ON THE VOLUME RATIO OF PROJECTIONS OF CONVEX BODIES

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Abstract. We study the volume ratio between *projections* of two convex bodies. Given a high-dimensional convex body K we show that there is another convex body L such that the volume ratio between any two projections of fixed rank of the bodies K and L is large. Namely, we prove that for every $1 \leq k \leq n$ and for each convex body $K \subset \mathbb{R}^n$ there is a centrally symmetric body $L \subset \mathbb{R}^n$ such that for any two projections $P, Q : \mathbb{R}^n \to \mathbb{R}^n$ of rank k one has

$$\mathrm{vr}(PK, QL) \geq c \min \left\{ \frac{k}{\sqrt{n}} \sqrt{\frac{1}{\log\log\log(\frac{n\log(n)}{k})}}, \frac{\sqrt{k}}{\sqrt{\log(\frac{n\log(n)}{k})}} \right\},$$

where $c > 0$ is an absolute constant. This general lower bound is sharp (up to logarithmic factors) in the regime $k \geq n^{2/3}$.

1. Introduction.

The problem of estimating Banach-Mazur distances between projections or sections of convex bodies had aroused considerable interest (see for example [4, 22, 20, 29] and references therein). Recall that this distance, for two centrally symmetric convex bodies K and L in $\mathbb{R}^n$, is defined as

$$d_{BM}(K, L) = \inf\{a \cdot b \ \mid \ \frac{1}{a}K \subset TL \subset bK\}, \tag{1}$$

where the infimum is taken over all invertible linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$ and all $a, b > 0$.

Note that two convex bodies can be far apart but they may have their projections or sections quite close. This happens, for example, if the bodies are Gluskin's polytopes (absolute convex hulls of, say, $3n$ random points on the standard Euclidean sphere in $\mathbb{R}^n$). Indeed, Gluskin [9] proved that with high probability the Banach–Mazur distance between two such polytopes is at least cn, where c is an absolute positive constant. On the other hand it is known that "most" sections of a Gluskin polytope are nearly Euclidean, thus "most" sections of two Gluskin's polytopes are quite close to each other. This follows from results on sections of convex bodies having bounded volume ratio [33, 32], see below for the precise definitions.

A more general question was studied by Mankiewicz and Tomczak-Jaegermann in [22]. They estimated average distance between random k-dimensional projections of two given centrally symmetric convex bodies. It turns out that such an average is bounded below by the product of averages of distances of $((\frac{1}{2} - \varepsilon)k)$-dimensional projections of these bodies to the Euclidean ball. Note here that a Gluskin's polytope can be viewed as a random projection of, say, $(3n)$-dimensional octahedron to $\mathbb{R}^n$.

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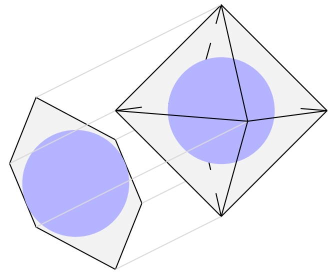

FIGURE 1. A projection of the octahedron and the Euclidean ball.

Rudelson [29] studied the problem of estimating extremal distances between projections of centrally symmetric convex bodies. For $k < n$ define the distance $\delta_k(K, L)$ as the minimal Banach–Mazur distance between k-dimensional projections of K and L. Rudelson was interested in estimating the diameter of the Banach–Mazur compactum for this distance; that is, in finding the asymptotic behaviour of

$$\Delta(k, n) := \sup \delta_k(K, L),$$

where the supremum is taken over all n-dimensional convex symmetric bodies K and L. He proved that

$$\Delta(k, n) \sim_{\log n} \begin{cases} \sqrt{k} & \text{if } k \leq n^{2/3} \\ \frac{k^2}{n} & \text{if } k > n^{2/3}, \end{cases} \tag{2}$$

where $A \sim_{\log n} B$ means that

$$\frac{1}{C \log^a n} A \leq B \leq (C \log^a n) A$$

for some absolute constants $C, a > 0$. In particular, Rudelson showed that there are two centrally symmetric convex bodies $K, L \subset \mathbb{R}^n$, such that for any $k < n$,

$$\delta_k(K, L) \gtrsim \frac{k^2}{n \log \log n},$$

where $A \gtrsim B$ means that $A \geq cB$ for some absolute constant $c > 0$. Note also the well-known fact (proved in [3, 6, 10, 11])

$$\delta_k(B_2^n, B_1^n) \gtrsim \sqrt{\frac{k}{\log(1 + \frac{n}{k})}}.$$

Another possible measure of how far two convex bodies $K, L \subset \mathbb{R}^n$ are from each other, is given by their *volume ratio*:

$$\mathrm{vr}(K, L) := \inf \left\{ \left(\frac{|K|}{|T(L)|} \right)^{\frac{1}{n}} \mid T : \mathbb{R}^n \to \mathbb{R}^n \text{ is an affine transformation, } T(L) \subset K \right\},$$

where $| \cdot |$ denotes n-dimensional volume. Note that the standard volume ratio $\mathrm{vr}(K)$ introduced in [33] is just $\mathrm{vr}(K, B_2^n)$.

In other words, $\mathrm{vr}(K, L)$ measures how well K can be approximated from inside by an affine image of L in terms of volume. This invariant goes back to the works of McBeath [21] and Levi [18]. It was further investigated by many authors. In particular, Giannopoulos and Hartzoulaki [8] proved that for every two convex bodies $K, L \subset \mathbb{R}^n$,

$$\mathrm{vr}(K,L) \leq C\sqrt{n}\,\log n, \tag{3}$$

where $C > 0$ is an absolute constant. On the other hand, it was proved in [7] that given a convex body $K \subset \mathbb{R}^n$ there is centrally symmetric body $L \subset \mathbb{R}^n$ such that the volume ratio $\mathrm{vr}(K,L)$ is large. Precisely, we have

$$\mathrm{vr}(K,L) \geq C\sqrt{n}, \tag{4}$$

where $C > 0$ is an absolute constant. This general lower estimate is sharp: by John's theorem and a reduction to the symmetric case we have, for example, that given any convex body $L \subset \mathbb{R}^n$, $\mathrm{vr}(B_2^n, L) \leq \sqrt{n}$. The lower bound in (4) is a refinement of a previous estimate obtained by Khrabrov in [15] of order $\sqrt{\frac{n}{\log\log(n)}}$.

We would also like to note that

$$d_m(K,L) = \mathrm{vr}(K,L)\,\mathrm{vr}(L,K)$$

is a weaker version of the Banach–Mazur distance, called *modified Banach–Mazur distance* (this name comes from [16]). Clearly, $d_m(K,L) \leq d_{BM}(K,L)$. It was introduced in [21] (in fact, the logarithm of it, see also [18]) and then implicitly used in [14, 9] in order to estimate the Banach–Mazur distance from below. Then it was investigated in a series of works by Khrabrov, see also Corollary 5.3 and Remark 5.4 in [13]. Moreover, Khrabrov [15] proved that for every centrally symmetric convex body $K \subset \mathbb{R}^n$ and every $1 \leq p \leq \infty$,

$$d_m(K, B_p^n) = \mathrm{vr}(K, B_p^n)\,\mathrm{vr}(B_p^n, K) \leq \sqrt{en}. \tag{5}$$

We extend the notion of volume ratio for two given bodies lying in different subspaces of $\mathbb{R}^n$ in the following natural way. Let $1 \leq k \leq n$ and let E, F be two k-dimensional subspaces of $\mathbb{R}^n$. Then for two convex bodies $K \subset E$ and $L \subset F$, we define

$$\mathrm{vr}(K,L) := \inf\left\{ \left(\frac{|K|}{|T(L)|}\right)^{\frac{1}{k}} \mid T : E \to F \text{ is an affine transformation, } T(L) \subset K \right\},$$

where $|\cdot|$ denotes k-dimensional volume.

Note that for a convex body K we have a collection of k-dimensional convex bodies given by $QK \subset \mathbb{R}^k$ for any given projection $Q : \mathbb{R}^n \to \mathbb{R}^n$ of rank k. Here we provide a lower bound for volume ratio in the spirit of Rudelson's approach. Namely, we show that for every high-dimensional convex body $K \subset \mathbb{R}^n$ there exists a centrally symmetric convex body $L \subset \mathbb{R}^n$ such that, for every pair of k-dimensional projections $P, Q : \mathbb{R}^n \to \mathbb{R}^n$, the volume ratio $\mathrm{vr}(PK, QL)$ is large. The following theorem is the main result of this work.

Theorem 1.1. *Let n be large enough and $k \leq n$. Then for every convex body $K \subset \mathbb{R}^n$ there is a centrally symmetric body $L \subset \mathbb{R}^n$ such that for any two k-dimensional projections $P, Q : \mathbb{R}^n \to \mathbb{R}^n$ one has*

$$\mathrm{vr}(PK, QL) \geq c\,\min\left\{ \frac{k}{\sqrt{n}} \cdot \sqrt{\frac{1}{\log\log\log(\frac{n\log(n)}{k})}},\ \frac{\sqrt{k}}{\sqrt{\log(\frac{n\log(n)}{k})}} \right\}, \tag{6}$$

where $c > 0$ is an absolute constant.

Moreover, in Corollary 5.2 below, we show that Theorem 1.1 is sharp (up to logarithmic factors) in the regime $k \geq n^{2/3}$. Remarkably, the phase transition in (2) is exactly $k \sim n^{2/3}$.

Although it is not directly related, we would like to mention the following result from [12]. Interestingly, the proof of this uses Gluskin's polytopes (as our theorem) and leads to the essentially same lower bound. For all $1 \leq k \leq n$ there exist a universal convex body $K \subset \mathbb{R}^n$ such that for every centrally symmetric convex body $L \subset \mathbb{R}^n$ and any k-dimensional projections P and Q one has

$$d_{BM}(PK, QL) \geq \frac{c\,k}{\sqrt{n \log n}},$$

where $c > 0$ is an absolute constant. Note that we cannot expect to have a universal convex body L in Theorem 1.1. That is, the existence of a body L such that for *every* body $K \subset \mathbb{R}^n$ inequality (6) holds. Indeed, $\mathrm{vr}(PL, PL) = 1$ for every $P \in \mathcal{P}^k(n)$.

Finally, we mention that using duality we can reformulate our theorem in terms of sections. For a convex body K with the origin in its interior and a subspace $E \subset \mathbb{R}^n$ one has $P_E(K^\circ) = (E \cap K)^\circ$, where P_E denotes the orthogonal projection onto E. Note that for centrally symmetric bodies it is enough to consider only linear operators in the definition of volume ratio. Therefore, every result concerning volume ratios of projections of a centrally symmetric bodies K and L has a dual version concerning volume ratios of sections of K° and L°. Thus, we can also state a dual version of our previous result.

Corollary 1.2. *Let $1 \leq k \leq n$. For each centrally symmetric convex body $K \subset \mathbb{R}^n$ there is a centrally symmetric body $L \subset \mathbb{R}^n$ such that for any two k-dimensional subspaces $E, F \subset \mathbb{R}^n$ one has*

$$\mathrm{vr}(F \cap L, E \cap K) \geq c \min\left\{ \frac{k}{\sqrt{n}} \cdot \sqrt{\frac{1}{\log\log\log(\frac{n\log(n)}{k})}}, \frac{\sqrt{k}}{\sqrt{\log(\frac{n\log(n)}{k})}} \right\},$$

where $c > 0$ is an absolute constant.

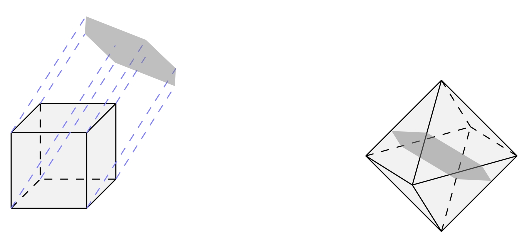

Figure 2. A projection of B_∞^n and a section of $B_1^n = (B_\infty^n)^\circ$.

2. Preliminaries.

Given two sequences of real numbers $(a_n)_n$ and $(b_n)_n$ we write $a_n \lesssim b_n$ (resp., $a_n \gtrsim b_n$) if there exists an absolute constant $C > 0$ (independent of n) such that $a_n \leq C b_n$ (resp., $C a_n \geq b_n$) for every n. We write $a_n \sim b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$. We denote by $e_1, \ldots, e_n$ the canonical vector basis in $\mathbb{R}^n$ and by B_2^n and S^{n-1}, the unit ball and unit sphere in $\mathbb{R}^n$.

Similarly, the unit ball of ℓ_p^n is denoted by B_p^n, where the norm in ℓ_p^n is defined by

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \quad \text{for } 1 \le p < \infty \quad \text{and} \quad \|x\|_\infty = \max_{i\le n} |x_i|.$$

Given $X_1, \dots, X_m \in \mathbb{R}^n$, we denote by $\mathrm{absconv}\{X_1, \dots, X_m\}$ their absolute convex hull, that is,

$$\mathrm{absconv}\{X_1, \dots, X_m\} := \left\{ \sum_{i=1}^m a_i X_i \ \Big| \ \sum_{i=1}^m |a_i| \le 1 \right\} \subset \mathbb{R}^n.$$

A convex body $K \subset \mathbb{R}^n$ is a compact convex set with non-empty interior. For K with the origin as an interior point its Minkowski functional is defined on $\mathbb{R}^n$ by

$$\|x\|_K = \inf\{\lambda > 0 \mid x \in \lambda K\}.$$

If K is centrally symmetric (i.e., $K = -K$), then $\|\cdot\|_K$ defines a norm and we denote by X_K the normed space $(\mathbb{R}^n, \|\cdot\|_K)$ that has K as its unit ball. By $|K|$ we denote the n-dimensional volume of K. Moreover, with slight abuse of notations, given a k-dimensional projection P on $\mathbb{R}^n$, by $|PK|$ we denote the k-dimensional volume of PK.

The polar set of a body K with 0 in its interior, denoted by K°, is defined as

$$K^\circ = \{x \in \mathbb{R}^n \mid \langle x, y\rangle \le 1 \quad \text{for all} \quad y \in K\}.$$

The following result relates the volume of a body with the volume of its polar and is due to Blaschke-Santaló and Bourgain-Milman [1, Theorem 1.5.10 and Theorem 8.2.2]:

There exists an absolute constant $c > 0$ such that for every centrally symmetric convex body $K \subset \mathbb{R}^n$,

$$c|B_2^n|^{2/n} \le |K|^{\frac{1}{n}} |K^\circ|^{\frac{1}{n}} \le |B_2^n|^{2/n}. \tag{7}$$

In other words, $|K|^{\frac{1}{n}}|K^\circ|^{\frac{1}{n}} \sim \frac{1}{n}$. We also use the support function of K defined on $\mathbb{R}^n$ by

$$h_K(x) = \sup_{y\in K} \langle x, y\rangle = \|x\|_{K^\circ}.$$

Given two normed spaces X and Y and an operator $T : X \to Y$, the operator norm is denoted by $\|T : X \to Y\|$. Similarly, given an operator $T : \mathbb{R}^n \to \mathbb{R}^n$, we denote $\|T\| := \|T : \ell_2^n \to \ell_2^n\|$.

We now recall some basic properties of the volume ratio, see e.g. [15].

Fact 2.1. For every pair of centrally symmetric convex bodies (K, L) in $\mathbb{R}^n$ the following holds:

(1)
$$\mathrm{vr}(K, L) = \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}} \cdot \inf_{T\in SL(n,\mathbb{R})} \|T : X_L \to X_K\|,$$
where the infimum runs all over the linear transformations T that lie on the special linear group of degree n (matrices of determinant one).

(2) $\mathrm{vr}(K, L) \sim \mathrm{vr}(L^\circ, K^\circ)$.

(3) If $T : X_L \to X_K$ is a linear operator we have that
$$\frac{1}{\|T : X_L \to X_K\|} \cdot T(L) \subset K \quad \text{and hence} \quad \mathrm{vr}(K, L) \le \frac{\|T : X_L \to X_K\| |K|^{\frac{1}{n}}}{|\det T|^{\frac{1}{n}} |L|^{\frac{1}{n}}}.$$

(4) $\mathrm{vr}(K, L) \le \mathrm{vr}(K, Z) \cdot \mathrm{vr}(Z, L)$ for every convex body Z in $\mathbb{R}^n$.

Remark 2.2. We finally discuss not necessarily symmetric convex bodies. Note that for every convex bodies K and L in $\mathbb{R}^n$ and for any affine transformations T and S one has

$$\mathrm{vr}(K, L) = \mathrm{vr}(T(K), S(L)).$$

In other words, the volume ratio between K and L depends exclusively on the affine classes of the bodies involved. By the Rogers-Shephards inequality (see e.g., [1, Theorem 1.5.2]), for every convex body $W \subset \mathbb{R}^n$ we have $\mathrm{vr}(W - W, W) \leq 4$. Clearly the last inequality in Fact 2.1 holds for any (not necessarily centrally symmetric) convex bodies K, L, Z. Therefore,

$$\mathrm{vr}(K - K, L) \leq \mathrm{vr}(K - K, K)\ \mathrm{vr}(K, L) \leq 4\ \mathrm{vr}(K, L). \tag{8}$$

3. Auxiliary results.

We start with recalling a standard result in geometric measure theory (see e.g., [24, Theorem 7.5]).

Theorem 3.1. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map with Lipschitz constant L_f, $0 \leq s \leq m$, and $A \subset \mathbb{R}^m$. Then*

$$\mathcal{H}^s(f(A)) \leq L_f^s \mathcal{H}^s(A),$$

where $\mathcal{H}^s$ is the s-Hausdorff measure.

Recall that for $k \in \mathbb{N}$ the k-Hausdorff measure is a multiple of the Lebesgue measure in $\mathbb{R}^k$. Namely, for every measurable set A, $\mathcal{H}^k(A) = \frac{2^k}{|B_2^k|}|A|$ (see [24, 4.3]).

We denote by $\mathcal{P}^k(n)$ the set of all orthogonal projections of rank k in $\mathbb{R}^n$. Given $Q \in \mathcal{P}^k(n)$, $|QK|$ denotes the k-dimensional Lebesgue measure of QK. As an application of the last theorem we prove the following lemma, that relates the k-dimensional volume of two different projections of K with their distance in the canonical operator metric.

Lemma 3.2. *Let $1 \leq k \leq n$ and let $P, Q \in \mathcal{P}^k(n)$ be such that $\|P - Q\| \leq \frac{1}{2\sqrt{n}}$. Then for every centrally symmetric convex body $K \subset \mathbb{R}^n$ in the John position,*

$$\frac{1}{2}|QK|^{\frac{1}{k}} \leq |PK|^{\frac{1}{k}} \leq 2|QK|^{\frac{1}{k}}.$$

Proof. Note that

$$P = P^2 = PQ + P(P - Q).$$

Since K is in John's position, $B_2^n \subset K \subset \sqrt{n}B_2^n$. Using that $\|P - Q\| \leq \frac{1}{2\sqrt{n}}$, we observe

$$(P - Q)K \subset \sqrt{n}(P - Q)B_2^n \subset \frac{1}{2}\, B_2^n \subset \frac{1}{2}\, K.$$

This implies

$$PK \subset PQ\, K + \frac{1}{2}\, PK.$$

Therefore for every $x \in \mathbb{R}^n$ we have

$$h_{PK}(x) \leq h_{PQK}(x) + \frac{1}{2}\, h_{PK}(x),$$

so $h_{PK}(x) \leq 2h_{PQK}(x)$. This implies that $PK \subset 2PQ\, K$.

Finally we apply Theorem 3.1 with $m := n$, $s := k$, $f := P$ and $A := QK$ to obtain

$$|PK|^{\frac{1}{k}} \leq 2|PQK|^{\frac{1}{k}} \leq 2|QK|^{\frac{1}{k}},$$

using that the Lipschitz constant of the mapping P is obviously one and simplifying the constants to pass from the Hausdorff to the Lebesgue measure. □

Next we introduce a variant of Gluskin's random polytopes. Instead of considering the absolute convex hull of points taken uniformly on the unit sphere we are going to work with Gaussian random vectors. The reason for doing this is that we want to deal with projections of these bodies, and the Gaussian measure is more suitable for this purpose. Let $N > n$ and $g_1, \dots, g_N$ be standard independent Gaussian vectors in $\mathbb{R}^n$. We consider the symmetric polytope

$$Z_N = Z_N(\omega) = \mathrm{absconv}\{\sqrt{n}e_1, \dots, \sqrt{n}e_n, g_1, \dots, g_N\}.$$

For basic properties of Gaussian polytopes we refer to [23]. It is well known that the Euclidean norm of a Gaussian vector in $\mathbb{R}^n$ is well concentrated about its average, which is essentially $\sqrt{n}$. We will need the following lemma, the standard proof of which is provided for the sake of completeness (for simplicity we write just $\|\cdot\|$ for $\|\cdot\|_2$).

Lemma 3.3. *Let $n \geq 1$ and let g be a standard Gaussian vector in $\mathbb{R}^n$. Then for every $\lambda \geq 2\sqrt{n}$,*

$$\mathbb{P}\left\{\|g\| \geq \lambda\right\} \leq \exp(-\lambda^2/8).$$

In particular,

$$\mathbb{P}\left\{\|g\| \geq 2\sqrt{n}\right\} \leq \exp(-n/2).$$

Moreover, for $n \geq 50$,

$$\mathbb{P}\left\{\|g\| \leq \sqrt{n}/4\right\} \leq \exp(-n/4).$$

Proof. The Gaussian concentration inequality (see [5] or inequality (2.35) in [19]) states for every $s > 0$,

$$\max\Big\{\mathbb{P}\left\{\|g\| - \mathbb{E}\|g\| \geq s\right\}, \mathbb{P}\left\{\mathbb{E}\|g\| - \|g\|) \geq s\right\}\Big\} \leq \exp\left(-s^2/2\right).$$

Since, $\mathbb{E}\|g\| \leq (\mathbb{E}\|g\|^2)^{1/2} = \sqrt{n}$, this yields the first and the second bounds. To obtain the third bound, denote $a = \mathbb{E}\|g\|$ and observe

$$n - a^2 = \mathbb{E}\left(\|g\| - a\right)^2 = \int_0^\infty 2t\mathbb{P}\left\{|\|g\| - a| \geq t\right\} dt \leq \int_0^\infty 4te^{-t^2/2}dt = 4.$$

Thus $a^2 \geq n-4$ and hence for $n \geq 50$, $a \geq \sqrt{n}(1/4 + 1/\sqrt{2})$. Applying the concentration inequality with $s = \sqrt{n/2}$, we obtain

$$\begin{aligned}\mathbb{P}\left\{\|g\| \leq \sqrt{n}/4\right\} &\leq \mathbb{P}\left\{a - \|g\| \geq a - \sqrt{n}/4\right\} \leq \mathbb{P}\left\{a - \|g\| \geq s\right\} \\ &\leq \exp\left(-s^2/2\right) = \exp\left(-n/4\right),\end{aligned}$$

which completes the proof. □

Remark 3.4. Below we denote

$$\Omega_0(n, N) := \{\omega \mid \forall i \leq N : \ \sqrt{n}/4 \leq \|g_i\| \leq 2\sqrt{n}\}.$$

Lemma 3.3 yields

$$\mathbb{P}(\Omega_0(n, N)) \geq 1 - 2Ne^{-n/4}. \tag{9}$$

Note that on $\Omega_0(n, N)$ we have

$$B_2^n \subset Z_N(\omega) \subset 2\sqrt{n}B_2^n. \tag{10}$$

In fact, Gluskin proved that there exist absolute constants $C, c > 0$ such that for $Cn \leq N \leq e^n$ one has

$$c\sqrt{\log\left(\frac{N}{n}\right)}\, B_2^n \subset Z_N(\omega) \subset 2\sqrt{n} B_2^n$$

with probability at least $1 - e^{-n}$ (see [11, Theorem 2] or the remark following the proof of Theorem 2 in [10]).

The following theorem establishes a bound for the volume of projections of Gluskin's polytopes.

Theorem 3.5. *There exists an absolute constant $C > 0$ such that the following holds. Let $k \leq n$ and $2n \leq N \leq ne^k$. Then there exists a set $\Omega_1(n, N) \subset \Omega_0(n, N)$ such that for every $\omega \in \Omega_1(n, N)$ and every $Q \in \mathcal{P}^k(n)$ one has*

$$|QZ_N(\omega)|^{1/k} \leq C \max\left\{\frac{\sqrt{n}}{k}\sqrt{\log\log\log\left(\frac{N}{k}\right)},\ \frac{\sqrt{\log(\frac{N}{k})}}{\sqrt{k}}\right\} \tag{11}$$

and such that

$$\mathbb{P}(\Omega_1(n, N)) \geq 1 - 4Ne^{-n/4}.$$

To prove the theorem we will need two lemmas. The first one on the cardinality of ε-nets in $\mathcal{P}^k(n)$ is due to Szarek [31]. The second lemma bounds the volume of a polytope in terms of the lengths of the vertices.

Lemma 3.6. *There exists an absolute positive constant C_0 such that for every $0 < \varepsilon < 1$ the set $\mathcal{P}^k(n)$ admits an ε-net Π of cardinality at most*

$$|\Pi| \leq \left(\frac{C_0}{\varepsilon}\right)^{nk}.$$

Lemma 3.7. *Let $(w_i)_{i=1}^N \subset \mathbb{R}^n$ be a collection of vectors. For every $\alpha \geq \sqrt{2}\max_{i\leq N}\{\|w_i\|_2\}$ we have*

$$|\operatorname{absconv}\{w_1, \dots, w_N\}|^{1/n} \leq \frac{\sqrt{2\pi e}\alpha}{n}\exp\left(\frac{2}{n}\sum_{i=1}^N \exp\left(-\frac{\alpha^2}{2\|w_i\|_2^2}\right)\right).$$

Remark 3.8. Assuming that $\|w_i\|_2 \leq 1$ for every $i \leq N$ and letting $\alpha = \sqrt{2\log(2N/n)}$ we observe the well known bound (see [2, 6, 10])

$$|\operatorname{absconv}\{w_1, \dots, w_N\}|^{1/n} \leq \frac{C\sqrt{\log(2N/n)}}{n}. \tag{12}$$

Our proof of Lemma 3.7 follows the proof of this bound with corresponding adjustments. Note also, that using the standard estimate (12) instead of Lemma 3.7, would lead to the bound

$$|QZ_N(\omega)|^{1/k} \leq C\frac{\sqrt{n}}{k}\sqrt{\log\left(\frac{N}{k}\right)} \tag{13}$$

in Theorem 3.5, and thus, to the bound

$$\operatorname{vr}(PK, QL) \geq \frac{ck}{\sqrt{n\log\frac{n\log n}{k}}}$$

in Theorem 1.1.

Proof of Lemma 3.7. For simplicity we write $\|\cdot\|$ for $\|\cdot\|_2$. Fix $\alpha \geq \sqrt{2}\max_{i\leq N}\{\|w_i\|\}$ and set $P_i := \{x \in \mathbb{R}^n \ : \ |\langle x, w_i\rangle| \leq \alpha\}$. Consider

$$K := \frac{1}{\alpha}\bigcap_{i=1}^{N} P_i.$$

Note that $K^\circ = \text{absconv}\{w_1, \dots, w_N\}$ and that

$$\gamma_n(\alpha K) = \gamma_n\left(\bigcap_{i=1}^{N} P_i\right) \geq \prod_{i=1}^{N}\gamma_n(P_i),$$

where γ_n denotes the Gaussian measure on $\mathbb{R}^n$ and where the inequality follows from Šidák's lemma ([30], [10]) or from the Gaussian correlation inequality ([27], see also [17]).

Clearly,

$$\gamma_n(P_i) = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\alpha}{\|w_i\|}}^{\frac{\alpha}{\|w_i\|}} e^{-\frac{t^2}{2}}\,dt.$$

Considering intervals of increasing and decreasing (on $[0,\infty)$) of the function

$$f(s) := e^{-\frac{s^2}{2}} + \frac{1}{\sqrt{2\pi}}\int_{-s}^{s} e^{-\frac{t^2}{2}}\,dt,$$

it is not difficult to see that

$$\frac{1}{\sqrt{2\pi}}\int_{-s}^{s} e^{-\frac{t^2}{2}}\,dt \geq 1 - e^{-\frac{s^2}{2}}.$$

Therefore,

$$\gamma_n(\alpha K) \geq \prod_{i=1}^{N}\left(1 - e^{-\frac{\alpha^2}{2\|w_i\|^2}}\right).$$

Note that, for $x \in (0, \frac{3}{4})$, $1 - x \geq e^{-2x}$. Using that $\alpha^2 \geq 2\|w_i\|^2$ for all $i \leq N$, we obtain

$$\gamma_n(\alpha K) \geq \prod_{i=1}^{N}\exp\left(-2e^{-\frac{\alpha^2}{2\|w_i\|^2}}\right) = \exp\left(-2\sum_{i=1}^{N} e^{-\frac{\alpha^2}{2\|w_i\|^2}}\right).$$

Since $|\alpha K| = \alpha^n|K| \geq (2\pi)^{\frac{n}{2}}\gamma_n(\alpha K)$,

$$|K|^{\frac{1}{n}} \geq \frac{\sqrt{2\pi}}{\alpha}\exp\left(-\frac{2}{n}\sum_{i=1}^{N} e^{-\frac{\alpha^2}{2\|w_i\|^2}}\right).$$

Finally, the Blashke-Santaló inequality (7) implies

$$|K^\circ|^{\frac{1}{n}} \leq \frac{|B_2^n|^{\frac{2}{n}}}{|K|^{\frac{1}{n}}} \leq \frac{2\pi e\alpha}{\sqrt{2\pi}n}\exp\left(\frac{2}{n}\sum_{i=1}^{N} e^{-\frac{\alpha^2}{2\|w_i\|^2}}\right).$$

This completes the proof. □

We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. Note that if k is proportional to n (i.e., $k = cn$ where $c > 0$ is an absolute constant) the theorem follows directly from (12) and (9), so we can assume that $k < n/16$. For this case, fix $\varepsilon \in [2\sqrt{k/n}, 1/2]$ to be defined later. Let C_0 denote the constant from Lemma 3.6. Denote $C = 40C_0$ (without loss of generality we assume $C_0 \geq 5$, hence $C \geq 200$) and set

$$m_0 = \frac{Ck\log(1/\varepsilon)}{4} \quad \text{and} \quad m_1 = \frac{Ck\log(1/\varepsilon)}{4\varepsilon^2} = \frac{10C_0k\log(1/\varepsilon)}{\varepsilon^2}.$$

Without loss of generality, we just assume for simplicity that m_0 and m_1 are integers. Consider the sequence $(\lambda_m)_{m=1}^N$ defined by $\lambda_m = 2\sqrt{n}$ for $i \leq m_0$, $\lambda_m = 2\varepsilon\sqrt{n}$ for $i > m_1$, and

$$\lambda_m = \sqrt{\frac{Cnk\log(1/\varepsilon)}{m}} \quad \text{for } m_0 < i \leq m_1.$$

Let g denote a standard Gaussian vector in $\mathbb{R}^n$. Note that for any fixed projection $Q_0 \in \mathcal{P}^k(n)$, $Q_0(g)$ is a standard k-dimensional Gaussian vector. Thus, by Lemma 3.3, for every $t \geq 2\sqrt{k}$ we have

$$\mathbb{P}\left\{\omega \in \Omega \mid \|Q_0(g(\omega))\| \geq t\right\} \leq e^{-t^2/8}. \tag{14}$$

Let $g_1, \ldots, g_N$ be standard Gaussian independent vectors in $\mathbb{R}^n$. For a fixed projection Q_0 in $\mathcal{P}^k(n)$ and for $m \leq N$ consider the events

$$\mathbb{A}(m, Q_0) := \Big\{\omega \in \Omega_0(n,N) \mid \#\{i : \|Q_0(g_i(\omega))\| > \lambda_m\} \geq m\Big\}.$$

Note that on $\Omega_0(n,N)$ we have $\|g_i(\omega)\| \leq 2\sqrt{n}$, hence $A(m, Q_0) = \emptyset$ for $m \leq m_0$. By A_{Q_0} denote the union (over m) of $A(m, Q_0)$, that is

$$\begin{aligned}\mathbb{A}_{Q_0} &= \Big\{\omega \in \Omega_0(n,N) \mid \exists m \in \{1, \ldots, N\} : \ \#\{i : \|Q_0(g_i(\omega))\| > \lambda_m\} \geq m\Big\} \\ &= \Big\{\omega \in \Omega_0(n,N) \mid \exists m \in \{m_0+1, \ldots, N\} : \ \#\{i : \|Q_0(g_i(\omega))\| > \lambda_m\} \geq m\Big\}.\end{aligned}$$

To estimate the probability of A_{Q_0} we first note that

$$\frac{eN}{m_1}\, e^{-\varepsilon^2 n/2} \leq \frac{eN}{m_1}\, e^{-\varepsilon^2 n/4} \leq \frac{1}{2}. \tag{15}$$

Indeed, consider the function

$$f(\varepsilon) := \varepsilon^{-2}\, e^{\varepsilon^2 n/4}.$$

Since it is increasing on $[2/\sqrt{n}, \infty)$, using that $\varepsilon \geq 2\sqrt{k/n} \geq 2/\sqrt{n}$, we observe that

$$f(\varepsilon) \geq \frac{n}{4k} e^k.$$

Thus, using that $\varepsilon < 1/2$, $N \leq ne^k$ and $C \geq 200$,

$$\frac{eN}{m_1}\, e^{-\varepsilon^2 n/2} = \frac{4eN\varepsilon^2}{Ck\log(1/\varepsilon)}\, e^{-\varepsilon^2 n/2} \leq \frac{4ene^k}{Ck(\log 2)f(\varepsilon)} \leq \frac{16e}{C\log 2} \leq \frac{1}{2}.$$

Denote

$$p := \exp(-C_0 nk\log(1/\varepsilon)).$$

Using the union bound, the independence of g_i's, equations (14), (15), and the standard bound for $1 \leq \ell \leq N$

$$\sum_{m=0}^{\ell} \binom{N}{m} \leq \left(\frac{eN}{\ell}\right)^{\ell},$$

we obtain

$$\begin{aligned}
P(\mathbb{A}_{Q_0}) &\le \sum_{m=m_0+1}^{N} \mathbb{P}\left(\mathbb{A}(m,Q_0)\right) \le \sum_{m=m_0+1}^{N} \binom{N}{m} e^{-\lambda_m^2 m/8} \\
&\le \sum_{m=m_0+1}^{m_1} \binom{N}{m} \exp(-Cnk\log(1/\varepsilon)/8) + \sum_{m=m_1+1}^{N} \binom{N}{m} e^{-\varepsilon^2 nm/2} \\
&\le \left(\frac{eN}{m_1}\right)^{m_1} p^5 + \sum_{m=m_1+1}^{N} \left(\frac{eN}{m} e^{-\varepsilon^2 n/2}\right)^m \\
&\le \left(\frac{eN}{m_1}\right)^{m_1} p^5 + \left(\frac{eN}{m_1} e^{-\varepsilon^2 n/2}\right)^{m_1} \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \\
&= \left(\frac{eN}{m_1}\right)^{m_1} \left(p^5 + \exp(-Cnk\log(1/\varepsilon)/8)\right) \le 2\left(\frac{eN}{m_1}\right)^{m_1} p^5.
\end{aligned}$$

Using (15) again, we estimate

$$\left(\frac{eN}{m_1}\right)^{m_1} \le \exp\left(\frac{\varepsilon^2 n m_1}{4}\right) \le \exp\left(\frac{10C_0 kn\log(1/\varepsilon)}{4}\right) = p^{-2.5}.$$

Hence,

$$\mathbb{P}(\mathbb{A}_{Q_0}) \le 2p^2. \tag{16}$$

For each $\omega \in \Omega_0(n,N)$ define the vector $a = a(\omega) \in \mathbb{R}^N$ by $a_i := \|Q_0(g_i(\omega))\|$ for $i \le N$. Then, by definition, on $\Omega_0(n,N) \cap \mathbb{A}_{Q_0}^c$ we have

$$a_m^* \le \lambda_m,$$

where a^* stands for the decreasing rearrangement of a.

Similarly, given $Q \in \mathcal{P}^k(n)$ for each $\omega \in \Omega$ define the vector $b = b(\omega, Q) \in \mathbb{R}^N$ by $b_i := \|Q(g_i(\omega))\|$ for $i \le N$. Let

$$\mathbb{B} := \left\{\omega \in \Omega_0(n,N) \mid \exists Q \in \mathcal{P}^k(n)\, \exists m \le N :\ b_m^*(\omega, Q) > 2\lambda_m\right\}.$$

We now use an approximation argument. Let $Q \in \mathcal{P}^k(n)$ and consider Q_0 such that $\|Q - Q_0\| < \varepsilon$. Then,

$$\begin{aligned}
\|Q(g_i(\omega))\| &\le \|Q_0(g_i(\omega))\| + \|Q - Q_0\|\|g_i(\omega)\| \\
&\le \|Q_0(g_i(\omega))\| + \varepsilon \max \|g_i(\omega)\|_2.
\end{aligned}$$

Therefore, for $\omega \in \Omega_0(n,N) \cap \mathbb{A}_{Q_0}^c$ and for every $m \le N$ we have

$$b_m^*(\omega, Q) \le a_m^* + 2\varepsilon\sqrt{n} \le \lambda_m + 2\varepsilon\sqrt{n} \le 2\lambda_m.$$

Let $\Pi \subset \mathcal{P}^k(n)$ be an ε-net of cardinality at most $\left(\frac{C_0}{\varepsilon}\right)^{nk} \le \frac{1}{p}$ given by Lemma 3.6. Then,

$$\mathbb{B} \subset \bigcup_{Q_0 \in \Pi} A_{Q_0},$$

and therefore, by (16),

$$\mathbb{P}(\mathbb{B}) \le 2p.$$

Therefore, defining $\Omega_1(n,N) := \mathbb{B}^c \cap \Omega_0(n,N)$, by (9), we obtain

$$\mathbb{P}(\Omega_1(n,N)) \geq 1 - 2Ne^{-n/4} - 2p \geq 1 - 4Ne^{-n/4}.$$

It remains to estimate volumes of corresponding polytopes for $\omega \in \Omega_1(n,N)$. They can be written as $Q(Z_N(\omega)) = \operatorname{absconv}\{w_1, \dots, w_N\}$ with $\|w_m\| \leq 2\lambda_m$ for every $m \leq N$. We first estimate

$$\begin{aligned} A := \sum_{m=1}^{N} e^{-\frac{\alpha^2}{8\lambda_m^2}} &\leq m_0 e^{-\frac{\alpha^2}{32n}} + \sum_{m=m_0+1}^{m_1} e^{-\frac{\alpha^2 m}{8Cnk\log\left(\frac{1}{\varepsilon}\right)}} + (N-m_1)e^{-\frac{\alpha^2}{32\varepsilon^2 n}} \\ &\leq m_0 e^{-\frac{\alpha^2}{32n}} + \left(1 - e^{-\frac{\alpha^2}{8Cnk\log\left(\frac{1}{\varepsilon}\right)}}\right)^{-1} e^{-\frac{\alpha^2(m_0+1)}{8Cnk\log\left(\frac{1}{\varepsilon}\right)}} + Ne^{-\frac{\alpha^2}{32\varepsilon^2 n}} \\ &\leq \left(m_0 + \max\left\{2, \frac{16Cnk\log\left(\frac{1}{\varepsilon}\right)}{\alpha^2}\right\}\right) e^{-\frac{\alpha^2}{32n}} + Ne^{-\frac{\alpha^2}{32\varepsilon^2 n}} \\ &= \frac{Ck\log(1/\varepsilon)}{4}\left(1 + \max\left\{1, \frac{16n}{\alpha^2}\right\}\right) e^{-\frac{\alpha^2}{32n}} + Ne^{-\frac{\alpha^2}{32\varepsilon^2 n}}, \end{aligned}$$

where we used that $e^{-x} \leq \max\{1 - x/2, 1/2\}$ for $x > 0$. We choose

$$\alpha = 6\sqrt{n}\max\left\{\sqrt{\log\left(C\log\frac{1}{\varepsilon}\right)}, \sqrt{\log\frac{N}{k}}\cdot\varepsilon\right\},$$

then $\frac{2}{k}A \leq 2$. Furthermore, we choose

$$\varepsilon = \max\left\{\frac{\sqrt{\log\log\log\frac{N}{k}}}{\sqrt{\log\frac{N}{k}}}, 2\sqrt{\frac{k}{n}}\right\}$$

(recall that $k \leq n/16 \leq N/32$), then

$$\alpha \leq C_1 \max\left\{\sqrt{n\ \log\log\log\frac{N}{k}},\ 2\sqrt{k\ \log\frac{N}{k}}\right\},$$

where $C_1 > 0$ is an absolute constant. Applying Lemma 3.7 for $Q(Z_N(\omega))$ (note that $Q(Z_N(\omega))$ is k-dimensional) we obtain

$$|Q(Z_N(\omega))|^{1/k} \leq \frac{\sqrt{2\pi}e^3\alpha}{k} \leq C_2 \max\left\{\frac{\sqrt{n}}{k}\sqrt{\log\log\log\frac{N}{k}}, \frac{\sqrt{\log\frac{N}{k}}}{\sqrt{k}}\right\},$$

where $C_2 > 0$ is an absolute constant. This completes the proof. □

4. Proof of the main theorem.

We first prove a series of lemmas. Given two k-dimensional subspaces of $\mathbb{R}^n$, E and F, we denote by $\mathcal{S}(E,F)$ the set of all linear operators $T : E \to F$ preserving the volume (the k-dimensional Lebesque measure). If $E = Q_0\mathbb{R}^n$ and $F = Q_1\mathbb{R}^n$ for some $Q_0, Q_1 \in \mathcal{P}^k(n)$ we simply write $\mathcal{S}(Q_0, Q_1)$.

Lemma 4.1. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body, $Q_0, Q_1 \in \mathcal{P}^k(n)$ be fixed orthogonal projections of rank k, and $A > 0$. Let $T_0 \in \mathcal{S}(Q_0, Q_1)$ be a fixed linear operator. Then*

$$\mathbb{P}\{\omega \in \Omega \mid \|T_0 : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| \le A\} \le (A/\sqrt{2\pi})^{kN} |Q_1 K|^N.$$

Proof. Observe that

$$\begin{aligned} T_0 Q_0(Z_N(\omega)) \subset A Q_1 K &\iff Q_0 Z_N(\omega) \subset A T_0^{-1}(Q_1 K) \\ &\iff \forall i \le N \,:\, Q_0 g_i(\omega) \in A T_0^{-1}(Q_1 K). \end{aligned}$$

Note that for every k-dimensional convex body L one has $\gamma_k(L) \le (2\pi)^{-k/2}|L|$. Using this, the rotational invariance of the Gaussian measure, and the fact that T_0 preserves the Lebesgue measure in $Q_0\mathbb{R}^n$, we observe that for every $i \le N$,

$$\mathbb{P}\{\omega \in \Omega \mid Q_0 g_i(\omega) \in A T_0^{-1}(Q_1 K)\} \le (2\pi)^{-k/2} |A T_0^{-1}(Q_1 K)| = (2\pi)^{-k/2} A^k |Q_1 K|.$$

The result follows by the indepenence of g_i's. □

Lemma 4.2. *There exists and absolute constant $C > 0$ such that the following holds. Let $k \le n$, $Cn \le N \le ne^k$, and $A > 0$. Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body and $Q_0, Q_1 \in \mathcal{P}^k(n)$ be fixed orthogonal projections of rank k. Then*

$$\begin{aligned} \mathbb{P}\{\omega \in \Omega_0(n, N) \mid \exists T \in \mathcal{S}(Q_0, Q_1) \ \text{ such that } \ \|T : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| \le A\} \\ \le (5\sqrt{n})^{k^2} (A/\sqrt{2\pi})^{Nk} |Q_1 K|^N. \end{aligned}$$

Proof. Let $E := \ell_2^n \cap Q_0\mathbb{R}^n$ and let $U := B_{\mathcal{L}(E, X_{Q_1 K})}$ be the unit ball of $\mathcal{L}(E, X_{Q_1 K})$. Denote

$$a := \frac{A}{2\sqrt{n}}.$$

Let $\mathcal{N}$ be a maximal a-separated set in $AU \cap \mathcal{S}(Q_0, Q_1)$ in the metric $\|\cdot\|_{\mathcal{L}(E, X_{Q_1 K})}$. By the maximality of $\mathcal{N}$, the set $\mathcal{N}$ is an a-net for $AU \cap \mathcal{S}(Q_0, Q_1)$ and moreover, the following inclusion for the disjoint union holds,

$$\bigcup_{\eta \in \mathcal{N}} \left(\eta + \frac{a}{2} U\right) \subset \left(A + \frac{a}{2}\right) U.$$

Identifying the space with $\mathbb{R}^{k^2}$ and computing volumes we conclude that

$$\#\mathcal{N} \le \left(\frac{A + a/2}{a/2}\right)^{k^2} \le (5\sqrt{n})^{k^2}.$$

Take $\omega \in \Omega_0(n, N)$ such that there exists $T \in \mathcal{S}(Q_0, Q_1)$ with

$$\|T : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| \le A.$$

Recall that by Remark 3.4 we have on $\Omega_0(n, N)$,

$$B_2^n \subset Z_N(\omega) \subset 2\sqrt{n} B_2^n,$$

hence $T \in AU$. Since $\mathcal{N}$ is an a-net for $AU \cap \mathcal{S}(Q_0, Q_1)$ there is $S \in \mathcal{N}$ such that

$$\|S - T : E \to X_{Q_1 K}\| \le a.$$

Using that, $Z_N(\omega) \subset 2\sqrt{n} B_2^n$,

$$\begin{aligned} \|S : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| &\le \|S - T : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| + \|T : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| \\ &\le 2\sqrt{n}\, \|S - T : E \to X_{Q_1 K}\| + A \le 2\sqrt{n} a + A = 2A. \end{aligned}$$

This shows

$$\{\omega \in \Omega_0(n,N) \mid \exists T \in \mathcal{S}(Q_0,Q_1) \text{ such that } \|T : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| \le A\} \subset \bigcup_{S\in\mathcal{N}} \{S \mid \|S : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| \le 2A\}.$$

Using the union bound and applying Lemma 4.1, we obtain the desired bound. □

Given bases $B = \{v_1, \dots, v_k\}$ and $B' = \{v_1', \dots, v_k'\}$ of vector spaces F and F' and a vector $x \in F$ we denote by $(x)_B$ the coordinates of x in the basis B (similarly, $(y)_{B'}$ for $y \in F'$). That is, $(x)_B = (\alpha_1, \dots, \alpha_k)$ if $x = \sum_{i=1}^k \alpha_i v_i$. Also for an operator $T : F \to F'$ we denote by $[T]_{B,B'}$ the matrix $(a_{i,j})_{1\le i,j\le k}$ such that $T(v_\ell) = \sum_{i=1}^k a_{i,\ell} v_i'$, for every $1 \le \ell \le k$ (i.e., the ℓ-column of $[T]_{B,B'}$ is $(Tv_\ell)_{B'}^t$).

Lemma 4.3. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body in John's position. Then for every $\beta > 0$ one has*

$$\mathbb{P}\left\{\omega \in \Omega_0(n,N) \,|\, \exists Q_0, Q_1 \in \mathcal{P}^k(n)\, \exists T \in \mathcal{S}(Q_0,Q_1) \,:\, \|T : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| \le \frac{\beta}{|Q_1 K|^{\frac{1}{k}}}\right\} \le C^{kN} (\sqrt{n})^{2nk+k^2} \beta^{kN},$$

where $C > 0$ is an absolute constant.

Proof. By Lemma 3.6 there is a $\frac{1}{2\sqrt{n}}$-net, say Π, for $\mathcal{P}^k(n)$ of cardinality $\#\Pi \le (C_0\sqrt{n})^{nk}$. By Lemma 4.2 and the union bound, it is enough to show that

$$\left\{\omega \in \Omega_0(n,N) \,|\, \exists Q_0, Q_1 \in \mathcal{P}^k(n),\ \exists T \in \mathcal{S}(Q_0,Q_1) \,:\, \|T : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| \le \frac{\beta}{|Q_1 K|^{\frac{1}{k}}}\right\}$$
$$\subset \bigcup_{Q_0', Q_1' \in \Pi} \left\{\omega \in \Omega_0(n,N) \,|\, \exists S \in \mathcal{S}(Q_0',Q_1') \,:\, \|S : X_{Q_0' Z_N(\omega)} \to X_{Q_1' K}\| \le C' \frac{\beta}{|Q_1' K|^{\frac{1}{k}}}\right\}.$$

Let $\omega \in \Omega_0(n,N)$ be such that there are $Q_0, Q_1 \in \mathcal{P}^k(n)$ and $T \in \mathcal{S}(Q_0,Q_1)$ with

$$\|T : X_{Q_0 Z_N(\omega)} \to X_{Q_1 K}\| \le \frac{\beta}{|Q_1 K|^{\frac{1}{k}}}. \tag{17}$$

Take $Q_0', Q_1' \in \Pi$ such that $\|Q_i - Q_i'\| \le \frac{1}{2\sqrt{n}}$, $i = 1, 2$. Fix orthonormal bases

$$B = \{v_1, \dots, v_k\} \qquad \text{and} \qquad B' = \{v_1', \dots, v_k'\}$$

of $Q_0\mathbb{R}^n$ and $Q_1\mathbb{R}^n$ respectively. It is easy to see that the collections

$$B_0 = \{Q_0' v_1, \dots, Q_0' v_k\} \qquad \text{and} \qquad B_1 = \{Q_1' v_1', \dots, Q_1' v_k'\}$$

are bases of $Q_0'\mathbb{R}^n$ and $Q_1'\mathbb{R}^n$ respectively. Let $S : Q_0'\mathbb{R}^n \to Q_1'\mathbb{R}^n$ be such that

$$[S]_{B_0,B_1} = [T]_{B,B'},$$

in particular, $S \in \mathcal{S}(Q_0', Q_1')$.

It is enough to show that

$$\|S : X_{Q_0' Z_N(\omega)} \to X_{Q_1' K}\| \le \frac{C'\beta}{|Q_1' K|^{\frac{1}{k}}},$$

which by Lemma 3.2 reduces to

$$\|S : X_{Q_0' Z_N(\omega)} \to X_{Q_1' K}\| \leq \frac{C'\,\beta}{2|Q_1 K|^{\frac{1}{k}}}.$$

Take $x \in Z_N(\omega)$ and note

$$SQ_0'x = \underbrace{SQ_0'(Q_0'x - Q_0 x)}_{(1)} + \underbrace{SQ_0'Q_0 x}_{(2)}.$$

We check that both terms (1) and (2) are contained in a multiple of $\frac{\beta}{|Q_1K|^{\frac{1}{k}}} Q_1' K$.

We start with the second term, $SQ_0'Q_0x$. Write $Q_0x = \sum \alpha_i v_i$, so $Q_0'Q_0x = \sum \alpha_i Q_0' v_i$. We have,

$$\begin{aligned}(SQ_0'Q_0x)^t_{B_1} &= [T]_{B,B'}(Q_0'Q_0x)^t_{B_0}\\ &= [T]_{B,B'}(Q_0x)^t_{B}\\ &= (TQ_0x)^t_{B'}.\end{aligned}$$

Therefore $SQ_0'Q_0x = Q_1'TQ_0x$. Since $x \in Z_N(\omega)$, by (17) we have $TQ_0x \in \frac{\beta}{|Q_1K|^{\frac{1}{k}}} Q_1 K$, hence

$$Q_1'TQ_0x \in \frac{\beta}{|Q_1K|^{\frac{1}{k}}} Q_1'Q_1K.$$

Since K is in John's position, $B_2^n \subset K \subset \sqrt{n}B_2^n$. Using $\|Q_1 - Q_1'\| \leq \frac{1}{2\sqrt{n}}$, we obtain

$$\begin{aligned}Q_1'Q_1K &\subset Q_1'K + Q_1'((Q_1 - Q_1')K)\\ &\subset Q_1'K + Q_1'((Q_1 - Q_1')\sqrt{n}B_2^n)\\ &\subset Q_1'K + Q_1'B_2^n\\ &\subset 2Q_1'K.\end{aligned} \tag{18}$$

This implies

$$SQ_0'Q_0x = Q_1'TQ_0x \in \frac{\beta}{|Q_1K|^{\frac{1}{k}}} Q_1'Q_1K \subset \frac{2\beta}{|Q_1K|^{\frac{1}{k}}} Q_1'K,$$

proving the inclusion for term (2).

Next we deal with the first term, $SQ_0'(Q_0'x - Q_0x)$. Recall that $Z_N(\omega) \subset 2\sqrt{n}B_2^n$ on $\Omega_0(n, N)$ and $\|Q_0 - Q_0'\| \leq \frac{1}{2\sqrt{n}}$. Therefore $(Q_0' - Q_0)x \in B_2^n$ and then $Q_0'(Q_0' - Q_0)x \in B_2^k$. Thus, it is enough to show that

$$\|S : X_{Q_0'B_2^n} \to X_{Q_1'K}\| \leq C'' \frac{\beta}{|Q_1K|^{\frac{1}{k}}}.$$

Take $y \in Q_0'\mathbb{R}^n$ with $\|y\|_2 = 1$. Write $(y)_{B_0} = (\beta_1, \ldots, \beta_k)$. Then,

$$(\gamma_1, \ldots, \gamma_k) := (Sy)_{B_1} = [T]_{B,B'}(\beta_1, \ldots, \beta_k)^t = (T(\sum \beta_i v_i))_{B'}. \tag{19}$$

Notice that

$$\begin{aligned}\|\sum \beta_i v_i\|_2 &\leq \|\sum \beta_i Q_0 v_i - \sum \beta_i Q_0' v_i\|_2 + \|\sum \beta_i Q_0' v_i\|_2\\ &\leq \frac{1}{2\sqrt{n}} \|\sum \beta_i v_i\|_2 + 1,\end{aligned}$$

which implies

$$\|\sum \beta_i v_i\|_2 \leq \frac{1}{1-\frac{1}{2\sqrt{n}}} \leq 2.$$

Since by (10) we have $B_2^n \subset Z_N(\omega)$ on $\Omega_0(n,N)$, using (17), we observe

$$T(\sum \beta_i v_i) \in \frac{2\beta}{|Q_1K|^{\frac{1}{k}}} Q_1K. \tag{20}$$

On the other hand, by (19),

$$Sy = \sum \gamma_i Q_1' v_i' = Q_1'(\sum \gamma_i v_i') = Q_1' T(\sum \beta_i v_i).$$

Therefore, using (20) and (18),

$$Sy \in \frac{2\beta}{|Q_1K|^{\frac{1}{k}}} Q_1' Q_1 K \subset \frac{4\beta}{|Q_1K|^{\frac{1}{k}}} Q_1K.$$

This completes the proof. □

We are now ready to prove our main result.

Proof of Theorem 1.1. First note that it is enough to consider orthogonal projections only. Indeed, let P be a projection of rank k and Q be the orthogonal projection with the same kernel as P. Then $QP = Q$, hence $Q(PK) = QK$ and $\mathrm{vr}(PK, W) = \mathrm{vr}(QK, W)$ for every convex body W.

Furthermore, by (8) for every $Q_0, Q_1 \in \mathcal{P}^k(n)$ we have

$$\mathrm{vr}(Q_0(K-K), Q_1Z_N) \leq \mathrm{vr}(Q_0(K-K), Q_0K)\,\mathrm{vr}(Q_0K, Q_1Z_N) \leq 4\,\mathrm{vr}(Q_0K, Q_1Z_N).$$

Thus it is enough to estimate from below $\mathrm{vr}(Q_0(K-K), Q_1Z_N)$. In other words, without loss of generality, we may assume that K is centrally symmetric. Moreover, since volume ratio is an affine invariant, we may also assume that K is in John's position.

Let $N = n\log n$ and β a positive constant to be chosen later. Let $\Omega_1(n,N)$ be the set given by Theorem 3.5 (note that for $k \leq \sqrt{n}$ the result is trivial, so we may assume that $k \geq \sqrt{n}$ and thus the assumption on N in Theorem 3.5 is satisfied). Consider the event

$$\mathcal{E}_\beta := \left\{ \exists Q_0, Q_1 \in \mathcal{P}^k(n),\ \exists T \in \mathcal{S}(Q_0,Q_1)\ :\ \|T : X_{Q_0Z_N(\omega)} \to X_{Q_1K}\| \leq \frac{\beta}{|Q_1K|^{\frac{1}{k}}} \right\}.$$

Since $\Omega_1(n,N) \subset \Omega_0(n,N)$, Lemma 4.3 yields that for some absolute constant $C > 0$,

$$\mathbb{P}\left(\mathcal{E}_\beta \bigcap \Omega_1(n,N)\right) \leq C^{kN} (\sqrt{n})^{2nk+k^2} \beta^{kN} \leq (C\beta)^{kN}\, n^{2nk}.$$

Choose β as $C^{-1}e^{-3} := \tilde{c}$. Then, using Theorem 3.5 we obtain

$$\mathbb{P}(\mathcal{E}_\beta) \leq e^{-Nk} + \mathbb{P}(\Omega_1(n,N)) \leq e^{-Nk} + 4Ne^{-n/4} \leq 5n\log(n)\, e^{-n/4} < 1.$$

Thus there is $\omega \in \Omega_1(n, n\log(n))$ such that for every $Q_0, Q_1 \in \mathcal{P}^k(n)$ and $T \in \mathcal{S}(Q_0,Q_1)$,

$$\|T : X_{Q_0Z_N(\omega)} \to X_{Q_1K}\| \geq \frac{\tilde{c}}{|Q_1K|^{\frac{1}{k}}}.$$

Using Fact 2.1 (1) and Theorem 3.5, we conclude that

$$\begin{aligned}\mathrm{vr}(Q_1K, Q_0Z_N) \geq & \frac{|Q_1K|^{\frac{1}{k}}}{|Q_0Z_N|^{\frac{1}{k}}}\ \frac{\tilde{c}}{|Q_1K|^{\frac{1}{k}}} \\ \geq & c \min\left\{\frac{k}{\sqrt{n}\sqrt{\log\log\log\left(\frac{N}{k}\right)}}, \frac{\sqrt{k}}{\sqrt{\log(\frac{N}{k})}}\right\}\end{aligned}$$

which proves the desired result. □

Finally, for the sake of completeness, we prove Corollary 1.2.

Proof of Corollary 1.2. Let E, F be k-dimensional subspaces $\mathbb{R}^n$. Applying Theorem 1.1 and Fact 2.1 (2) for K°, there is a centrally symmetric body W such that

$$\begin{aligned}\min\left\{\frac{k}{\sqrt{n}}\cdot\sqrt{\frac{1}{\log\log\log(\frac{n\log(n)}{k})}}, \frac{\sqrt{k}}{\sqrt{\log(\frac{n\log(n)}{k})}}\right\} & \lesssim \mathrm{vr}\left(P_EK^\circ, P_FW\right) \\ & \sim \mathrm{vr}\left((P_FW)^\circ, (P_EK^\circ)^\circ\right) \\ & = \mathrm{vr}\left(F\cap W^\circ, E\cap K\right),\end{aligned}$$

where we used that $(P_EK^\circ)^\circ = E\cap K$ and $(P_FW)^\circ = E\cap W^\circ$. This completes the proof. □

5. Sharpness.

We will use the following Rudelson's result proved in [29] (see the first page of the Section 4 in that paper, p. 1077).

Theorem 5.1. *Let $1 \leq k \leq n/16$. Let $L \subset \mathbb{R}^n$ be a centrally symmetric convex body. Then there are a parameter $t = t(L)$ and a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ of rank k such that*

$$c\left(B_1^k + \frac{t}{\sqrt{n}\,\log n}B_2^k\right) \subset TL \subset C\left(tB_1^k + \sqrt{\frac{k}{n}}B_2^k\right) \subset C\left(t + \frac{k}{\sqrt{n}}\right)B_1^k,$$

where $C > c > 0$ are absolute constants.

As a consequence of Rudelson's theorem we obtain the following bound.

Corollary 5.2. *Let $1 \leq k \leq n$. Let $L \subset \mathbb{R}^n$ be a convex body. Then there is a k-dimensional projection Q such that*

$$\mathrm{vr}(B_1^k, QL) \leq C\max\left\{\frac{k}{\sqrt{n}}, \sqrt{\frac{n}{k}}\log n\right\},$$

where $C > 0$ is an absolute constant.

Before providing the proof of Corollary 5.2 we make two important remarks.

Remark 5.3. Recall that by (5) and Remark 2.2, for every convex body $L \subset \mathbb{R}^n$ and every projection Q of rank k,

$$\mathrm{vr}(B_1^k, QL) \leq \mathrm{vr}(B_1^k, QL - QL)\,\mathrm{vr}(QL - QL, QL) \leq 4\sqrt{ek}. \tag{21}$$

Thus Corollary 5.2 implies that for every convex body $L \subset \mathbb{R}^n$ there exists a k-dimensional projection Q such that

$$\mathrm{vr}(B_1^k, QL) \leq C \begin{cases} \frac{k}{\sqrt{n}} & \text{if } k \geq n^{2/3}(\log n)^{2/3} \\ \sqrt{\frac{n}{k}} \log n & \text{if } \sqrt{n} \log n < k \leq n^{2/3}(\log n)^{2/3} \\ \sqrt{k} & \text{if } k \leq \sqrt{n} \log n. \end{cases}$$

Remark 5.4. Clearly, B_1^k can be realized as a (coordinate) projection of $K = B_1^n$. Thus Corollary 5.2 shows sharpness of Theorem 1.1 (up to logarithmic factors) in the regime $k \geq n^{2/3}$. Note that Rudelson's bound (2) has the same phase transition $k \sim n^{2/3}$.

Proof of Corollary 5.2. Clearly we may assume that $k \leq n/16$ (otherwise we may use (21)).

Using again Remark 2.2, which implies that for every projection Q of rank k,

$$\mathrm{vr}(B_1^k, QL) \leq \mathrm{vr}(B_1^k, QL - QL)\, \mathrm{vr}(QL - QL, QL) \leq 4\, \mathrm{vr}(B_1^k, QL - QL),$$

without lost of generality we assume that L is centrally symmetric.

Let T be a projection given by Theorem 5.1. We consider two cases. First assume that $t(L) \leq k/\sqrt{n}$. In this case

$$c\, B_1^k \subset TL \subset 2C\, \frac{k}{\sqrt{n}} B_1^k.$$

This implies

$$\mathrm{vr}(B_1^k, TL) \leq \frac{2C\, k}{c\sqrt{n}}.$$

The second case is $t(L) > k/\sqrt{n}$. In this case we have

$$\frac{c\, t}{\sqrt{n}\, \log n} B_2^n \subset TL \subset 2tC B_1^k.$$

This implies

$$\mathrm{vr}(B_1^k, TL) \leq \frac{2C\sqrt{n}\, \log n}{c} \left(\frac{|B_1^k|}{|B_2^k|} \right)^{1/k} \leq \frac{2C\sqrt{n}\, \log n}{c\sqrt{k}}.$$

Finally, note that, similarly to the beginning of the proof of Theorem 1.1, any image of a convex body under a linear operator of rank k is on the Banach–Mazur distance 1 to an image of the body under a projection having the same kernel. Since volume ratio is an affine invariant, this completes the proof. □

6. Concluding remarks.

In fact our results can be interpreted in terms of the following parameter of convex bodies.

Let $1 \leq k \leq n$, for a given convex body $K \subset \mathbb{R}^n$ define its k-projection volume ratio as

$$\mathrm{pvr}_k(K) = \sup_L \inf_{P,Q} \mathrm{vr}(PK, QL).$$

where the supremum is taken over all convex bodies $L \subset \mathbb{R}^n$ and the infimum is taken over all projections P, Q of rank k. Given two bodies K and L, note that the quantity $\inf_{P,Q} \mathrm{vr}(PK, QL)$ measures how close k-dimensional projections of the bodies can be (in terms of the volume ratio). So, $\mathrm{pvr}_k(K)$ provides the worst estimate of this measure that works for any body L. In this terminology, Theorem 1.1 says that

$$\mathrm{pvr}_k(K) \gtrsim \min\left\{\frac{k}{\sqrt{n}}\cdot\sqrt{\frac{1}{\log\log\log(\frac{n\log(n)}{k})}}, \frac{\sqrt{k}}{\sqrt{\log(\frac{n\log(n)}{k})}}\right\},$$

while Remark 5.3 states

$$\mathrm{pvr}_k(B_1^n) \lesssim \begin{cases} \frac{k}{\sqrt{n}} & \text{if } k \geq n^{2/3}(\log n)^{2/3} \\ \sqrt{\frac{n}{k}}\log n & \text{if } \sqrt{n}\log n < k \leq n^{2/3}(\log n)^{2/3} \\ \sqrt{k} & \text{if } k \leq \sqrt{n}\log n. \end{cases}$$

Note also that (3) implies that for every convex body $K \subset \mathbb{R}^n$,

$$\mathrm{pvr}_k(K) \lesssim \sqrt{k}\,\log k,$$

while (12) implies that

$$\mathrm{pvr}_k(B_2^n) \geq \inf_{\mathrm{rk}\ Q=k} \mathrm{vr}(B_2^k, QB_1^n) \gtrsim \sqrt{\frac{k}{\log(2n/k)}},$$

which in particular shows that up to logarithmic factors B_2^n maximizes $\mathrm{pvr}_k(\cdot)$ and that, in general, $\mathrm{pvr}_k(K)$ could be significantly larger than $k/\sqrt{n}$ even for $k \geq n^{2/3}$.

Finally let us note that it would be natural to consider the following counterpart of the previous parameter. For $1 \leq k \leq n$ and a convex body $K \subset \mathbb{R}^n$ we define the k-projection outer volume ratio as

$$\mathrm{povr}_k(K) = \sup_L \inf_{P,Q} \mathrm{vr}(PL, QK),$$

where as before the supremum is taken over all convex bodies $L \subset \mathbb{R}^n$ and the infimum is taken over all projections P, Q of rank k.

Note that by Dvoretzky theorem for $1 \leq k \leq c\log n$,

$$\mathrm{povr}_k(B_2^n) \leq 2.$$

In the next theorem we show that this quantity is also bounded when k is proportional to n.

Theorem 6.1. *Let $0 < \lambda \leq 1$ There exists a constant $C(\lambda) > 0$ depending only on λ such that if $k = \lambda n$ then*

$$\mathrm{povr}_k(B_2^n) \leq C(\lambda).$$

Proof. Let $L \in \mathbb{R}^n$ be a convex body. Applying an affine transformation if needed we can assume that L is in M-position, which means that $|L| = |B_2^n|$, that L can be covered by e^{Cn} translates of B_2^n and that B_2^n can be covered by e^{Cn} translates of L. We refer to [1, Chapter 8] and to [26, Chapter 7] for several equivalent definitions of M-position, its existence, and for basic properties of convex bodies in M-position. Note that the existence of M-position in the non-symmetric case was first established in [25, 28]. By [1, Theorem 8.5.4] such a position exists for every convex body.

Now, Theorem 8.6.1 in [1] implies that there exists a projection P of rank $k = \lambda n$ such that the body PL has its volume ratio bounded by a constant depending only on λ (in fact, it is true for "most" projections). This implies the desired result. □

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