsilon m\sqrt{2m}}\right)^m ,$$

which implies the desired result as $m \leq 1/\varepsilon^2$.

b) Fix $\varepsilon \in (0, 1)$. For any fixed $J \subset [m]$ of cardinality k, we cover S^J by an ε-net (of points in S^J) of cardinality at most $(1 + 2/\varepsilon)^k \leq (3/\varepsilon)^k$ and we take the union of these nets over all sets J of cardinality k. We conclude using that $\binom{m}{k} \leq (em/k)^k$. □

The next lemma is a classical consequence of estimates for covering numbers for evaluating operator norms of random matrices.

Lemma 3.4. *Let $B = \{b_{ij}\}_{\substack{1\leq i\leq N\\ 1\leq j\leq n}}$ be a fixed $N \times n$ matrix. Let $k \in [N]$ be such that $k \ln \frac{eN}{k} \geq n$. Let ε_{ij} be i.i.d. Rademacher random variables. Denote $B_\varepsilon = \{\varepsilon_{ij} b_{ij}\}_{\substack{1\leq i\leq N\\ 1\leq j\leq n}}$. Then for every $t \geq 1$ one has*

$$\mathbb{P}\left(\|B_\varepsilon : \ell_\infty^n \to X_{k,2}\| \geq 6\, t \sqrt{k \ln\left(\frac{eN}{k}\right)} \max_{i\leq N} \|R_i(B)\|_2\right) \leq e^{-t^2 k \ln(eN/k)},$$

where $R_i(B)$, $i \leq N$, are the rows of B.

Proof. Observe that for any $a \in \mathbb{R}^N$, we have

$$\|a\|_{k,2} = \sup_{\substack{J\subset[N]\\ |J|=k}} \sup_{b\in S^J} \sum_{i=1}^N a_i b_i.$$

Given $x \in \{\pm 1\}^n$, $y \in S^{N-1}$, consider the following random variable,

$$\xi_{x,y} = \sum_{j=1}^n \sum_{i=1}^N \varepsilon_{ij} b_{ij} x_j y_i.$$

Since $e^x + e^{-x} \leq 2\exp(x^2/2)$ for every real x, we observe for $\lambda > 0$,

$$\mathbb{E}\exp\left(\lambda \sum_{j=1}^n \sum_{i=1}^N \varepsilon_{ij} b_{ij} x_j y_i\right) = \prod_{j=1}^n \prod_{i=1}^N \mathbb{E}\exp\left(\lambda \varepsilon_{ij} b_{ij} x_j y_i\right) \leq \exp\left(\frac{\lambda^2}{2}\sum_{i=1}^N y_i^2 \|R_i(B)\|_2^2\right)$$
$$\leq \exp\left(\frac{\lambda^2}{2}\max_{i\leq N}\|R_i(B)\|_2^2\right).$$

Therefore, using the Laplace transform of $\xi_{x,y}$, we deduce that for any $u > 0$,

$$\mathbb{P}\left(\xi_{x,y} > u \max_{i \le N} \|R_i(B)\|_2\right) \le e^{-u^2/2}.$$

Note that

$$\|B_\varepsilon : \ell_\infty^n \to X_{k,2}\| = \sup_{x \in \{\pm 1\}^n} \sup_{\substack{J \subset [N] \\ |J| = k}} \sup_{y \in S^J} \xi_{x,y}. \tag{8}$$

Now we apply the classical net argument. Let $\mathcal{N}$ be the net defined by (7) with $\varepsilon = 1/2$. Then

$$\mathbb{P}\left(\sup_{x \in \{\pm 1\}^n} \sup_{z \in \mathcal{N}} \xi_{x,z} \ge u \max_{i \le N} \|R_i(B)\|_2\right) \le 2^n |\mathcal{N}| e^{-u^2/2}$$
$$\le 2^n \exp\left(-\frac{u^2}{2} + k \ln 6 + k \ln(eN/k)\right).$$

Taking $u = 3t\sqrt{k \ln(eN/k)}$ and using $k \ln(eN/k) \ge n$, we get for every $t \ge 1$,

$$\mathbb{P}\left(\sup_{x \in \{\pm 1\}^n} \sup_{z \in \mathcal{N}} \xi_{x,z} \ge 3t\sqrt{k \ln(eN/k)} \max_{i \le N} \|R_i(B)\|_2\right) \le e^{-t^2 k \ln(eN/k)}.$$

By definition of $\mathcal{N}$, for any $J \subset [N]$ of cardinality k and $y \in S^J$, there exists $z \in \mathcal{N} \cap S^J$ such that $\|z - y\|_2 \le 1/2$, hence, by the triangle inequality,

$$\sup_{x \in \{\pm 1\}^n} \sup_{\substack{J \subset [N] \\ |J| = k}} \sup_{y \in S^J} \xi_{x,y} \le 2 \sup_{x \in \{\pm 1\}^n} \sup_{z \in \mathcal{N}} \xi_{x,z}.$$

This completes the proof of the lemma. □

3.2 Auxiliary statements

By $\mathcal{D}_n$ we denote the set of all $n \times n$ diagonal matrices whose diagonal entries belong to the set $\{1\} \cup \{2^{-2^k}\}_{k \ge 0}$. The following theorem was proved in [25] in the square case. However the proof works as well in the rectangular case. One just needs to repeat the proof of Proposition 2.7 there for $N \times n$ matrices, to combine it with Remark 2.8 following the proposition, and to substitute the upper bound on the expectation with a probability bound using Markov's inequality.

Theorem 3.5. *Let $\Gamma = \{\xi_{ij}\}_{1 \le i \le N, 1 \le j \le n}$ be an $N \times n$ random matrix on a probability space Ω. Assume that entries of Γ are independent centered random variables with unit variances and that in each row the entries are identically distributed. Let $\delta \in (0, 1]$. Then there exists a random matrix D_Γ on Ω taking values in $\mathcal{D}_n$ such that*

(i) for every $\omega \in \Omega$, $D_\Gamma(\omega)$ depends only on the realization $\{|\xi_{ij}(\omega)|\}_{1 \le i \le N, 1 \le j \le n}$,

(ii) for every $\omega \in \Omega$ one has

$$\|R_i(\Gamma(\omega)D_\Gamma(\omega))\|_2 \le C\sqrt{n/\delta},$$

(iii)

$$\mathbb{P}\left(\det D_\Gamma \le e^{-4\delta N}\right) \le e^{-\delta N},$$

where C is an absolute positive constant.

As in [25], Theorem 3.5 has important consequences. It allows us to construct, with high probability, a diagonal matrix D such that the volume of DB_∞^n remains big enough and such that, according to Lemma 3.4, we have a good control of the operator norm of ΓD from ℓ_∞^n to $X_{k,2}$. Comparing to [25], Lemma 3.4 simplifies significantly the proof and allows to extend Theorem 3.1 from [25] to the case of rectangular matrices and to approximations with respect to $\|\cdot\|_{k,2}$ norms.

Theorem 3.6. *Let $1 \le n \le N$ be integers, $\delta \in (0,1]$. Let $k \in [N]$ such that $k \ln \frac{eN}{k} \ge n$. Let Γ be an $N \times n$ random matrix satisfying the hypothesis* (4). *Then*

$$\mathbb{P}\left(\exists D \in \mathcal{D}_n \,|\, \det D \ge e^{-\delta N} \quad \textit{and} \quad \|\Gamma D : \ell_\infty^n \to X_{k,2}\| \le C\sqrt{\frac{kn}{\delta}\ln\left(\frac{eN}{k}\right)}\right)$$
$$\ge 1 - e^{-\delta N/4} - e^{-k\ln(eN/k)},$$

where C is a positive absolute constant.

Proof. Let D_Γ be the matrix given by Theorem 3.5. By property (iii) of D_Γ it is enough to prove that

$$\mathbb{P}\left(\|\Gamma D : \ell_\infty^n \to X_{k,2}\| \le C\sqrt{\frac{kn}{\delta}\ln\left(\frac{eN}{k}\right)}\right) \ge 1 - e^{-k\ln(eN/k)}.$$

Consider two probability spaces – the original one $(\Omega, \mathbb{P}_\omega)$, where the matrix Γ is defined, and the auxiliary space $(E, \mathbb{P}_\varepsilon)$, where $E := \{-1,1\}^{N\times n}$ and $\mathbb{P}_\varepsilon$ is the uniform probability on E. Given a matrix $A = \{a_{ij}\}_{1\le i\le N, 1\le j\le n}$ and $\varepsilon \in E$, denote $A_\varepsilon = \{\varepsilon_{ij}a_{ij}\}_{1\le i\le N,1\le j\le n}$. Since entries of Γ are symmetric, for every fixed $\varepsilon \in E$ the matrix Γ_ε has the same distribution on Ω as Γ. By property (i) of D_Γ, we have $D_\Gamma = D_{\Gamma_\varepsilon}$ for every fixed $\varepsilon \in E$. Therefore, since D_Γ is diagonal, we have for every $\varepsilon \in E$

$$(\Gamma D_\Gamma)_\varepsilon = \Gamma_\varepsilon D_\Gamma = \Gamma_\varepsilon D_{\Gamma_\varepsilon}.$$

Then, by property (ii) of D_Γ from Theorem 3.5, there exists an absolute positive constant C_1 such that for every $i \le N$ and every $(\omega, \varepsilon) \in \Omega \times E$,

$$\|R_i\left((\Gamma(\omega)D_\Gamma(\omega))_\varepsilon\right)\|_2 \le C_1\sqrt{n/\delta}.$$

Fixing $\omega \in \Omega$ and applying Lemma 3.4 to the matrix $B = \Gamma(\omega)D_\Gamma(\omega)$, we obtain that for every fixed $\omega \in \Omega$ one has

$$\mathbb{P}_\varepsilon\left(\|\Gamma_\varepsilon(\omega)D_\Gamma(\omega) : \ell_\infty^n \to X_{k,2}\| > 6C_1\sqrt{\frac{kn}{\delta}\ln\left(\frac{eN}{k}\right)}\right) \le e^{-k\ln(eN/k)}.$$

Using that Γ_ε has the same distribution as Γ and the Fubini theorem, we obtain

$$\mathbb{P}_\omega\left(\|\Gamma D_\Gamma : \ell_\infty^n \to X_{k,2}\| > 6C_1\sqrt{\frac{kn}{\delta}\ln\left(\frac{eN}{k}\right)}\right) =$$
$$= \mathbb{P}_\varepsilon\mathbb{P}_\omega\left(\|\Gamma_\varepsilon(\omega)D_\Gamma(\omega) : \ell_\infty^n \to X_{k,2}\| > 6C_1\sqrt{\frac{kn}{\delta}\ln\left(\frac{eN}{k}\right)}\right)$$
$$\leq e^{-k\ln(eN/k)}.$$

□

As in Lemma 3.11 from [25], we need to estimate the cardinality of the set of diagonal matrices in $\mathcal{D}_n$ with not so small determinant.

Lemma 3.7. *Let $n, N \geq 1$ be integers, $\delta \in (0,1]$ and*

$$Q := \{D \in \mathcal{D}_n \mid \det D \geq \exp(-\delta N)\}.$$

Then $|Q| \leq F(\delta, n, N)$, where $F(\delta, n, N)$ is defined by formula (5).

Proof. Note that if $D \in \mathcal{D}_n$ and d_1, ..., d_n its diagonal elements then for every $k \geq 0$ the set

$$Q_D(k) = \left\{i \leq n \mid d_i = 2^{-2^k}\right\}$$

has cardinality at most $m_k := \min\{n, \lfloor 2^{-k}2\delta N\rfloor\}$. Thus there are at most

$$\sum_{\ell=0}^{m_k}\binom{n}{\ell} \leq \left(\frac{en}{m_k}\right)^{m_k}$$

choices of $\sigma_k \subset [n]$, where matrices from $\mathcal{D}_n$ may have such coordinates. Note also that the trivial bound for the number of subsets is 2^n. Denote $a := 4\delta N/n$. Note that $m_k \leq n/2$ if and only if $2^k \geq a$.

Case 1. $a \geq 2$. Set $m := \lfloor \log_2 a\rfloor \geq 1$. By above we have

$$|Q| \leq \prod_{k<m} 2^n \prod_{k\geq m}\left(\frac{en}{m_k}\right)^{m_k} \leq 2^{nm}\prod_{k\geq m}\left(\frac{en}{2\delta N}\right)^{2\delta N/2^k}\prod_{k\geq m} 2^{2k\delta N/2^k}$$
$$\leq a^n\left(\frac{2e}{a}\right)^{4\delta N/a} 2^{2\delta N(2m+1)/2^m} \leq (2e)^n\, a^{4\delta N/2^m}\, 2^{4\delta N/2^m} \leq (8a)^n.$$

Case 2. $a \leq 2$. Similarly we have

$$|Q| \leq \prod_{k\geq 0}\left(\frac{en}{m_k}\right)^{m_k} \leq \prod_{k\geq 0}\left(\frac{en}{2\delta N}\right)^{2\delta N/2^k}\prod_{k\geq 0} 2^{2k\delta N/2^k} \leq \left(\frac{en}{2\delta N}\right)^{4\delta N} 2^{3\delta N},$$

which implies the desired result. □

3.3 Proof of Theorem 3.1

Let Q be as in Lemma 3.7. Note that every $D \in Q$ is diagonal with reciprocal of integers on the diagonal. Therefore, there exists a set $\mathcal{N}_D \subset T$ of cardinality

$$|\mathcal{N}_D| \le N(T, \varepsilon D B_\infty^n) \le N(T, \varepsilon B_\infty^n) N(B_\infty^n, D B_\infty^n) \le M \det D^{-1} \le M e^{\delta N}$$

which satisfies that for any $x \in T$ there exists $y \in \mathcal{N}_D$ such that $x - y \in \varepsilon D B_\infty^n$. Let

$$\mathcal{P} = \{ y + \varepsilon D B_\infty^n \mid D \in Q, y \in \mathcal{N}_D \}.$$

Then, by Lemma 3.7, $|\mathcal{P}| \le M e^{\delta N} F(\delta, n, N)$ and for any $x \in T$ and for any $D \in Q$ there exists $P = y_{x,D} + \varepsilon D B_\infty^n \in \mathcal{P}$ such that $x \in P$.

Theorem 3.6 implies that with probability at least $1 - e^{-k \ln(eN/k)} - e^{-\delta N/4}$ there exists $D \in Q$ such that

$$\Gamma(\varepsilon D B_\infty^n) \subset C \, \varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)} \, \mathbf{B}_{k,2}.$$

Therefore, for such D,

$$\Gamma(x - y_{x,D}) \in \Gamma(\varepsilon D B_\infty^n) \subset C \, \varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)} \, \mathbf{B}_{k,2},$$

hence,

$$\Gamma(P) \subset \Gamma x + \Gamma(y_{x,D} - x) + \Gamma(\varepsilon D B_\infty^n) \subset \Gamma x + 2C \, \varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)} \, \mathbf{B}_{k,2}.$$

This proves the existence of a "good" collection $\mathcal{P}$.

Finally, let $\mathcal{P}'$ be the set of all $P \in \mathcal{P}$ such that $P \cap T \ne \emptyset$. For every $P \in \mathcal{P}'$ choose an arbitrary $z_P \in P \cap T$ and let $\mathcal{N} = \{z_P\}_{P \in \mathcal{P}'}$. By above, for every $x \in T$ there exists $D \in Q$ and $P = y_{x,D} + \varepsilon D B_\infty^n \in \mathcal{P}$ such that $x \in P$, in particular $P \in \mathcal{P}'$, and

$$\Gamma(P) \subset \Gamma x + 2C \, \varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)} \, \mathbf{B}_{k,2}.$$

Thus, $\Gamma z_P \in \Gamma x + 2C \, \varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)} \, \mathbf{B}_{k,2}$. This implies the desired result. □

Remark 3.8. We apply Theorem 3.1 for $T \subset t B_2^n$, $t \ge 1$, $\varepsilon \le 1/\sqrt{n}$, and $\delta \ge n/(2N)$ so that, $F(\delta, n, N) = (32\delta N/n)^n$. Then Theorem 3.1 combined with Lemma 3.3 implies that there exists $\mathcal{N} \subset T$ with cardinality at most

$$\left(\frac{224\delta t N}{\varepsilon n^{3/2}}\right)^n e^{\delta N}$$

such that with probability at least $1 - e^{-k \ln(eN/k)} - e^{-\delta N/4}$ one has

$$\forall x \in T \; \exists y \in \mathcal{N} \quad \text{such that} \quad \Gamma(x - y) \in C \, \varepsilon \sqrt{\frac{kn}{\delta} \ln\left(\frac{eN}{k}\right)} \, \mathbf{B}_{k,2}.$$

4 Geometry of Random Polytopes

In this section, we study some classical geometric parameters associated to random polytopes of the form $K_N := \Gamma^* B_1^N$, where $\Gamma = \{\xi_{ij}\}_{1\le i\le N, 1\le j\le n}$ is an $N \times n$ random matrix. In other words, K_N is the absolute convex hull of the rows of Γ. We provide estimates on the asymptotic behavior of the volume and the mean widths of K_N and its polar. In this section, the random operator Γ satisfies the hypothesis (4): the random variables ξ_{ij} are independent symmetric with unit variances such that in each row of Γ the entries are identically distributed. Moreover, we assume that the random variables ξ_{ij} satisfy a uniform small ball probability condition which means that we can fix $u, v \in (0,1)$ such that

$$\forall i, j \quad \sup_{\lambda\in\mathbb{R}} \mathbb{P}\big\{|\xi_{ij} - \lambda| \le u\big\} \le v.$$

4.1 Inclusion Theorem

We start by showing that for an $N \times n$ random matrix Γ satisfying conditions described above, the body $K_N = \Gamma^* B_1^N$ contains a large "regular" body with high probability.

Theorem 4.1. *Let $\beta \in (0,1)$. There are two positive constants $M = M(u,v,\beta)$ and $C(u,v,\beta)$ which depend only on u,v,β and an absolute constant $c > 0$, such that the following holds. For every positive integers n, N satisfying $N \ge Mn$ one has*

$$\mathbb{P}\left(K_N \supset C(u,v,\beta)\left(B_\infty^n \cap \sqrt{\ln(N/n)}\, B_2^n\right)\right) \ge 1 - 4\exp\left(-cn^\beta N^{1-\beta}\right).$$

Remark 4.2. It is known that for a Gaussian random matrix one has

$$\mathbb{P}\left(K_N \supset C\sqrt{\beta\ln(N/n)}\ B_2^n\right) \ge 1 - 3\exp\left(-cn^\beta N^{1-\beta}\right),$$

where C, c are absolute positive constants (see e.g. [10]). Moreover, the probability estimate cannot be improved. Indeed, for a Gaussian random matrix and $\beta \in (0, c'')$ one has

$$\mathbb{P}\left(K_N \supset C'\sqrt{\beta\ln(N/n)} B_2^n\right) \le 1 - \exp\left(-c'n^\beta N^{1-\beta}\right),$$

where $C', c' > 0$ and $0 < c'' \le 1$ are absolute constants.

Since $B_\infty^n \subset \sqrt{n}B_2^n$, Theorem 4.1 has the following consequence.

Corollary 4.3. *Under the assumptions and notations of Theorem 4.1, for $Mn < N \le e^n$ one has*

$$\mathbb{P}\left(K_N \supset C(u,v,\beta)\sqrt{\frac{\ln(N/n)}{n}}\, B_\infty^n\right) \ge 1 - 4\exp\left(-cn^\beta N^{1-\beta}\right).$$

In fact, our proof of Theorem 4.1 gives that if

$$N \ge n \max\left\{\exp(4C_v/\beta), \left(\frac{C\ln(e/(1-\beta))}{c_{uv}\,(1-\beta)}\right)^{1/(1-\beta)}\right\},$$

where $C > 1$ is an absolute positive constant, $c_{uv} = cuv\sqrt{1-v}$ is the constant from Lemma 4.4 below, and $C_v = 5\ln(2/(1-v))$, then

$$\mathbb{P}\left(K_N \supset \frac{c_{uv}}{2\sqrt{2}}\left(B_\infty^n \cap RB_2^n\right)\right) \geq 1 - 4\exp\left(-\frac{n^\beta N^{1-\beta}}{40}\right) \tag{9}$$

with $R = \sqrt{\beta\ln(N/n)/C_v}$. Note that $K_N = \text{abs conv}\{x_j\}_{j\leq N}$, where $x_j = \Gamma^* e_j$ are the columns of Γ^*. Hence for every $z \in \mathbb{R}^n$,

$$h_{K_N}(z) = \sup_{j\leq N} |\langle z, x_j\rangle| = \|\Gamma z\|_\infty.$$

Let $L = c_{uv}(B_\infty^n \cap RB_2^n)$. To prove (9), we show that

$$\mathbb{P}\left(\exists z \in \partial L^0 \ \mid\ \|\Gamma z\|_\infty < \frac{1}{4}\right) \leq 4\exp\left(-\frac{n^\beta N^{1-\beta}}{40}\right). \tag{10}$$

The proof of this statement will be divided into two steps. First, we will show an individual estimate for a fixed $z \in \partial L^0$. Then we use the net introduced in Theorem 3.1 to get a global estimate for any point of this net, using that this net is a subset of ∂L^0. A crucial point is that this net is a good covering of $\Gamma(\partial L^0)$ in $\|\cdot\|_{k,2}$-metric.

4.1.1 Basic facts about small ball probabilities.

Recall that for a (real) random variable ξ its *Lévy concentration function* $\mathcal{Q}(\xi,\cdot)$ is defined on $(0,\infty)$ as

$$\mathcal{Q}(\xi,t) := \sup_{\lambda\in\mathbb{R}} \mathbb{P}\big\{|\xi-\lambda| \leq t\big\}.$$

For any centered random variable with unit variance, there exist $u, v \in (0,1)$ such that

$$\mathcal{Q}(\xi,u) \leq v. \tag{11}$$

The following lemma is a consequence of Rogozin's theorem [26] that was used for example in [25] (see Lemma 4.7 there).

Lemma 4.4. *Let $\xi_1, ..., \xi_m$ be independent random variables satisfying (11) with the same $u, v \in (0,1)$. Then for every $x \in S^{m-1}$ one has*

$$\mathcal{Q}\Big(\sum_{i=1}^m x_i\xi_i, c_{uv}\Big) \leq v,$$

where $c_{uv} = cuv\sqrt{1-v}$ and $c \in (0,1]$ is an absolute constant.

Remark 4.5. If we have a bounded moment of order larger than 2, then we could use a consequence of the Paley-Zygmund inequality, which also provides a lower bound on the small ball probability of a random sum. The following statement was proved in [19, Lemma 3.1] following the lines of [18, Lemma 3.6] with appropriate modifications to deal with centered random variables (rather than symmetric):

Let $2 < r \le 3$ and $\mu \ge 1$. Suppose $\xi_1, \dots, \xi_m$ are independent centered random variables such that $\mathbb{E}|\xi_i|^2 \ge 1$ and $\mathbb{E}|\xi_i|^r \le \mu^r$ for every $i \le m$. Let $x = (x_i) \in \ell_2$ be such that $\|x\|_2 = 1$. Then for every $\lambda \ge 0$

$$\mathbb{P}\left(\Big|\sum_{i=1}^m \xi_i x_i\Big| > \lambda\right) \ge \left(\frac{1-\lambda^2}{8\mu^2}\right)^{r/(r-2)}. \tag{12}$$

Proof of Lemma 4.4. Fix $x \in S^{m-1}$. We clearly have $\mathcal{Q}(x_i\xi_i, |x_i|u) \le v$ for every $x_i \ne 0$. Applying Theorem 1 of [26] to random variables $x_i\xi_i$, $i \le m$, we observe there exists and absolute constant $C \ge 1$ such that for every $w \ge u\|x\|_\infty/2$,

$$\mathcal{Q}\big(\sum_{i=1}^m x_i\xi_i, w\big) \le \frac{Cw}{\sqrt{\sum_{i=1}^m |x_i|^2 u^2\big(1-\mathcal{Q}(x_i\xi_i, |x_i|u)\big)}} \le \frac{Cw}{u\sqrt{1-v}}.$$

Take $w = uv\sqrt{1-v}/C$. If $\|x\|_\infty \le 2v\sqrt{1-v}/C$ then $w \ge u\|x\|_\infty/2$. Therefore for such x we have

$$\mathcal{Q}\left(\sum_{i=1}^m x_i\xi_i, w\right) \le v.$$

If there exists $\ell \le m$ such that $|x_\ell| > 2v\sqrt{1-v}/C$, then we have

$$\mathcal{Q}\left(\sum_{i=1}^m x_i\xi_i, w\right) \le \mathcal{Q}\big(x_\ell\xi_\ell, w\big) = \mathcal{Q}\big(\xi_\ell, w/|x_\ell|\big) \le \mathcal{Q}\big(\xi_\ell, u\big) \le v,$$

which completes the proof. □

4.1.2 The individual small ball estimate.

To prove Theorem 4.1 we need to extend a result by Montgomery-Smith [23], which originally was proved for Rademacher random variables. Note that this lemma does not require any conditions on the moments of random variables.

Lemma 4.6. *Let ξ_i, $i \le n$, be independent symmetric random variables satisfying condition (11). Let $\alpha \ge 1$ and $L = c_{uv}(B_\infty^n \cap \alpha B_2^n)$, where c_{uv} is the constant from Lemma 4.4. Then for every non-zero $z \in \mathbb{R}^n$ one has*

$$\mathbb{P}\left(\sum_{i=1}^n \xi_i z_i > h_L(z)\right) > ((1-v)/2)^{5\alpha^2}.$$

We postpone the proof of this lemma to the end of this section. Note that if our variables satisfy $1 \le \mathbb{E}\xi_i^2 \le \mathbb{E}|\xi_i|^r \le \mu^r$ for some $r > 2$ then using (12) and repeating the proof of Lemma 4.3 from [18] we could consider $L = (1-\delta)(B_\infty^n \cap \alpha B_2^n)$ and estimate the corresponding probability from below by $\exp(-C_{\mu,\delta,r}\alpha^2)$, where $C_{\mu,\delta,r}$ depends only on μ, δ, r.

Lemma 4.6 has the following consequence.

Lemma 4.7. *Under assumptions of Lemma 4.6 for every* $z \in \mathbb{R}^n$ *and every* $\sigma \subset [N]$ *one has*

$$\mathbb{P}\left(\|P_\sigma \Gamma z\|_\infty < h_L(z)\right) < \exp\left(-|\sigma| \exp(-C_v \alpha^2)\right),$$

where $P_\sigma : \mathbb{R}^N \to \mathbb{R}^\sigma$ *is the coordinate projection and* $C_v = 5\ln(2/(1-v)))$.

Proof. Applying Lemma 4.6 to the $|\sigma| \times n$ random matrix $P_\sigma \Gamma = (\xi_{ij})_{i\in\sigma, j\le n}$ we have for every $z = \{z_j\}_{j=1}^n \in \mathbb{R}^n$ and every $i \in \sigma$

$$\mathbb{P}\left(\sum_{j=1}^n z_j \xi_{ij} < h_L(z)\right) \le 1 - \exp(-C_v\alpha^2) \le \exp\left(-\exp(-C_v\alpha^2)\right).$$

Thus

$$\mathbb{P}\left(\|P_\sigma \Gamma z\|_\infty < h_L(z)\right) = \mathbb{P}\left(\sup_{i\in\sigma}\left|\sum_{j=1}^n z_j \xi_{ij}\right| < h_L(z)\right)$$
$$= \prod_{i\in\sigma} \mathbb{P}\left(\left|\sum_{j=1}^n z_j \xi_{ij}\right| < h_L(z)\right) < \exp\left(-|\sigma|\exp(-C_v\alpha^2)\right).$$

□

We can now state the main individual small ball estimate.

Lemma 4.8. *Let* $\beta \in (0,1)$ *and define* $m = 8\lceil (N/n)^\beta \rceil$ *(if the latter number is larger than or equal to* $N/4$ *we take* $m = N$*) and* $k = \lfloor N/m \rfloor$. *Let* $L = c_{uv}(B_\infty^n \cap RB_2^n)$, *where* $R = \sqrt{\beta \ln(N/n)/C_v}$. *Then for any* $z \in \partial L^o$ *one has*

$$\mathbb{P}\left(\frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2} < \frac{1}{2}\right) \le \exp(-0.3\; n^\beta\, N^{1-\beta}).$$

Proof. Below we assume $m < N/4$ (then $k \ge 4$, hence $km > 4N/5$); the proof in the case $m = N$, $k = 1$ repeats the same lines with simpler calculations. Let $\sigma_1, \dots, \sigma_k$ be a partition of $[N]$ such that $m \le |\sigma_i|$ for every $i \le k$. Then, for any $a \in \mathbb{R}^N$

$$\frac{1}{\sqrt{k}}\|a\|_{k,2} \ge \frac{1}{\sqrt{k}}\left(\sum_{i=1}^k \|P_i a\|_\infty^2\right)^{1/2} \ge \frac{1}{k}\sum_{i=1}^k \|P_i z\|_\infty,$$

where $P_i = P_{\sigma_i} : \mathbb{R}^N \to \mathbb{R}^{\sigma_i}$ is the coordinate projection. Define $|||\cdot|||$ on $\mathbb{R}^N$ by

$$|||z||| = \frac{1}{k}\sum_{i=1}^k \|P_i z\|_\infty$$

for every $z \in \mathbb{R}^N$. Note that if for some $z \in \mathbb{R}^n$ we have $|||\Gamma z||| < h_L(z)/2$ then there exists $I \subset [k]$ of cardinality at least $k/2$ such that for every $i \in I$ one has $\|P_i \Gamma z\|_\infty < h_L(z)$.

Applying Lemma 4.7 with $\alpha = R$ (note that $\alpha \geq 2$, by the condition on n and N), we obtain for every $z = \{z_i\}_{i=1}^n \in \mathbb{R}^n$,

$$\begin{aligned}
\mathbb{P}\left(|||\Gamma z||| < h_L(z)/2\right) &\leq \sum_{|I|=[(k+1)/2]} \mathbb{P}\left(\|P_i \Gamma z\|_\infty < h_L(z) \text{ for every } i \in I\right) \\
&\leq \sum_{|I|=[(k+1)/2]} \prod_{i\in I} \mathbb{P}\left(\|P_i \Gamma z\|_\infty < h_L(z)\right) \\
&\leq \sum_{|I|=[(k+1)/2]} \prod_{i\in I} \exp\left(-|\sigma_i| \exp(-C_v \alpha^2)\right) \\
&\leq \binom{k}{[k/2]} \exp\left(-(km/2)\exp(-C_v\alpha^2)\right) \\
&\leq \exp\left(k \ln 2 - (km/2)\exp(-C_v \alpha^2)\right),
\end{aligned}$$

where $C_v = 5\ln(2/(1-v))$. By our choice of k and m we have $km > 4N/5$, therefore $(km/2)\exp(-C_v\alpha^2) \geq 2N^{1-\beta}n^\beta/5$. We also have $k \leq N^{1-\beta}n^\beta/8$. Thus

$$\mathbb{P}\left(|||\Gamma z||| < h_L(z)/2\right) \leq \exp\left(-0.3 \; N^{1-\beta}n^\beta\right).$$

This completes the proof. □

Finally we prove Lemma 4.6. For a positive integer m, define $|||\cdot|||_m$ on $\mathbb{R}^n$ by

$$|||z|||_m = \sup \sum_{i=1}^m \left(\sum_{k\in B_i} |z_k|^2\right)^{1/2},$$

where the supremum is taken over all partitions $B_1, \ldots, B_m$ of $[n]$. We will need the following lemma, which was essentially proved in [23] (see Lemma 2 there).

Lemma 4.9. *Let $\alpha \geq 1$ and $m \geq 1 + 4\alpha^2$ be an integer. For all $x \in \mathbb{R}^n$ one has*

$$h_{B_\infty^n \cap \alpha B_2^n}(x) \leq |||x|||_m.$$

Proof. Fix $x \in \mathbb{R}^n$ and choose $y \in B_\infty^n \cap \alpha B_2^n$ so that $h(x) = \sum_i x_i y_i$. For every k with $y_k^2 \geq 1/2$ choose $B_{1,k} = \{k\}$. Since $|y| \leq \alpha$ there are at most $2\alpha^2$ such sets. Denote $B := \cup_k B_{1,k}$. Now let z_i denote y_i if $|y_i| \leq 1/\sqrt{2}$ and $z_i = 0$ otherwise. Let $n_0 = 0$ and define $n_0 < n_1 < n_2 < \ldots$ by

$$n_{k+1} = 1 + \sup\left\{\ell \in [n_k + 1, n-1] \mid \sum_{i=n_k+1}^{\ell} z_i^2 \leq 1/2\right\}$$

(if $n_k = n$ we stop the procedure). Denote $B_{2,k} := [n_{k-1}+1, n_k] \setminus B$. Since $|y| \leq \alpha$ we have at most $2\alpha^2 + 1$ such sets. Moreover, we have

$$\sum_{i\in B_{2,k}} z_i^2 = \sum_{i\in B_{2,k}} y_i^2 \leq 1.$$

Since $y \in B_\infty^n$ and $m \geq 4\alpha^2 + 1$, we obtain

$$h(x) = \sum_{i=1}^n x_i y_i \leq \sum_{j=1}^2 \sum_k \Big(\sum_{i \in B_{j,k}} x_i^2 \Big)^{1/2} \Big(\sum_{i \in B_{j,k}} y_i^2 \Big)^{1/2} \leq \sum_{j \leq 2, k} \Big(\sum_{i \in B_{j,k}} x_i^2 \Big)^{1/2} \leq |||x|||_m.$$

□

Proof of Lemma 4.6: We folow the lines of Montgomery-Smith's proof. Let $m = \lceil 1 + 4\alpha^2 \rceil$. Given $z \in \mathbb{R}^n$, let $m' \leq m$ and $B_1, \ldots, B_{m'}$ be a partition of $[n]$ such that

$$\forall i \leq m' \ \sum_{k \in B_i} |z_k|^2 \neq 0 \qquad \text{and} \qquad |||z|||_m = \sum_{i=1}^{m'} \Big(\sum_{k \in B_i} |z_k|^2 \Big)^{1/2}.$$

Then, using Lemma 4.9, we have

$$\begin{aligned} p := \mathbb{P}\left(\sum_{i=1}^n \xi_i z_i > h_L(z) \right) &\geq \mathbb{P}\left(\sum_{i=1}^n \xi_i z_i > c_{uv} \, |||z|||_m \right) \\ &= \mathbb{P}\left(\sum_{i=1}^{m'} \sum_{k \in B_i} \xi_k z_k > c_{uv} \sum_{i=1}^{m'} \Big(\sum_{k \in B_i} |z_k|^2 \Big)^{1/2} \right) \\ &\geq \mathbb{P}\left(\bigcap_{i \leq m'} \Big(\sum_{k \in B_i} \xi_k z_k \geq c_{uv} \, (\sum_{k \in B_i} |z_k|^2)^{1/2} \Big) \right). \end{aligned}$$

Since ξ_i's are independent we obtain

$$p \geq \prod_{i=1}^{m'} \mathbb{P}\left(\sum_{k \in B_i} \xi_k z_k > c_{uv} \ \Big(\sum_{k \in B_i} |z_k|^2 \Big)^{1/2} \right).$$

For $i \leq m'$ set

$$f_i = \Big(\sum_{k \in B_i} \xi_k z_k \Big) \cdot \Big(\sum_{k \in B_i} |z_k|^2 \Big)^{-1/2}.$$

Using that ξ_i's are symmetric and applying Lemma 4.4 we get

$$\mathbb{P}(f_i > c_{uv}) \ = \ \frac{1}{2} \, \mathbb{P}(|f_i| > c_{uv}) \geq \frac{1-v}{2}.$$

Therefore,

$$p \geq ((1-v)/2)^{m'} \geq ((1-v)/2)^m \geq ((1-v)/2)^{5\alpha^2},$$

which implies the desired result. □

4.1.3 The global small ball estimate.

In this section, we prove Theorem 4.1. As we mentioned after its statement, our goal is to prove (10) for $N \geq Mn$, where M depends only on β, u and v.

Let $\beta \in (0,1)$ and, as in Lemma 4.8, define $m = 8\lceil (N/n)^\beta \rceil$ and $k = \lfloor N/m \rfloor$ so that $N^{1-\beta}n^\beta/10 \leq k \leq N^{1-\beta}n^\beta/8$. By the choice of M, we obviously have $k\ln(eN/k) \geq n$. Let $T = \partial L^o$ and set

$$\delta = 0.1(n/N)^\beta \quad \text{ and } \quad \varepsilon = \frac{1}{c_{uv}\sqrt{n}\,\exp((N/n)^{1-\beta}/20)}.$$

Since

$$T \subset L^0 = c_{uv}^{-1}\big(\operatorname{conv}\, B_1^n \cup (B_2^n/R)\big) \subset c_{uv}^{-1}B_2^n,$$

we use Theorem 3.1 (see Remark 3.8) to construct a set $\mathcal{N} \subset T$ of cardinality at most

$$\left(\frac{224\delta N}{\varepsilon c_{uv} n^{3/2}}\right)^n e^{\delta N}$$

such that with probability at least $1 - e^{-k\ln(eN/k)} - e^{-\delta N/4}$ one has

$$\forall x \in T\ \exists z \in \mathcal{N} \quad \text{such that} \quad \|\Gamma(x-z)\|_{k,2} \leq C_1\,\varepsilon\sqrt{\frac{kn}{\delta}\ln\left(\frac{eN}{k}\right)}, \tag{13}$$

where $C_1 > 0$ is an absolute constant. Since

$$\exp\left(n\ln(224\delta N/n) + n\ln(1/(\varepsilon c_{uv}n^{1/2})) + \delta N - 0.3\ N^{1-\beta}n^\beta\right) \leq \exp\left(-0.1\ N^{1-\beta}n^\beta\right),$$

provided that $(N/n)^{1-\beta}$ is large enough, and $\mathcal{N} \subset T$, we deduce from Lemma 4.8 that

$$\mathbb{P}\left(\exists z \in \mathcal{N}\ :\ \frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2} < 1/2\right) \leq \sum_{z\in\mathcal{N}} \mathbb{P}\left(\frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2} < 1/2\right) \leq \exp\left(-0.1\ N^{1-\beta}n^\beta\right).$$

Let $\overline{\Omega}$ be the subset of Ω, where (13) holds. Then, on $\overline{\Omega}$, for every $x \in T$ there exists $z \in \mathcal{N}$ such that

$$\begin{aligned}\frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2} &\leq \frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2} + \frac{1}{\sqrt{k}}\|\Gamma(z-x)\|_{k,2} \leq \frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2} + C_1\varepsilon\sqrt{\frac{n}{\delta}\ln\left(\frac{eN}{k}\right)}.\\ &\leq \frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2} + \frac{C_2\sqrt{\left(\frac{N}{n}\right)^\beta \ln\left(10e\left(\frac{N}{n}\right)^\beta\right)}}{c_{uv}\ \exp((N/n)^{1-\beta}/20)},\end{aligned}$$

where C_2 is an absolute positive constant. Since $N \geq Mn$ (for large enough M depending only on u, v and β), we observe

$$c_{uv}^2\exp((N/n)^{1-\beta}/10) > 16\,C_2^2\left(\frac{N}{n}\right)^\beta \ln\left(10e\left(\frac{N}{n}\right)^\beta\right).$$

Therefore,

$$\mathbb{P}\left(\left\{\omega\in\overline{\Omega}\mid \exists x\in\partial L^o\,:\,\frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2}<\frac14\right\}\right)$$
$$\le \mathbb{P}\left(\left\{\omega\in\overline{\Omega}\mid \exists z\in\mathcal{N}\,:\,\frac{1}{\sqrt{k}}\|\Gamma z\|_{k,2}<\frac12\right\}\right)\le \exp\left(-0.1\ N^{1-\beta}n^{\beta}\right).$$

The desired result follows since $h_{K_N}(x)=\|\Gamma x\|_\infty\ge\frac{1}{\sqrt{k}}\|\Gamma x\|_{k,2}$ for every $x\in\mathbb{R}^n$ and since

$$\mathbb{P}(\overline{\Omega})\ge 1-e^{-k\ln(eN/k)}-e^{-\delta N/4}\ge 1-2\exp(N^{1-\beta}n^{\beta}/40).$$

□

4.2 Volumes and mean widths of K_N and K_N^0

In this section we apply the results of the previous subsection to obtain asymptotically sharp estimates for the volumes and the mean widths of K_N and K_N^0. We refer to [24] for general knowledge about these parameters. We recall that by Santaló inequality and Bourgain-Milman [4] inverse Santaló inequality there exists an absolute positive constant c such that for every convex symmetric body K one has

$$c^n|B_2^n|^2\le|K||K^0|\le|B_2^n|^2. \tag{14}$$

Below we fix constants $M=M(u,v,\beta)$ and $C(u,v,\beta)$ from Theorem 4.1.

We start estimating the volumes of K_N and K_N^0. For convenience we separate upper and lower estimates (some bounds require an additional condition on the matrix Γ). Corollary 4.3 and (14) imply the following volume estimates for K_N and K_N^0.

Theorem 4.10. *Let $Mn<N\le e^n$, $\beta\in(0,1)$. There exists absolute positive constants C and c such that with probability at least $1-\exp\left(-cn^{\beta}N^{1-\beta}\right)$ one has*

$$|K_N|^{1/n}\ge 2C(u,v,\beta)\sqrt{\frac{\ln(N/n)}{n}}\qquad\text{and}\qquad |K_N^0|^{1/n}\le\frac{C}{C(u,v,\beta)\sqrt{n\ln(N/n)}}.$$

To prove the remaining bounds on volumes of K_N and K_N^0 we introduce one more condition on the matrix Γ, namely we require that

$$\mathbb{P}\left(\max_{i\le N}|\Gamma^*e_i|>\lambda\sqrt{n}\right)\le p_0 \tag{15}$$

for some $0<p_0<1$ and $\lambda\ge1$. Such condition holds for example when entries of Γ are i.i.d. centered random variables with finite p-th moment for some $p>4$, provided that $N\le C_pn^{p/4}$ (this can be proved using Rosenthal's inequality, see Corollary 6.4 in [15]).

The lower bound on $|K_N|$ (and the upper bound on $|K_N^0|$) follows from (14) and a well known estimate on the volume of the convex hull of k points ([2], [6], [10]):

Let $2n\le k\le e^n$ and $z_1,\ldots,z_k\in S^{n-1}$, then

$$|\text{abs conv}\{z_i\}_{i\le k}|^{1/n}\le c\sqrt{\ln(k/n)}/n,$$

where $c>0$ is an absolute constant.

Theorem 4.11. *Let $Mn < N \le e^n$ and $\beta \in (0,1)$. Assume that the matrix Γ satisfies (15). There exist absolute positive constants c and C such that one has*

$$|K_N|^{1/n} \le C\lambda\sqrt{\frac{\ln(N/n)}{n}} \qquad \text{and} \qquad |K_N^0|^{1/n} \ge c/(\lambda\sqrt{n\ln(N/n)})$$

with probability at least $1 - p_0$.

An important geometric parameter associated to a convex body is the (half of) mean width of K^0 defined by

$$M_K = M(K) = \int_{S^{n-1}} \|x\|_K \, d\nu,$$

where ν is the normalized Lebesgue measure on S^{n-1}. It is well known that there exists a constant $c_n > 1$ ($c_n \to 1$ as $n \to \infty$) such that

$$M_K = \frac{c_n}{\sqrt{n}}\mathbb{E}\|\sum_{i=1}^n e_i g_i\|_K,$$

for every $K \subset \mathbb{R}^n$. The (half of) mean width of K, $M(K^0)$, we denote by $M_K^* = M^*(K)$. Observe that

$$M^*(K) = \frac{c_n}{\sqrt{n}}\mathbb{E}\|\sum_{i=1}^n e_i g_i\|_{K^0} = \frac{c_n}{\sqrt{n}}\mathbb{E}\sup_{t\in K}\sum_{i=1}^n t_i g_i = \frac{c_n}{\sqrt{n}}\ell_*(K),$$

where $\ell_*(K) = \mathbb{E}\sup_{t\in K}\sum_{i=1}^n t_i g_i$ is the Gaussian complexity measure of the convex body K. We recall the following inequality, which holds for every convex body K (see e.g. [24])

$$M_K^* \ge (|K|/|B_2^n|)^{1/n} \ge 1/M_K. \tag{16}$$

Now we calculate the mean widths $M(K_N)$ and $M(K_N^0)$.

Theorem 4.12. *Let $Mn < N \le e^n$ and $\beta \in (0,1)$. Then*

$$M(K_N) \le CC^{-1}(u,v,\beta)\left(\sqrt{(\ln(2n))/n} + 1/\sqrt{\ln(N/n)}\right)$$

with probability at least $1 - \exp\left(-cn^\beta N^{1-\beta}\right)$, where C and c are absolute positive constants. Moreover, if the matrix Γ satisfies (15), then there exists an absolute positive constant c_1 such that with probability at least $1 - p_0$ one has

$$M(K_N) \ge c_1/(\lambda\sqrt{\ln(N/n)}).$$

Proof. By Theorem 4.1 we have

$$\begin{aligned} M(K_N) &\le M\left(C(u,v,\beta)\left(B_\infty^n \cap \sqrt{\ln(N/n)}B_2^n\right)\right) \\ &\le (1/C(u,v,\beta))\left(M\left(B_\infty^n\right) + M\left(\sqrt{\ln(N/n)}B_2^n\right)\right), \end{aligned}$$

which proves the upper bound.

By (16) and Theorem 4.11 there exists an absolute positive constant c_1 such that

$$M(K_N) \ge (|B_2^n|/|K_N|)^{1/n} \ge c_1/(\lambda\sqrt{\ln(2N/n)}),$$

with probability larger than or equal to $1 - p_0$. This proves the lower bound. $\square$

Remark 4.13. Note that by Theorem 4.12, for $N \le \exp(n/\ln n)$ we have

$$M(K_N) \approx 1/\sqrt{\ln(N/n)}.$$

If $N \ge \exp(n/\ln(2n))$ there is a gap between lower and upper estimates. Both estimates could be asymptotically sharp as was shown in [18].

Theorem 4.14. *There exist positive absolute constants c, c_0, and C such that the following holds. Let $Mn < N \le e^n$. Then*

$$M(K_N^0) \ge c_0\sqrt{\ln(N/n)}$$

with probability at least $1 - \exp\left(-cn^\beta N^{1-\beta}\right)$. Moreover, assuming that the matrix Γ satisfies (15), with probability at least $1 - p_0$ one has

$$M(K_N^0) \le C\lambda\sqrt{\ln N}.$$

Proof. By (16) we have

$$M(K_N^0) \ge \left(|B_2^n|/|K_N^0|\right)^{1/n}.$$

Therefore, the lower bound follows by Theorem 4.10.

Let $G = \sum_{i=1}^n g_i e_i$. Recall that K_N is the absolute convex hull of N vertices $\Gamma^* e_i$. Thus we have

$$M(K_N^0) \le \frac{c_1}{\sqrt{n}}\mathbb{E}\|G\|_{K_N^0} = \frac{c_1}{\sqrt{n}}\mathbb{E}\max_{i\le N}\langle G, \Gamma^* e_i\rangle,$$

where c_1 is an absolute constant. Since with probability at least $1 - p_0$ we have $|\Gamma^* e_i| \le \lambda\sqrt{n}$ for every $i \le N$, using standard estimate for the expectation of maximum of Gaussian random variables (see, e.g., [24]), we obtain that there is an absolute constant c_2 such that

$$M(K_N^0) \le c_2\lambda\sqrt{\ln N}$$

with probability larger than or equal to $1 - e^{-n}$. □

Finally we note that the bounds of Theorem 4.14 are sharp, whenever $\ln N$ and $\ln(N/n)$ are comparable, for example if $N > n^2$. However, when N is close to n we have a gap between upper and lower bounds. Below we provide a better lower bound for $M(K_N^0)$ in the case $N \le n^2$, which closes this gap. We will need two more conditions on the matrix Γ, namely

$$\mathbb{P}\left(\|\Gamma\|_{HS} < \sqrt{Nn}/2\right) \le p_1, \tag{17}$$

for some $p_1 \in (0,1)$ and where $\|\Gamma\|_{HS}$ denotes the Hilbert–Schmidt norm of Γ; and

$$\mathbb{P}\left(\|\Gamma\| > \mu\sqrt{N}\right) \le p_2, \tag{18}$$

for some $p_2 \in (0,1)$, $\mu \ge 1$ and where $\|\Gamma\|$ denotes the operator (spectral) norm of Γ. Both conditions are satisfied for example when entries of Γ are i.i.d. centered random variables with finite p-th moment for some $p > 4$. Indeed, Rosenthal's inequality (see proof of Corollary 6.4 in [15]) implies (17) with $p_1 \le (C_p \mathbb{E}|\xi|^p)/(Nn)^{p/4}$; while Theorem 2.1 combined with Corollary 6.4 in [15] implies (18) with $\mu = C_p'$ and

$$p_2 \le 1/N^{c_p} + (C_p \mathbb{E}|\xi|^p) N / n^{p/4}$$

(to make $p_2 < 1$ we have to ask $C_p \mathbb{E}|\xi|^p N \le n^{p/4}$). We would like also to note that the proof below works also for $N \le n^{\alpha}$ for some $\alpha \in (1, 2]$ if we substitute the condition (18) with

$$\mathbb{P}\left(\|\Gamma\| > \mu (Nn)^{\gamma}\right) \le p_2$$

for some $\gamma \in (0, 1/2)$, which could be the case in the absence of 4-th moment (see for example Corollary 2 in [1] and Remark 2 in [20]). Note also that the condition (18) implies (15), since $\|\Gamma\| \ge \max_{i \le N} |\Gamma^* e_i|$.

Theorem 4.15. *Let $\mu \ge 1$, $n \ge 16\mu^2$, and $2n < N \le n^2$ and assume that the matrix Γ satisfies conditions (17) and (18) for some $p_1, p_2 \in (0, 1)$. Then with probability at least $1 - p_1 - p_2$*

$$M(K_N^0) \ge c\sqrt{\ln(n/(8\mu^2))}.$$

Proof. We apply Vershynin's extension [31] of Bourgain-Tzafriri theorem [5]. Denote $A = \|\Gamma^*\|_{HS}$, $B = \|\Gamma^*\|$. Vershynin's theorem implies that there exists $\sigma \subset \{1, \dots, N\}$ of cardinality at least $A^2/(2B^2)$ such that for all $i \in \sigma$ one has $|\Gamma^* e_i| \ge c_3 A/\sqrt{N}$, where c_3 is an absolute positive constant, and vectors $\Gamma^* e_i$, $i \in \sigma$, are almost orthogonal (up to an absolute positive constant). Since Γ satisfies conditions (17) and (18), with probability at least $1 - p_1 - p_2$ we have $A \ge \sqrt{Nn}/2$ and $B \le \mu\sqrt{N}$. Therefore, there exists $\sigma \subset \{1, \dots, n\}$ of cardinality at least $n/(8\mu^2)$ such that $|\Gamma^* e_i| \ge c_3\sqrt{n}/2$ for $i \in \sigma$ and $\{\Gamma^* e_i\}_{i \in \sigma}$ are almost orthogonal. Then,

$$M(K_N^0) \ge \frac{1}{\sqrt{n}} \mathbb{E}\|G\|_{K_N^0} = \frac{1}{\sqrt{n}} \mathbb{E} \max_{i \le N} \langle G, \Gamma^* e_i \rangle \ge \frac{1}{\sqrt{n}} \mathbb{E} \max_{i \in \sigma} \langle G, \Gamma^* e_i \rangle .$$

Since $\{\Gamma^* e_i\}_{i \in \sigma}$ are almost orthogonal, by Sudakov inequality (see, e.g., [24]), the last expectation is greater than $c_4\sqrt{\ln(n/(8\mu^2))}$, where c_4 is an absolute constant. This completes the proof. □

5 Smallest singular value

In this section we provide a simple short proof of a weaker inclusion, namely, we obtain a lower bound on the radius of the largest ball inscribed into K_N. It is based on a lower bound for the smallest singular value for tall matrices. Although such bounds are known with possibly better constants (see the last remark in [16] or the main theorem of [28]), we would like to emphasize a simple short proof, based on our Theorem 3.1. In fact our proof is close to the corresponding proofs in [18] and [19], however it is somewhat cleaner and it uses Theorem 3.1 instead of a standard net argument via the norm of an operator. We would also like to mention that very recently G. Livshyts has extended such results to rectangular random matrices with arbitrarily small aspect ratio [21].

Recall that for an $N \times n$ matrix Γ with $N \ge n$, its smallest singular value $s_n(\Gamma)$ can be defined by

$$s_n(\Gamma) = \inf_{x \in S^{n-1}} \|\Gamma x\|_2.$$

In this section we assume that the random matrix Γ satisfies conditions described at the beginning of Section 4 with fixed $u, v \in (0,1)$. Recall that $c_{uv} = cuv\sqrt{1-v}$ is the constant from Lemma 4.4. It will be also convenient to fix two more constants depending on v,

$$\gamma_1 = \gamma_1(v) := \begin{cases} \sqrt{\ln 2} & \text{if } v \ge 1/2, \\ \sqrt{\ln \frac{1}{v}} & \text{if } v < 1/2 \end{cases} \quad \text{and} \quad \gamma_2 = \gamma_2(v) := \begin{cases} \ln \frac{2}{1+v} & \text{if } v \ge 1/2, \\ \ln \frac{1}{2v - v^2} & \text{if } v < 1/2. \end{cases}$$

Theorem 5.1. *There exist an absolute constant $C_0 > 1$ such that for $N \ge \left(\frac{C_0}{\gamma_2} \ln \frac{1}{c_{uv}}\right) n$ one has*

$$\mathbb{P}\left(s_n(\Gamma) \le \frac{c_{uv}\sqrt{\gamma_2}}{4\gamma_1}\sqrt{N}\right) \le 3\exp\left(-\min\{2, \gamma_2\} N/8\right).$$

Since

$$h_{\Gamma^* B_1^N}(x) = \|\Gamma^* x\|_\infty \quad \text{and} \quad K_N = \Gamma^* B_1^N \supset \frac{1}{\sqrt{N}} \Gamma^* B_2^N,$$

this theorem immediately implies the following inclusion.

Corollary 5.2. *For $N \ge \left(\frac{C_0}{\gamma_2} \ln \frac{1}{c_{uv}}\right) n$ one has*

$$\mathbb{P}\left(K_N \supset \frac{c_{uv}\sqrt{\gamma_2}}{4\gamma_1}\sqrt{N} B_2^n\right) \ge 1 - 3\exp\left(-\min\{2, \gamma_2\} N/8\right).$$

To prove Theorem 5.1 we first provide the individual bounds.

Proposition 5.3. *Let $1 \le n < N$. Then for every $x \in S^{n-1}$ one has*

$$\mathbb{P}\Big(\|\Gamma x\|_2 \le \frac{c_{uv}\sqrt{\gamma_2}}{2\gamma_1}\sqrt{N}\Big) \le \exp\left(-3\gamma_2 N/4\right).$$

Proof. Fix $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $\|x\|_2 = 1$. Denote $f_j := |\sum_{i=1}^n \xi_{ji} x_i|$, so that

$$\|\Gamma x\|_2^2 = \sum_{j=1}^N f_j^2.$$

Clearly $f_1, \ldots, f_N$ are independent. Therefore, for any $t, \tau > 0$ one has

$$\mathbb{P}\big(\|\Gamma x\|_2^2 \le t^2 N\big) = \mathbb{P}\Big(\sum_{j=1}^N f_j^2 \le t^2 N\Big) = \mathbb{P}\Big(\tau N - \frac{\tau}{t^2}\sum_{j=1}^N f_j^2 \ge 0\Big)$$
$$\le \mathbb{E}\exp\Big(\tau N - \frac{\tau}{t^2}\sum_{j=1}^N f_j^2\Big) = e^{\tau N}\prod_{j=1}^N \mathbb{E}\exp\Big(-\frac{\tau f_j^2}{t^2}\Big).$$

Lemma 4.4 implies that $\mathbb{P}(f_j < c_{uv}) \le v$ for every $j \le N$. Write $\tau = t^2\eta/c_{uv}^2$ for some $\eta > 0$. Then

$$\begin{aligned}\mathbb{E}\exp\Big(-\frac{\tau f_j^2}{t^2}\Big) &= \int_0^1 \mathbb{P}\Big(\exp\Big(-\frac{\eta f_j^2}{c_{uv}^2}\Big) > s\Big)\, ds \\ &= \int_0^{e^{-\eta}} \mathbb{P}\Big(\exp\Big(\frac{\eta f_j^2}{c_{uv}^2}\Big) < \frac{1}{s}\Big)\, ds + \int_{e^{-\eta}}^1 \mathbb{P}\Big(\exp\Big(\frac{\eta f_j^2}{c_{uv}^2}\Big) < \frac{1}{s}\Big)\, ds \\ &\le e^{-\eta} + \mathbb{P}(f_j < c_{uv})(1 - e^{-\eta}) \le e^{-\eta} + v(1 - e^{-\eta}).\end{aligned}$$

Choose $\eta = \gamma_1^2 = \ln \max\{2, 1/v\}$. Then the right hand side is $e^{-\gamma_2}$. Therefore

$$\mathbb{P}\big(\|\Gamma x\|_2^2 \le t^2 N\big) \le e^{\tau N} e^{-\gamma_2 N} = \exp\left(-N(\gamma_2 - t^2\gamma_1^2/c_{uv}^2)\right).$$

Choosing $t = \sqrt{\gamma_2}c_{uv}/(2\gamma_1)$ we complete the proof. □

Proof of Theorem 5.1. Let $\delta = \min\{1, \gamma_2/2\}$. Note that $n/(2N) \le \delta \le 1$. Let $C \ge 1$ be the absolute constant from Theorem 3.1. Set

$$\varepsilon := \frac{c_{uv}\sqrt{\gamma_2\delta}}{4C\gamma_1\sqrt{n}} < \frac{1}{\sqrt{n}}.$$

By Theorem 3.1 (see Remark 3.8), applied with $T = S^{n-1}$ and $k = N$, there exists a net $\mathcal{N} \subset B_2^n$ with cardinality at most

$$\left(\frac{224\delta N}{\varepsilon n^{3/2}}\right)^n e^{\delta N} \le \left(\frac{896C\gamma_1\sqrt{\delta}N}{c_{uv}\sqrt{\gamma_2}n}\right)^n e^{\delta N}$$

such that with probability at least $1 - e^{-\delta N/4} - e^{-N}$ one has

$$\forall x \in B_2^n\ \exists y_x \in \mathcal{N} \quad \text{such that} \quad \Gamma(x - y_x) \in C\varepsilon\sqrt{Nn/\delta}\, B_2^n = (c_{uv}\sqrt{\gamma_2}/(4\gamma_1))\sqrt{N}\, B_2^n.$$

Condition on the corresponding event, denoted below by Ω_0. Assume that $x \in S^{n-1}$ satisfies $\|\Gamma x\|_2 \le (c_{uv}\sqrt{\gamma_2}/(4\gamma_1))\sqrt{N}$. Then for the corresponding $y_x \in \mathcal{N}$ we have

$$\|\Gamma y_x\|_2 \le \|\Gamma x\|_2 + \|\Gamma(y_x - x)\|_2 \le (c_{uv}\sqrt{\gamma_2}/(2\gamma_1))\sqrt{N}.$$

This implies

$$q_0 := \mathbb{P}\left(\exists x \in S^{n-1} \mid \|\Gamma x\|_2 \le \frac{c_{uv}\sqrt{\gamma_2}}{4\gamma_1}\sqrt{N}\right) \le \mathbb{P}(\Omega_0^c) + \mathbb{P}\left(\exists y \in \mathcal{N} \mid \|\Gamma y\|_2 \le \frac{c_{uv}\sqrt{\gamma_2}}{2\gamma_1}\sqrt{N}\right).$$

Applying Proposition 5.3 and using $\delta \le \gamma_2/2$,

$$q_0 \le 2e^{-\delta N/4} + \left(\frac{896C\gamma_1\sqrt{\delta}N}{c_{uv}\sqrt{\gamma_2}n}\right)^n \exp\left(-\gamma_2 N/4\right).$$

Using formulas for c_{uv}, γ_1, γ_2, and δ, it is not difficult to check that there exists an absolute constant $C_1 > 0$ such that

$$\ln\frac{896C\gamma_1\sqrt{\delta}}{c_{uv}\sqrt{\gamma_2}} \le C_1 \ln\frac{1}{c_{uv}}.$$

Therefore there exists another absolute constant $C_2 > 0$ such that

$$\left(\frac{896C\gamma_1\sqrt{\delta}N}{c_{uv}\sqrt{\gamma_2}n}\right)^n \exp\left(-\gamma_2 N/4\right) \leq \exp\left(-\gamma_2 N/8\right),$$

provided that

$$N/n \geq (C_2/\gamma_2)\ \ln(1/c_{uv}).$$

This completes the proof. □

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Olivier Guédon
Université Paris-Est Marne-La-Vallée
Laboratoire d'Analyse et de Mathématiques Appliquées,
5, boulevard Descartes, Champs sur Marne,
77454 Marne-la-Vallée, Cedex 2, France
`e-mail: olivier.guedon@univ-mlv.fr`

A. E. Litvak and K. Tatarko,
Dept. of Math. and Stat. Sciences,
University of Alberta,
Edmonton, AB, Canada, T6G 2G1.
`e-mails: aelitvak@gmail.com and ktatarko@gmail.com`