JOHN'S DECOMPOSITION IN THE GENERAL CASE AND APPLICATIONS

Y. Gordon, A. E. Litvak, M. Meyer & A. Pajor

Abstract

We give a description of an affine mapping T involving contact pairs of two general convex bodies K and L, when $T(K)$ is in *a position of maximal volume in* L. This extends the classical John's theorem of 1948, and is applied to the solution of a problem of Grünbaum; namely, any two convex bodies K and L in $\mathbb{R}^n$ have non-degenerate affine images K' and L' such that $K' \subset L' \subset -nK'$. As a corollary, we obtain that if L has a center of symmetry, then there are non-degenerate affine images K'' and L'' of K and L such that $K'' \subset L'' \subset nK''$. Other applications to volume ratios and distance estimates are given. In particular, the Banach-Mazur distance between the n-dimensional simplex and any centrally symmetric convex body is equal to n.

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1. Introduction

The ellipsoid of maximal volume inside a convex body was characterized in term of contact points by F. John in [**J**], using an optimization theorem (Theorem 3.3 below). This ellipsoid is commonly called the John ellipsoid whereas the ellipsoid of minimal volume containing a convex body is called Loewner ellipsoid. These ellipsoids sometimes called John-Loewner ellipsoids play a central role in the study of distances between convex bodies, see [**T**] for many applications. We refer to [**Gru**] for an extensive survey on this subject. When the John ellipsoid is the unit Euclidean ball, then as shown by John ([**J**]), the identity may be written as a positive combination of rank one projections $x_i \otimes x_i$, where the x_i are some contact points between the two bodies. Such

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decomposition led to many important results in the asymptotic theory of finite-dimensional normed spaces (see e.g. [**B2**], [**Pi**], [**T**]). It was noticed by Lewis (Theorem 1.3 in [**L**]) and Milman (Theorem 14.5 in [**T**]) that such a result also holds for arbitrary centrally-symmetric convex bodies. In [**Gi-P-T**], the authors proved a version of Theorem 3.5 below, for two smooth enough convex bodies and of Theorem 3.8, when one body is a polytope and the second has a smooth boundary with positive curvature. Another version of Theorem 3.5, removing smoothness conditions but with assumptions of connectedness was proved in [**B-R**], with applications to quasi-convex bodies.

Let A be a subset of a finite dimensional affine space E and B a subset of affine forms on E. We study the problem of maximizing the volume of $T(\text{conv}(A))$, among all affine tranformations such that $T(A) \subset P(B)$ where $P(B) = \{x \in E \mid f(x) \leq 0, \text{ for all } f \in B\}$. We get a description of an operator T involving contact pairs of two general convex bodies K and L, when $T(K)$ is in *a position of maximal volume in* L (Theorems 3.5 and 3.8). Using methods of optimization, which can be applied to other convexity results, we give a general John's theorem for arbitrary convex bodies. One nice and new feature of Theorems 3.1, and 3.8 is that they are self dual.

We apply these results to show that any two convex bodies K and L in $\mathbb{R}^n$ have positions, i.e. non-degenerate affine images K' and L', such that $K' \subset L' \subset -nK'$ (Theorem 5.1). This gives a positive answer to an old problem raised by B. Grünbaum ([**Gr**]). As a corollary, we obtain that the Banach-Mazur distance between an arbitrary convex body and an arbitrary centrally-symmetric body is bounded by n (Theorem 5.5), improving the previous result of Lassak to the best possible one. It also follows that the Banach-Mazur distance between a non-degenerate simplex in $\mathbb{R}^n$ and any centrally-symmetric convex body is exactly n (Corollary 5.8). More precisely, if K and L are convex bodies in $\mathbb{R}^n$, and L has a center of symmetry, there exist affine images K'' and L'' and $b \in \mathbb{R}^n$, such that $K'' - b \subset L'' - b \subset n(K'' - b)$. Note that generally and unexpectedly, b is not the center of symmetry of L''; as the example of the triangle and the square in the plane shows. This means that the simplex is, in some sense (see Remark 5.9), the center of the set of centrally symmetric convex bodies in the space of convex bodies equipped with the Banach-Mazur distance. Other applications to volume ratio estimates are given.

2. Definition and notation

We use the following standard notation. The space $\mathbb{R}^n$ is equipped with the canonical Euclidean scalar product $\langle \cdot, \cdot \rangle$. We denote by I_n the identity mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. The space of linear mappings $\mathcal{L}(\mathbb{R}^n)$ is equipped with the corresponding scalar product defined by $\langle S, T\rangle = \text{trace}(S^*T)$, for every $S, T \in \mathcal{L}(\mathbb{R}^n)$.

Let $K \subset \mathbb{R}^n$ be a compact convex body with non-empty interior (below we consider only such convex bodies) such that $0 \in K$. We denote by $|K|$ the volume of K, and by K° the polar of K, i.e.

$$K^\circ = \{x \mid \langle x, y\rangle \le 1 \ \text{ for every } \ y \in K\}.$$

Let $x, y \in \mathbb{R}^n$, we denote by $y \otimes x$ the rank one projection defined by $y \otimes x\,(t) = \langle y, t\rangle\, x$ for all $t \in \mathbb{R}^n$.

If $A,\ B \subset \mathbb{R}^n$, let $A + B$ be their *Minkowski sum*, i.e.

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

The convex hull of a set $A \subset \mathbb{R}^n$ is denoted by $\text{conv}(A)$. If $z \in \mathbb{R}^n$, let

$$K_z = K - z.$$

By K_z° we denote the polar of K_z, i.e. $K_z^\circ = (K_z)^\circ$. Finally, let $\text{int}\, K$ denote the interior of K, ∂K its boundary, $\text{Ext}\, K$ the set of its extreme points and $\overline{\text{Ext}}\, K$ the closure of $\text{Ext}\, K$.

We say that a convex body K' is *a position of* K if $K' = TK + a$, for some non-degenerate linear mapping $T \in GL_n$ and some $a \in \mathbb{R}^n$.

Given two convex bodies K and L in $\mathbb{R}^n$, we denote the *volume ratio of the pair* (L, K) to be

$$vr(L, K) = \inf \left(\frac{|L|}{|K'|}\right)^{1/n},$$

where the infimum is taken over all positions K' of K such that $K' \subset L$. We define the *geometric distance between* K *and* L by

$$\widetilde{d}(K, L) := \inf\{\alpha\,\beta \mid \alpha > 0, \beta > 0, (1/\beta)L \subset K \subset \alpha L\}.$$

The *Banach-Mazur distance* is defined by

$$\begin{aligned} d(K, L) &= \inf\left\{\widetilde{d}(uK_z, L_x)\right\} \\ &= \inf\{\alpha\,\beta \mid \alpha > 0, \beta > 0, (1/\beta)L_x \subset uK_z \subset \alpha L_x\}, \end{aligned}$$

where the infimum is taken over all $z,\ x \in \mathbb{R}^n$ and all $u \in GL_n$. For a convex body K in $\mathbb{R}^n$, we define *the asymmetry constant* δ_K by

$$\delta_K := \inf\{d(K, B) \mid B = -B \text{ is a convex body in } \mathbb{R}^n\}.$$

This constant is one of the possible ways to measure the asymmetry of a given body. We refer to [**Gr**] for a detailed discussion on various measures of asymmetry (see also [**Gl-L-T**], [**Gl-L**] for related results). We shall also use the following weaker version of the Banach-Mazur distance suggested in Grünbaum's paper ([**Gr**])

$$\bar{d}(K,L) = \inf\{|\alpha\,\beta| \ \mid \alpha,\beta \in \mathbb{R},\ (1/\beta)L_x \subset uK_z \subset \alpha L_x\},$$

where the infimum is taken over all $z,\ x \in \mathbb{R}^n$ and all $u \in GL_n$. In other words this definition allows to multiply the body by -1. For related problems concerning uniqueness of position of maximal volume for symmetric bodies see [**Go-R**].

We say that K is in *a position of maximal volume in* L if $K \subset L$ and for any position K' of K such that $K' \subset L$ one has $|K'| \leq |K|$. Moreover, we say that K is in *a position of maximal volume in* L *with respect to* $a \in \mathbb{R}^n$ if $a \in K \cap L$ and for every $T \in GL_n$ such that $TK_a \subset L_a$, one has $|\det T| \leq 1$ (here shifts are not allowed).

3. John's decomposition

Let A be a subset of an n-dimensional affine space E and B a subset of affine forms on E. We study the problem of maximizing the volume of $T(\mathrm{conv}(A))$, among all affine tranformations such that $T(A) \subset P(B)$, where $P(B) = \{x \in E \mid f(x) \leq 0, \text{ for all } f \in B\}$. After choosing an origin and a basis, we may consider the problem in $\mathbb{R}^n$. Also, if $P(B)$ has a non-empty interior, after a translation, we may assume that $0 \in \mathrm{int}\, P(B)$. In this position, $P(B)$ is the polar of a body. This representation of $P(B)$ depends on the underline Euclidean structure. The problem of maximizing the volume of $T(\mathrm{conv}(A))$, among all affine tranformations such that $T(A) \subset P(B)$, is an affine invariant problem, but the representation of $P(B)$ as the polar of a body, depends in a non-linear way on the Euclidean structure, especially after translation. We choose to state the results in the Euclidean space $\mathbb{R}^n$: Theorems 3.1 and 3.5 have an intrinsic affine version given by Theorem 3.7, whereas Theorem 3.8 is of Euclidean nature.

Let C_1 be a compact subset of vectors in $\mathbb{R}^n$ and C_2 be a compact subset of linear forms on the Euclidean space $\mathbb{R}^n$, identified as a subset of $\mathbb{R}^n$. Assume that $C_1 \subset C_2^\circ$.

We call (x, y) a *contact pair* of (C_1, C_2), if it satisfies

(i) $\langle x, y\rangle = 1$,
(ii) $x \in C_1 \cap \partial C_2^\circ$,
(iii) $y \in C_2 \cap \partial C_1^\circ$.

Let C_1 and C_2 be two compact subsets of $\mathbb{R}^n$ such that $\operatorname{conv}(C_1)$ has a non-empty interior (or equivalently the linear span of C_1 is $\mathbb{R}^n$) and such that $0 \in \operatorname{int}\operatorname{conv}(C_2)$. It follows that C_2° is bounded, and since $\operatorname{conv}(C_1)$ has a non-empty interior in $\mathbb{R}^n$, the function det(T) is bounded on the set of $(T, a) \in GL_n \times \mathbb{R}^n$, such that $T(\operatorname{conv}(C_1)) + a \subset C_2^\circ$. It follows that there exists a position of $\operatorname{conv}(C_1)$ which is of maximal volume inside C_2°. The following result is an extension of the classical John's theorem ([**J**]) on the maximal volume ellipsoid.

Theorem 3.1. *Let C_1 and C_2 be two compact subsets of $\mathbb{R}^n$ such that* $\operatorname{conv}(C_1)$ *has a non-empty interior and* $0 \in \operatorname{int}\operatorname{conv}(C_2)$. *If* $\operatorname{conv}(C_1)$ *is in a position of maximal volume in C_2°, then there exist $m \le n^2 + n$ contact pairs $(x_i, y_i)_{1 \le i \le m}$ of (C_1, C_2), and $c_1, \dots, c_m > 0$ such that*
(i) $I_n = \sum_{i=1}^m c_i\, x_i \otimes y_i$,
(ii) $\sum_{i=1}^m c_i y_i = 0$.

Remark 3.2. a) Theorem 3.1 was proved in [**B-R**] with the assumption that C_1 is connected.
b) Note that in this theorem, we have

$$\operatorname{trace}(I_n) = n = \sum_{i=1}^m c_i$$

because $\langle x_i, y_i \rangle = 1$ for $1 \le i \le m$.

We postpone to the next section another proof of Theorem 3.1. The proof given below uses the original optimization result of F. John:

Theorem 3.3. [**J**] *Let $F : \mathbb{R}^N \mapsto \mathbb{R}$ be a C^1-function. Let S be a compact metric space and $G : \mathbb{R}^N \times S \mapsto \mathbb{R}$ be continuous. Suppose that for every $s \in S$, $\nabla_z G(z, s)$ exists and is continuous on $\mathbb{R}^N \times S$.*

Let $A = \{z \in \mathbb{R}^N \mid G(z, s) \ge 0$ for all $s \in S\}$ and $z_0 \in A$ satisfy

$$F(z_0) = \min_{z \in A} F(z).$$

Then, either $\nabla_z F(z_0) = 0$, or, for some $1 \le m \le N$, there exist $s_1, \cdots, s_m \in S$ and $\lambda_1, \cdots, \lambda_m \in \mathbb{R}$ such that $G(z_0, s_i) = 0$, $\lambda_i \ge 0$ for $1 \le i \le m$, and

$$\nabla_z F(z_0) = \sum_{i=1}^m \lambda_i \nabla_z G(z_0, s_i).$$

Proof of Theorem 3.1: Let $N = n^2 + n$, $\mathbb{R}^N = \mathbb{R}^{n^2} \times \mathbb{R}^n$, and $F : \mathbb{R}^N \mapsto \mathbb{R}$ be defined by

$$F(T, a) = -\det T,$$

where $a \in \mathbb{R}^n$ and $T \in \mathbb{R}^{n^2}$ is viewed as the linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. Clearly F is C^1. We define $S = C_1 \times C_2$, which is compact. Let $G : \mathbb{R}^N \times S \mapsto \mathbb{R}$ be defined by

$$G((T,a),(x,y)) = 1 - \langle a + Tx, y \rangle .$$

Define the set A as in Theorem 3.3. Observe that $(T,a) \in A$ if and only if $a + TC_1 \subset C_2^\circ$ or equivalently if and only if $a + T(\text{conv}(C_1)) \subset C_2^\circ$. Now, if $\text{conv}(C_1)$ is in a position of maximal volume in C_2°, then F attains its minimum on A at $(I_n, 0)$.

It is easy to see that for non-degenerate T one has

$$\begin{aligned} \nabla_{(T,a)}F &= (-(\det T)\cdot (T^{-1})^*, 0), \\ \nabla_{(T,a)}G(\,\cdot\,,(x,y)) &= (-\nabla_T \langle Tx, y\rangle, -\nabla_a \langle a, y\rangle) = -(x \otimes y, y). \end{aligned}$$

Thus, since F attains its minimum on A at $z_0 = (I_n, 0)$, by Theorem 3.3, we obtain that for some $m \le N$, there exist $\lambda_i \ge 0$, $s_i \in S$, $s_i = (x_i, y_i)$, $1 \le i \le m$, such that

$$\langle x_i, y_i \rangle = 1 - G((I_n,0),(x_i,y_i)) = 1, \ 1 \le i \le m$$

and

$$\begin{aligned} \nabla_{(T,a)}F(I_n,0) = (-I_n, 0) &= \sum_{i=1}^m \lambda_i \nabla_{(T,a)} G((I_n,0),(x_i,y_i)) \\ &= -\sum_{i=1}^m \lambda_i (x_i \otimes y_i, y_i). \end{aligned}$$

Since $\langle x_i, y_i \rangle = 1$, $x_i \in C_1 \subset C_2^\circ$, $y_i \in C_2 \subset C_1^\circ$, we obtain $x_i \in \partial C_2^\circ$ and $y_i \in \partial C_1^\circ$. Taking trace in the last equality above, we get

$$n = \sum_{i=1}^m \lambda_i .$$

Therefore, one has

$$\lambda_i \ge 0, \ 1 \le i \le m, \ \sum_{i=1}^m \lambda_i y_i = 0 \text{ and } I_n = \sum_{i=1}^m \lambda_i \, x_i \otimes y_i .$$

By duality, one has also $I_n = \sum_{i=1}^m \lambda_i \, y_i \otimes x_i$. To conclude, we get the c_i by choosing the non-zero λ_i. q.e.d.

Remark 3.4. In order to state a necessary condition of order 2 for maximizing the determinant in the proof of Theorem 3.1, let

$$\mathcal{C} := \{H \in GL_n \mid \exists t > 0, (I + tH)(C_1) \subset C_2^\circ\}$$

and

$$\mathcal{C}_0 = \{H \in \mathcal{C} \mid \text{trace } H = 0\}.$$

Then, it is easy to see that under the hypothesis of Theorem 3.1, the following condition:

$$\forall H \in \mathcal{C}_0 \ , \ \text{trace } (H^2) \geq 0$$

is necessary for $\text{conv}(C_1)$ to be in a position of maximal volume in C_2°.

Theorem 3.5. *Let K and L be two convex bodies (with non-empty interiors) in $\mathbb{R}^n$, such that K is in a position of maximal volume in L, and $0 \in \operatorname{int} L$. Then there exist $m \leq n^2 + n$ contact pairs $(x_i, y_i)_{1 \leq i \leq m}$ of $(\overline{\operatorname{Ext}} K, \overline{\operatorname{Ext}} L^\circ)$, and $c_1, \dots, c_m > 0$ such that*

(i) $I_n = \sum_{i=1}^m c_i \, x_i \otimes y_i$,
(ii) $\sum_{i=1}^m c_i y_i = 0$.

Proof : We apply Theorem 3.1 with $C_1 = \overline{\operatorname{Ext}} K$ and $C_2 = \overline{\operatorname{Ext}} L^\circ$.

Remark 3.6. Theorem 3.5 was proved in [**Gi-P-T**] with some smoothness assumption on the boundaries of K and L. In the case of centrally symmetric bodies, it was proved by D. Lewis (Theorem 1.3 in [**L**]) and V. Milman (Theorem 14.5 in [**T**]). If K is in a position of maximal volume in L with respect to 0 and $0 \in \operatorname{int} L$, then as it can be seen from the proof of Theorem 3.1, Theorem 3.5 holds with $1 \leq m \leq n^2$, but without formula (*ii*). As observed in Remark 5 in [**B-R**], the hypothesis of convexity on L is essential to get a decomposition of the identity.

Theorems 3.1 and 3.5 have an intrinsic geometric formulation.

Theorem 3.7. *Let A be a compact subset of a finite dimensional affine space E and B a compact set of affine forms on E. Assume that the identity maximizes the volume of $T(\text{conv}(A))$ among all affine tranformations T such that $T(A) \subset P(B)$ where*

$$P(B) = \{M \in E \mid f(M) \leq 0, \textit{for all } f \in B\}.$$

Then, there exist pairs $(M_i, f_i)_{1 \leq i \leq m} \in A \times B$ and positive real numbers $(c_i)_{1 \leq i \leq m}$ satisfying:

(i) $f_i(M_i) = 0$ *for all* $1 \leq i \leq m$, *i.e.* (M_i) *are a contact points between* A *and* $P(B)$,
(ii) $\sum_{1 \leq i \leq m} c_i f_i$ *is constant on* E,
(iii) $f = \sum_{i=1}^m c_i \, f(M_i) \, f_i$ *for every affine form* f *on* E.

When the space is equipped with an Euclidean structure, a change of origin modifies the representation of $P(B)$, as a polar body, in a non-linear way. The following result takes into account, the new polar representation of $P(B)$ after a translation. It depends on the underline Euclidean structure.

Theorem 3.8. *Let K and L be two convex bodies (with non-empty interiors) in $\mathbb{R}^n$ such that K is in a position of maximal volume in L, and $0 \in \operatorname{int} L$. Then there exist $z \in \operatorname{int}(K)$ and $m \leq n^2 + n$ contact pairs $(u_i, v_i)_{1 \leq i \leq m}$ of $(\overline{\operatorname{Ext}}\, K_z, \overline{\operatorname{Ext}}\, L_z^\circ)$, and $a_1, , \dots, a_m > 0$ such that*

(i) $I_n = \sum_{i=1}^m a_i\, u_i \otimes v_i$,
(ii) $\sum_{i=1}^m a_i u_i = \sum_{i=1}^m a_i v_i = 0$.

Proof: Without lost of generality, we assume that the origin is in the interior of K. Let $(x_i, y_i)_{1 \leq i \leq m}$ be $m \leq n^2 + n$ contact pairs of $(\overline{\operatorname{Ext}}\, K, \overline{\operatorname{Ext}}\, L^\circ)$, and $c_1, \dots, c_m > 0$ satisfy (i)–(ii) of Theorem 3.5. Let

$$z = \frac{1}{n+1} \sum_{i=1}^m c_i x_i.$$

Since $\sum_{i=1}^m c_i = n$ (see Remark 3.2), one has

$$z \in \frac{n}{n+1} K \subset \frac{n}{n+1} L.$$

Recalling that $0 \in \operatorname{int} K$, we get that

$$z \in \operatorname{int} K \text{ and } \langle y_i, z \rangle \leq \frac{n}{n+1} < 1 \text{ for } 1 \leq i \leq m.$$

Define

$$u_i = x_i - z \text{ and } v_i = \gamma_i y_i, \text{ where } \gamma_i = (1 - \langle y_i, z \rangle)^{-1}.$$

Clearly, one has $u_i \in \overline{\operatorname{Ext}}\, K_z \cap \partial L_z$.
Since $z \in \operatorname{int} L$, we define a mapping $\Phi : L^\circ \longrightarrow \mathbb{R}^n$, by

$$\Phi(y) = \frac{y}{1 - \langle y, z \rangle}.$$

It is easy to check that Φ is a one-to-one mapping from L° onto L_z°, with $\Phi^{-1}(y') = y'/(1 + \langle y', z \rangle)$. Moreover, for every $u, v \in L^\circ$, $u \neq v$, one can show that

$$\Phi(]u, v[) =]\Phi(u), \Phi(v)[,$$

where $]u, v[$ denotes the interval $\{(1-\theta)u + \theta v \mid \theta \in (0,1)\}$. This implies that $\Phi(\overline{\operatorname{Ext}}\, L^\circ) = \overline{\operatorname{Ext}}\, L_z^\circ$. Therefore $v_i = \Phi(y_i) \in \overline{\operatorname{Ext}}\, L_z^\circ$ for every $1 \leq i \leq m$. We have also

$$\langle u_i, v_i \rangle = \langle x_i - z, \gamma_i y_i \rangle = \gamma_i(\langle x_i, y_i \rangle - \langle z, y_i \rangle) = \gamma_i(1 - \langle z, y_i \rangle) = 1.$$

This relation and the fact that $u_i \in K_z$ and $v_i \in L_z^\circ \subset K_z^\circ$, imply that $v_i \in \partial K_z^\circ$. It follows that $(u_i, v_i)_{1 \leq i \leq m}$ are contact pairs of $(\overline{\operatorname{Ext}}\, K_z, \overline{\operatorname{Ext}}\, L_z^\circ)$.

Let now $a_i = c_i/\gamma_i = c_i(1 - \langle z, y_i\rangle) > 0$, $1 \le i \le m$. It follows from *(i)-(ii)* of Theorem 3.5, that for every $x \in \mathbb{R}^n$, we have

$$\left(\sum_{i=1}^{m} a_i u_i \otimes v_i\right) x = \sum_{i=1}^{m} a_i \langle u_i, x\rangle v_i = \sum_{i=1}^{m} a_i \gamma_i \langle x_i - z, x\rangle y_i$$
$$= \sum_{i=1}^{m} c_i \langle x_i, x\rangle y_i - \sum_{i=1}^{m} c_i y_i \langle z, x\rangle = x.$$

This implies that $I_n = \sum_{i=1}^{m} a_i\, u_i \otimes v_i$, and taking the dual operator that $I_n = \sum_{i=1}^{m} a_i\, v_i \otimes u_i$.

Finally, one has

$$\sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} c_i y_i = 0$$

and since $\sum_{i=1}^{m} c_i = n$ (see Remark 3.2), one has

$$\begin{aligned}
\sum_{i=1}^{m} a_i u_i &= \sum_{i=1}^{m} c_i (1 - \langle z, y_i\rangle)(x_i - z) \\
&= \sum_{i=1}^{m} c_i x_i - \left(\sum_{i=1}^{m} c_i\right) z - \sum_{i=1}^{m} c_i \langle z, y_i\rangle x_i + \langle z, \sum_{i=1}^{m} c_i y_i\rangle z \\
&= (n+1)z - nz - z = 0.
\end{aligned}$$

This concludes the proof. q.e.d.

Remark 3.9. Theorem 3.8 was proved in [**Gi-P-T**] (Theorem 3.1) using Brouwer fixed point theorem, under the assumption that L is a polytope and K has a C^2 boundary with positive curvature.

Remark 3.10. Our proof gives that z can be chosen in $\frac{n}{n+1}K$. Note that if K is in a position of maximal volume in L, then for every $w \in \mathbb{R}^n$, K_w is in a position of maximal volume in L_w. Thus, Theorem 3.5 holds for every shift of K and L. So, first we can choose some "good" center w inside K and then z will be inside $\frac{n}{n+1}K_w$.

Remark 3.11. Observe that Theorem 3.8 is self dual in the sense that $(u_i, v_i)_{1\le i\le m}$ are contact pairs of $(\overline{\operatorname{Ext}}\, K_z, \overline{\operatorname{Ext}}\, L_z^\circ)$. This is because if $0 \in \operatorname{int} L$ and if K_z is in a position of maximal volume in L_z, then L_z° is in a position of maximal volume in K_z° with respect to z. A similar remark is valid for Theorem 3.1.

Observe that in Theorems 3.5 and 3.8, when $(u_i, v_i)_{1\le i\le m}$ are contact pairs of $(\overline{\operatorname{Ext}}\, K_z, \overline{\operatorname{Ext}}\, L_z^\circ)$, the (u_i) need not to be distinct, and similarly the (v_i). The case when K is a simplex of maximal volume in a cube L shows that to get a decomposition of the identity as in Theorem 3.8, repetitions are needed. But, if for instance K is strictly convex (∂K does not contain a segment) and is in a position of maximal volume in L, then the (u_i) may be chosen to be distinct extreme points of K. Similarly, if L is smooth (every point of ∂L has a unique supporting hyperplane), and K is in a position of maximal volume in L, then the (v_i) may be chosen to be distinct extreme points of L°.

Remark 3.12. As it was noticed by many authors, if K is in a position of maximal volume in L and both K and L have a center of symmetry, say $K_a = -K_a$, $L_b = -L_b$ for some $a, b \in \mathbb{R}^n$, then K_a is in a position of maximal volume in L_b with respect to the origin. Indeed, one has

$$K_a \subset L_a = L_b + b - a$$

and

$$K_a = -K_a \subset -L_a = -(L_b + b - a) = L_b + a - b,$$

which implies

$$K_a = (K_a + K_a)/2 \subset (L_b + b - a + L_b + a - b)/2 = L_b.$$

In particular, if $K = -K$ and $L = -L$, it is enough to consider the positions of maximal volume with respect to the origin.

4. John's decomposition: particular cases

In this section we first give a general optimization result. This allows to get yet another proof of Theorem 3.1 and characterizations of maximum volume positions in some particular cases. We will use the following easy lemma, the proof of which is included for completeness.

Lemma 4.1. *Let A be a compact Hausdorff space and $f, f_0 : A \to \mathbb{R}$ be upper semi-continuous functions. Suppose that $f_0 \le 1$ and that $f(x) \le 0$ for every $x \in A$ such that $f_0(x) = 1$. Then for every $\varepsilon > 0$ there exists M_ε such that $f - \varepsilon f_0 \le M_\varepsilon(1 - f_0)$.*

Proof. Let $\varepsilon > 0$. By the upper-continuity of f_0 and the compactness of A, there exists $0 < \eta \le 1$ such that $f(x) \le \varepsilon$ whenever $x \in A$ satisfies $f_0(x) \ge 1 - \eta$. Let $\beta = \sup_{x \in A} f(x)$. Then, for $x \in A$,

- if $f_0(x) \ge 1 - \eta$, and $M \ge \varepsilon$, one has

$$f(x) + (M - \varepsilon) f_0(x) \le \varepsilon + (M - \varepsilon) \le M.$$

- if $f_0(x) \le 1 - \eta$, and $M \ge \frac{\beta}{\eta}$, one has

$$f(x) + (M - \varepsilon) f_0(x) \le f(x) + (M - \varepsilon)(1 - \eta) \le \beta + M - M\eta - \varepsilon(1 - \eta) \le M.$$

We obtain the desired result with $M_\varepsilon = \max\{\varepsilon, \beta/\eta\}$. q.e.d.

In this section, we use the following notation: given a normed space X, its dual X^*, and $A \subset X$, $B \subset X^*$, we denote

$$A^\circ = \{y \in X^* \mid y(x) \le 1 \ \text{ for every } \ x \in A\},$$

$$B^\circ = \{x \in X \mid y(x) \le 1 \ \text{ for every } \ y \in B\}.$$

Theorem 4.2. *Let U be a non-empty open subset of a normed space X, X^* be the dual of X, A be a $\sigma(X^*, X)$-compact subset of X^*, and $F : X \mapsto \mathbb{R}$ be a Gateaux-differentiable function. Let A° be the polar of A in X, and for $x_0 \in A^\circ \cap U$, let $B(x_0) = \operatorname{conv}\{x' \in A \mid x'(x_0) = 1\}$. The following assertions hold:*

1. *If x_0 is a local maximum of F on the set $A^\circ \cap U$, then either $dF(x_0) = 0$ or there exists $\lambda > 0$ such that $dF(x_0) \in \lambda\overline{B(x_0)}$.*
2. *If conversely either $dF(x_0) = 0$ or there exists $\lambda > 0$ such that $dF(x_0) \in \lambda\overline{B(x_0)}$, and if moreover U is convex and F is concave on U, then x_0 is a global maximum of F on $A^\circ \cap U$ (which is a strict maximum if F is strictly concave on U).*

Proof. Let

$$\mathcal{C}_1 = \{x \in X \mid x_0 + tx \in A^\circ \cap U \text{ for some } t > 0\}.$$

Then, since U is open and A° is convex,

$$\mathcal{C}_1 = \mathcal{C} := \{x \in X \mid x_0 + tx \in A^\circ \text{ for some } t > 0\}.$$

Let $x \in \mathcal{C}$ and $x_0' = dF(x_0) \in X^*$. One has

$$F(x_0 + tx) = F(x_0) + tx_0'(x) + t\varepsilon_x(t),$$

where $\varepsilon_x(t) \to 0$ when $t \to 0$.

1) For the first part, if $x_0 \in X$ is a local maximum of F on A°, one has

$$x_0'(x) \le 0 \text{ for every } x \in \mathcal{C}.$$

Let $\mathcal{D} = \{x \in X \mid x'(x) \le 0 \text{ for every } x' \in A \text{ such that } x'(x_0) = 1\}$. We claim that $\overline{\mathcal{C}} = \mathcal{D}$. One can see that $\mathcal{C} \subset \mathcal{D}$ and that $\mathcal{D}$ is $\sigma(X, X^*)$-closed in X. To prove the reverse inclusion, apply the previous lemma with $f(x') = x'(x)$ and $f_0(x') = x'(x_0)$, for every $x' \in A$. It follows that for every $x \in \mathcal{D}$ and $\varepsilon > 0$, $x - \varepsilon x_0 \in \mathcal{C}$.

Now, if $\{x' \in A \mid x'(x_0) = 1\} = \emptyset$, then $\mathcal{D} = X$ and $x_0' = 0$. Otherwise $B(x_0) \neq \emptyset$. Let $\mathcal{B}$ be the convex cone generated by $\overline{B(x_0)}$ in X^*. Observe that $\mathcal{B}$ is $\sigma(X^*, X)$-closed, because if $(y_i') \subset \mathcal{B}$, with $y_i' = \lambda_i x_i'$, $(x_i') \subset \overline{B(x_0)}$ and $\lambda_i \ge 0$, are such that $y_i' \to y' \in X^*$ for $\sigma(X^*, X)$, then $y_i'(x_0) = \lambda_i \to y'(x_0)$, and

- if $y'(x_0) = 0$, then $\lambda_i \to 0$, and since $\overline{B(x_0)}$ is bounded, $y'_i \to 0 = y'$;
- if $y'(x_0) \neq 0$, then (x'_i) is $\sigma(X^*, X)$ convergent in $\overline{B(x_0)}$ and $y' \in \mathcal{B}$.

By definition, we have $\mathcal{D} = \mathcal{B}^\circ$ and since $\overline{\mathcal{C}} = \mathcal{D}$ and $x'_0(x) \leq 0$ for every $x \in \mathcal{C}$, we end up with

$$x'_0(x) \leq 0 \text{ for every } x \in \mathcal{B}^\circ.$$

By the Hahn-Banach theorem, we have thus

$$x'_0 \in \mathcal{B}^{\circ\circ} = \overline{\mathcal{B}} = \mathcal{B}.$$

2) Let $x_1 \in A^\circ \cap U$. Then $x_1 - x_0 \in \mathcal{C}_1 = \mathcal{C} \subset \mathcal{B}^\circ$. By the concavity of F on U, we have

$$\begin{aligned} F(x_1) = F(x_0 + x_1 - x_0) &\geq F(x_0) + dF(x_0)(x_1 - x_0) \\ &= F(x_0) + x'_0(x_1 - x_0) \geq F(x_0) \end{aligned}$$

since $x_1 - x_0 \in \mathcal{B}^\circ$ and by hypothesis $x'_0 \in \mathcal{B}$. The strict concavity of F implies strict inequality in the first inequality. q.e.d.

Proof of Theorem 3.1.

Let $X = X^* = M_n(\mathbb{R}) \times \mathbb{R}^n$, with the duality

$$\langle (S, s), (T, t) \rangle = \operatorname{trace}(S^*T) + \langle s, t \rangle,$$

where $M_n(\mathbb{R})$ is the space of $n \times n$ matrices with real entries equipped with the trace duality and $\langle s, t \rangle$ denotes the canonical scalar product on $\mathbb{R}^n$.

We define the compact set

$$A = \{(y \otimes x, y) \mid x \in C_1,\ y \in C_2\} \subset X^*.$$

Let $F : X \mapsto \mathbb{R}$ be defined by

$$F(T, t) = \det(T) \quad \text{for} \quad T \in M_n(\mathbb{R}) \quad \text{and} \quad t \in \mathbb{R}^n.$$

If we suppose that F reaches its maximum on A° at $(I_n, 0)$, then the conclusion follows from Theorem 4.2, using Caratheodory's theorem for cones in $\mathbb{R}^{n^2+n}$. q.e.d.

The existence of contact pairs together with a decomposition of the identity as in Theorem 3.5, is not sufficient to ensure that $(I_n, 0)$ is a local maximum of the determinant on the set A of $(T, a) \in GL_n \times \mathbb{R}^n$ such that $a + TK \subset L$. This can be seen from the example of the octahedron inscribed in a cube.

Moreover, it may happen that a local maximum is not a global one (see Example 5.7). But if we restrict the optimization problem on a convex subset of matrices over which the determinant is for instance strictly log-concave, we get a convex programming for which a local maximum is a global maximum and is unique.

Theorem 4.3. *Let C_1 and C_2 be two compact subsets of $\mathbb{R}^n$ such that* $\mathrm{conv}(C_1)$ *has a non-empty interior and* $0 \in \mathrm{int\,conv}(C_2)$*. Let Y be a N-dimensional linear subspace of the space of $n \times n$ real matrices and U be a relatively open convex subset of Y such that $T \mapsto \det(T)$ is positive and log-concave on U. Then $(I_n, 0) \in U \times \mathbb{R}^n$ is a maximum of the determinant of T, under the constrains,*

$$T \in U,\ a \in \mathbb{R}^n \quad \textit{and} \quad a + T(C_1) \subset C_2^\circ,$$

if and only if, there exist $m \leq N + n$ contact pairs $(x_i, y_i)_{1 \leq i \leq m}$ of (C_1, C_2), and $c_1, \dots, c_m > 0$ such that

(i) $I_n = \sum_{i=1}^m c_i\, x_i \otimes y_i$,

(ii) $\sum_{i=1}^m c_i y_i = 0$.

Proof: We use Theorem 4.2 and Caratheodory's theorem as in the preceding proof of Theorem 3.1.

Since the determinant is log-convave on the open subset of positive-definite matrices, we get as a corollary the following result, which is essentially Theorem 4 from [**B-R**].

Corollary 4.4. *Let K and L be two convex bodies (with non-empty interiors) in $\mathbb{R}^n$, such that $0 \in \mathrm{int}\, L$. Then the following assertions are equivalent:*

a. $|K| = \max\{|a + TK| \mid T \in D, a \in \mathbb{R}^n\}$ *where D is the set of all symmetric positive-definite matrices such that $a + T(K) \subset L$.*

b. *There exist $m \leq \frac{n^2+3n}{2}$ contact pairs $(x_i, y_i)_{1 \leq i \leq m}$ of $(\overline{\mathrm{Ext}}\, K, \overline{\mathrm{Ext}}\, L^\circ)$, and $c_1, , \dots, c_m > 0$ such that*

(i) $I_n = \sum_{i=1}^m c_i\, x_i \otimes y_i$,

(ii) $\sum_{i=1}^m c_i y_i = 0$.

Moreover, if **a.** *holds, then for any $(T, a) \in D \times \mathbb{R}^n$ such that $a + T(K) \subset L$, and $(T, a) \neq (I_n, 0)$, one has* $\det(T) < 1$.

Remark 4.5. For other applications, we may consider for Y the n-dimensional subspace of diagonal positive $n \times n$ real matrices or the $\frac{n(n+1)}{2}$-dimensional subspace of upper-triangular $n \times n$ matrices with non-negative diagonal, and maximize $\det(T)$ over the $(T, a) \in X \times \mathbb{R}^n$ such that $\det(T) > 0$ and $a + T(K) \subset L$.

Remark 4.6. We refer to [**B1**] and [**B2**] for detailed discussion on John's ellipsoid and its unicity and to [**P-S**] for the case when L is a parallelepiped. For a discussion on how to decrease the number of contact pairs see [**R2**].

5. Applications

In [**Gr**] the following problem was risen: what is the maximal possible value of $\bar{d}(K, L)$ for two convex bodies K and L in $\mathbb{R}^n$ and it was conjectured that the answer is n. The theorem below gives affirmative answer to this conjecture. Let us note that the similar problem about Banach-Mazur distance d seems much more difficult and is still open. The best known bound for d has been given by M. Rudelson ([**R1**]): for some absolute positive constant α, one has $d(K, L) \leq n^{4/3}(\ln(n+2))^{\alpha}$.

Theorem 5.1. *Let K and L be two convex bodies in $\mathbb{R}^n$ with non-empty interiors. Then $\bar{d}(K, L) \leq n$, that is there exists a linear mapping $T \in GL_n$ and $x, z \in \mathbb{R}^n$ such that*

$$K_z \subset T(L_x) \subset -nK_z.$$

More precisely, if K is in a position of maximal volume in L, then there exist $z \in \mathbb{R}^n$ such that $K_z \subset L_z \subset -nK_z$.

Proof. After an affine transformation, we may assume that K is in a position of maximal volume in L and that $0 \in \operatorname{int} L$. Applying Theorem 3.8, we get that exist $z \in \operatorname{int}(K)$ and $m \leq n^2 + n$ contact pairs $(u_i, v_i)_{1 \leq i \leq m}$ of $(\overline{\operatorname{Ext}}\, K_z, \overline{\operatorname{Ext}}\, L_z^{\circ})$, and $a_1, , \ldots, a_m > 0$ such that

$$I_n = \sum_{i=1}^{m} a_i u_i \otimes v_i = \sum_{i=1}^{m} a_i v_i \otimes u_i$$

and

$$\sum_{i=1}^{m} a_i u_i = \sum_{i=1}^{m} a_i v_i = 0; \; \sum_{i=1}^{m} a_i = n.$$

For every $x \in \mathbb{R}^n$, denote

$$c(x) = \max_{1 \leq i \leq m} \langle v_i, x \rangle .$$

Since $0 = \sum_{i=1}^{m} a_i u_i$, one has $c(x) \geq 0$ for every $x \in \mathbb{R}^n$. Then, for $x \in \mathbb{R}^n$,

$$-x = \sum_{i=1}^{m} a_i \langle v_i, -x \rangle u_i,$$

and using the fact that $\sum_{i=1}^{m} a_i u_i = 0$, we get

$$-x = c(x) \sum_{i=1}^{m} a_i u_i - \sum_{i=1}^{m} a_i \langle v_i, x \rangle u_i = \sum_{i=1}^{m} a_i \left(c(x) - \langle v_i, x \rangle\right) u_i.$$

Since $c(x) - \langle v_i, x\rangle \geq 0$ and $u_i \in K_z$, $1 \leq i \leq m$, the last equality together with the convexity of K_z, gives that

$$-x \in \left(\sum_{i=1}^{m} a_i \left(c(x) - \langle v_i, x\rangle\right)\right) K_z.$$

Using the fact that $\sum_{i=1}^m a_i = n$ and $\sum_{i=1}^m a_i v_i = 0$, we arrive at

$$-x \in \left(c(x)\sum_{i=1}^{m} a_i - \langle \sum_{i=1}^{m} a_i v_i, x\rangle\right) K_z = nc(x)K_z.$$

To conclude, recall that $c \geq 0$, $L_z \subset \{x \in \mathbb{R}^n \mid c(x) \leq 1\}$ and $0 \in K_z$. It follows that $-L_z \subset nK_z$, which is equivalent to $L_z \subset -nK_z$. q.e.d.

Remark 5.2. If $K = -K$, and $L = -L$, and K is in a position of maximal volume in L with respect to the origin, then our proof gives $K \subset L \subset nK$ (see also [**Gi-P-T**] and [**Las**]).

Corollary 5.3. *Let K and L be two convex bodies in $\mathbb{R}^n$. Then*

$$vr(K, L)\, vr(L, K) \leq n.$$

This corollary follows immediately from the previous theorem. Note that recently Giannopoulos and Hartzoulaki ([**Gi-H**]) proved that for any two convex bodies K and L in $\mathbb{R}^n$ one has

$$vr(K, L) \leq c\sqrt{n}\ln(n + 1)$$

for some absolute constant $c > 0$.

Remark 5.4. It follows immediately from a result of Gluskin [**Gl**], that there exists a constant $c > 0$, such that for every dimension n, there exist centrally symmetric convex bodies K and L (the so-called Gluskin bodies) such that $vr(K, L)\, vr(L, K) \geq cn$. The paramater $vr(K, L)\, vr(L, K)$ is investigated by Khrabrov in [**K**] and called modified Banach-Mazur distance. Corollary 5.3 gives the precise upper estimate attained for the simplex and the Euclidean ball.

Theorem 5.5. *Let K and L be two convex bodies in $\mathbb{R}^n$, and suppose that L has a center of symmetry. Then $d(K, L) \leq n$. More precisely, if we assume that K and L are in a position given by Theorem* 3.8 *and that L is centrally symmetric with respect to some $a \in L$, then there exists $b \in \mathbb{R}^n$ such that $K - b \subset L - b \subset n(K - b)$.*

Proof. We assume that K is in a position of maximum volume in L. By Theorem 5.1, there exists $z \in \mathbb{R}^n$ such that we have $K_z \subset L_z \subset -nK_z$. After a translation of K and L, we suppose that $z = 0$. By the condition of the theorem, there exists $a \in \mathbb{R}^n$, such that $L-a = -(L-a)$, i.e. $L = -L + 2a$. Take $b = -2a/(n-1)$. Then

$$K - b \subset L - b = -L + 2a + 2a/(n-1) \subset nK + 2\,\frac{n}{n-1}a = n(K-b),$$

which proves that $d(K, L) \le n$. q.e.d.

Remark 5.6. Theorem 5.5 is sharp, as the example of a regular simplex and the circumscribed Euclidean ball shows. It improves a result of Lassak ([**Las**]) who proved that there are positions K' and L' of K and $L = -L$ such that $K' \subset L' \subset (2n-1)K'$.

It may happen that $(I_n, 0)$ is a local maximum of $\det(T)$ on the set $\{(T, a)|\ a + T(K) \subset L\}$ without being a global one. In this situation, it is impossible to deduce any good distance estimates from the contact pairs. We denoted here by $|\cdot|$ the Euclidean norm and by $\|T - I\|$ the operator norm.

Example 5.7. Let K be a regular simplex in $\mathbb{R}^n$, with vertices $u_1, \dots, u_{n+1}$ on the Euclidean sphere. For $0 < \varepsilon < 1$, we define a body $L(\varepsilon)$ by

$$L(\varepsilon) = \mathrm{conv}\,(K, -(1-\varepsilon)nK).$$

It is easy to prove that there exists $\eta > 0$ such that if $z \in L(\varepsilon)$ satisfies $0 < |z - u_i| < \eta$ for some $1 \le i \le n+1$, then $|z| < 1$. We shall see that $(I_n, 0)$ is a strict local maximum of the determinant function $f(T, a) = \det(T)$ on

$$\{(T, a) \mid a + T(K) \subset L_\varepsilon\}.$$

For this, setting $z_i = a + Tu_i$, it suffices clearly to check that if $z_1, \dots, z_{n+1} \in L(\varepsilon)$ satisfy $|z_i - u_i| \le \eta$, $1 \le i \le n+1$, then

$$|\,\mathrm{conv}\,(z_1, \dots, z_{n+1})| \le |\,\mathrm{conv}\,(u_1, \dots, u_{n+1})| = |K|,$$

with equality if and only if $z_i = u_i$ for $1 \le i \le n+1$. But as we mentioned above, for $z_i \in L(\varepsilon)$, $0 \le |z_i - u_i| \le \eta$ implies $|z_i| \le 1$ with equality if an only if $z_i = u_i$. The result follows now from the well known fact that the regular simplex with vertices on the Euclidean sphere is, up to isometries of $\mathbb{R}^n$, the unique simplex of maximal volume inscribed in the Euclidean ball.

Now, it is clear that the simplex of maximal volume inside L_ε is $-(1-\varepsilon)nK$. Thus $(I_n, 0)$ is not the global maximum of f. Moreover, it is impossible to deduce any good distance estimate from contact pairs, because we have here:

$$\inf\{t > 0|\ K \subset L(\varepsilon) \subset tK\} = \frac{n^2}{1-\varepsilon} \to n^2 \text{ when } \varepsilon \to 0,$$

although

$$-(1-\varepsilon)nK \subset L_\varepsilon \subset -nK,$$

which indicates that $d(L_\varepsilon, K) = \frac{1}{1-\varepsilon} \to 1$ when $\varepsilon \to 0$.

Similarly, it is impossible to deduce any good estimate for $vr(L_\varepsilon, K)$ since $(|L_\varepsilon|/|K|)^{1/n} \geq n/(1-\varepsilon)$, but $vr(L_\varepsilon, K) \leq 1/(1-\varepsilon)$.

Corollary 5.8. *If S is a non-degenerate simplex and K is a centrally symmetric convex body in $\mathbb{R}^n$, then $d(S, L) = n$.*

Proof. The estimate $d(S, L) \leq n$ follows from the previous theorem. Now suppose that $S \subset L \subset tS$. Let b be the center of symmetry of L, i.e. $L = -L + 2b$. Then $-L + 2b \subset tS$ and therefore, $L \subset -tS + 2b$. Thus we obtain $S \subset -tS + 2b$, i.e. $S_a \subset -tS_a$, where $a = 2b/(t+1)$. It implies that $t \geq n$. q.e.d.

Remark 5.9. From a result of Palmon [**Pa**], the only convex body with extremal distance from the Euclidean ball is the simplex. Combining this result with Corollary 5.8, we conclude that if the set of equivalence classes of convex bodies (up to affine transformation) is equipped with the Banach-Mazur distance, then the class corresponding to the simplex is the unique center of the set of equivalence classes of symmetric convex bodies.

By the triangle inequality we immediately obtain the following Corollary.

Corollary 5.10. *Let K and L be two convex bodies in $\mathbb{R}^n$. Then*

$$d(K, L) \leq n \cdot \min\{\delta_K, \delta_L\}.$$

We conclude this section with two other consequences of Theorem 3.5.

Theorem 5.11. *Let K and L be centrally symmetric convex bodies in $\mathbb{R}^n$, $n \geq 2$, such that K is in a position of maximal volume in L with respect to the origin. Then there exists a parallelepiped P and a cross-polytope C such that*

$$C \subset K \subset L \subset P \quad \text{and} \quad \left(\frac{|P|}{|C|}\right)^{1/n} \leq \frac{1}{n}\left(\binom{n^2}{n} n!\right)^{1/n} < n.$$

Remark 5.12. For a similar result when K is the Euclidean ball, see [**P-S**], [**B1**], [**Ge**], [**Go-M-P**].

Proof. By Theorem 3.5 and Remark 3.6 and 3.12, there are $c_i \geq 0$, and $x_i, y_i \in \mathbb{R}^n$, $1 \leq i \leq m$, where $n \leq m \leq n^2$, such that $I_n = \sum_{i=1}^m c_i x_i \otimes y_i$. Let A, B be $m \times n$ matrices defined by

$$A = (c_i x_i)_{i\leq m} = (c_i x_{ij})_{i\leq m, j\leq n}, \quad B = (y_i)_{i\leq m} = (y_{ij})_{i\leq m, j\leq n},$$

where $(x_{ij})_{1\le j\le n}$, $(y_{ij})_{1\le j\le n}$ are the coordinates of the vectors x_i, y_i in the canonical basis of $\mathbb{R}^n$. Then $B^*A = I_n$.

If $I \subset \{1,\dots,m\}$, denote by $|I|$ its cardinality. Using the Cauchy-Binet formula, we obtain

$$\begin{aligned} 1 = \det I_n &= \sum_{I\subset\{1,\dots,m\},|I|=n} \det_{i\in I}(c_i x_i)\cdot\det_{i\in I}(y_i) \\ &\le \max_{|I|=n}\Big(|\det_{i\in I}(x_i)|\cdot|\det_{i\in I}(y_i)|\Big)\cdot\sum_{|I|=n}\prod_{i\in I} c_i \\ &\le \max_{|I|=n}\Big(|\det_{i\in I}(x_i)|\cdot|\det_{i\in I}(y_i)|\Big)\cdot\binom{m}{n}\cdot\left(\sum_{i=1}^m \frac{c_i}{m}\right)^n. \end{aligned}$$

Since $n \le m \le n^2$, one has

$$\binom{m}{n}\left(\sum_{i=1}^m \frac{c_i}{m}\right)^n = \left(\frac{n}{m}\right)^n\binom{m}{n} \le \left(\frac{1}{n}\right)^n\binom{n^2}{n} < n^n.$$

Therefore, if the maximum of $|\det_{i\in I}(x_i)|\cdot|\det_{i\in I}(y_i)|$ is attained at $I_0 \subset \{1,\dots,m\}$, we get

$$1 \le |\det_{i\in I_0}(x_i)|\cdot|\det_{i\in I_0}(y_i)|\cdot\left(\frac{1}{n}\right)^n\binom{n^2}{n}.$$

Let

$$P = \{x\in\mathbb{R}^n \mid |\langle x, y_i\rangle| \le 1 \text{ for all } i\in I_0\}$$

and

$$C = \left(\{x\in\mathbb{R}^n \mid |\langle x, x_i\rangle| \le 1 \text{ for all } i\in I_0\}\right)^\circ.$$

Since $y_i \in \partial L^\circ$, one has $P \supset L$, and since $x_i \in \partial K$, we get $C^\circ \supset K^\circ$. Thus $C \subset K$. One has

$$|C| = \frac{2^n}{n!}|\det_{i\in I_0}(x_i)| \text{ and } |P| = 2^n|\det_{i\in I_0}(y_i)|^{-1}.$$

The result follows. q.e.d.

Theorem 5.13. *Let K and L be convex bodies in $\mathbb{R}^n$, such that $0\in K\cap \operatorname{int} L$, and that K is in a position of maximal volume in L with respect to 0. Then there exist two simplices S_1 and S_2 such that $S_1\subset K$ and $S_2\subset L^\circ$ and*

$$(|S_1|\ |S_2|)^{1/n} \ge \frac{1}{n^2}.$$

Proof: By Theorem 3.5 and Remark 3.6, for some $n < m \le n^2$, there exist $c_i > 0$, $x_i \in \partial K$, $y_i\in\partial L^\circ$, $1\le i\le m$ such that $I_n = \sum_{i=1}^m c_i x_i\otimes y_i$ and $\sum_{i=1}^m c_i = n$ (if $m = n$ in the theorem we set $x_{n+1} = y_{n+1} = 0$, $c_{n+1} = 0$).

Let A, B be $m \times (n+1)$ matrices defined as above with $x_{ik} = y_{ik} = 1$ for every $1 \leq i \leq m$ and $k = n+1$. Then, since $\sum_{i=1}^{m} c_i = n$, $B^*A = (a_{ij})_{1 \leq i,j \leq n+1}$, where $a_{ij} = \delta_{ij}$ for $1 \leq i, j \leq n$ and $a_{kk} = n$ if $k = n+1$. Repeating the proof above we obtain

$$n = \det(B^*A) \leq \max_{|I|=n+1} \left(\left| \det_{i \in I} (x_i) \right| \cdot \left| \det_{i \in I} (y_i) \right| \right) \cdot \sum_{|I|=n+1} \prod_{i \in I} c_i$$

$$\leq \max_{|I|=n+1} \left(\left| \det_{i \in I} (x_i) \right| \cdot \left| \det_{i \in I} (y_i) \right| \right) \cdot \binom{m}{n+1} \cdot (n/m)^{n+1}.$$

Assuming that this maximum is attained on I_0, let $S_1 \subset K$ be the simplex with vertices x_i, $i \in I_0$, and $S_2 \subset L^\circ$ the simplex with vertices y_i, $i \in I_0$. Then

$$|S_1|\ |S_2| = (n!)^{-2} \left| \det_{i \in I_0} (x_i) \right| \cdot \left| \det_{i \in I_0} (y_i) \right|$$

$$\geq n(n!)^{-2} \binom{m}{n+1}^{-1} \cdot (m/n)^{n+1} > n^{-2n},$$

which proves the theorem. q.e.d.

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Y. Gordon, Department of Mathematics, Technion, Haifa 32000, Israel.
e-mail: gordon@techunix.technion.ac.il

A.E. Litvak, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, Canada, T6G 2G1.
e-mail: alexandr@math.ualberta.ca

M. Meyer and A. Pajor, Equipe d'Analyse et Mathmatiques Appliques, Université de Marne-la-Vallée, 5, boulevard Descartes, Champs sur Marne, 77454 Marne-la-Vallée Cedex 2, France.
emails : meyer@math.univ-mlv.fr - pajor@math.univ-mlv.fr