On the asymmetry constant of a body with few vertices.

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Abstract

In this note we show that a non-degenerated polytope in $I\!R^n$ with $n+k$, $1 \leq k < n$, vertices is far from any symmetric body. We provide the asymptotically sharp estimates for the asymmetry constant of such polytopes.

0 Introduction and notations

The canonical Euclidean inner product in $I\!R^n$ is denoted by $\langle \cdot, \cdot \rangle$, the norm in ℓ_p is denoted by $\| \cdot \|_p$, $1 \leq p \leq \infty$.

By a convex body $K \subset I\!R^n$ we shall always mean a compact convex set with the non-empty interior.

By a non-degenerated polytope we mean a convex polytope with the non-empty interior.

Given convex bodies K, L in $I\!R^n$, we define the geometric distance by

$$\widetilde{d}(K, L) = \inf\{\alpha\,\beta \mid \alpha > 0, \beta > 0, (1/\beta)L \subset K \subset \alpha L\}.$$

Denote by $\mathcal{C}^n$ the set of all centrally symmetric with respect to the origin convex bodies in $I\!R^n$. For a convex body K in $I\!R^n$ we define the asymmetry constant $\delta(K)$ of a convex body K as follows

$$\delta(K) := \widetilde{d}(K, \mathcal{C}^n) = \inf\left\{\widetilde{d}(K - a, B) \mid a \in I\!R^n, B \in \mathcal{C}^n\right\}.$$

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This constant is one of the possible ways to measure the asymmetry of a given body. We refer to [5] for a detail discussion on the problem of the measure of asymmetry.

By compactness, there exist $a \in K$ and a centrally symmetric convex body $B \subset I\!R^n$ such that $\delta(K) = \delta_a(K) := \widetilde{d}(K-a, B)$. Observe that we also have

$$\begin{aligned} \delta(K) &= \widetilde{d}(K-a, (K-a) \cap -(K-a)) \\ &= \widetilde{d}(K-a, \mathrm{conv}\,\{(K-a) \cup -(K-a)\}). \end{aligned}$$

That is, $(K-a) \cap -(K-a)$ and $\mathrm{conv}\,\{(K-a) \cup -(K-a)\}$ are two centrally symmetric bodies closest to K. Also note that for any $a \in K$ we have

$$\frac{1}{2}(K-K) \subset \mathrm{conv}\,\{(K-a) \cup -(K-a)\} \subset K-K.$$

During the last few years the main interest of the local theory of the Banach spaces trended to the study of high dimensional non-symmetric convex bodies (see e.g. [1]-[10]). Particularly Lassak ([6]) investigated the maximal distance between a centrally symmetric convex body and an arbitrary convex body in $I\!R^n$. From the other hand in [3], [7] the behavior of the asymmetry constant of non-symmetric bodies and their projections was investigated from the point of view of asymptotic Banach space theory. It is well known (and can be directly computed) that on the class of all n-dimensional convex bodies, the asymmetry constant obtains its maximum value n on the simplex which is a convex hull of $n+1$ affine independent points in $I\!R^n$ (see [5], cf. [9]). From the other hand the n-dimensional octahedron provide us with an example of a symmetric body with $2n$ vertices. It is very natural to ask now how small could be the asymmetry constant of n-dimensional convex polytope with $n+k$ vertices for k between 1 and n. However we couldn't find the discussion of this question in literature. The purpose of this note is to answer this question.

1 The estimate

Lemma 1.1 *Let $A = \{y_{ij}\}$ be an $m \times m$ matrix. Let R be the rank of A and T be the absolute value of trace of A. Then*

$$\sup_{i \le m} \sum_{j \le m} |y_{ij}| \ge T/R.$$

Proof: Let $\{\lambda_j\}_1^R$ be the non-zero eigenvalues of A. Then $|\sum \lambda_j| = T$ and hence $\max|\lambda_j| \geq T/R$. Clearly, for every Banach space X and every linear operator A on it the operator norm $\|A : X \longrightarrow X\|$ is not less than absolute value of any eigenvalue of A. In particular, $\|A : \ell_\infty \longrightarrow \ell_\infty\| \geq T/R$. But $\|A : \ell_\infty \longrightarrow \ell_\infty\| = \sup_i \sum_j |y_{ij}|$. That proves the lemma. □

Theorem 1.2 *Let $n \geq 1$, $1 \leq k < n$ be positive integers. Let K be a non-degenerated polytope with $n+k$ vertices in $I\!R^n$. Then*

$$\delta(K) \ \geq \frac{n}{k}.$$

From the other hand there exists a non-degenerated polytope with $n+k$ vertices K with

$$\delta(K) \ \leq \left[\frac{n}{k}\right] + \theta,$$

where as usual $[s]$ denotes the largest integer not exceeding s and $\theta = 0$ if $\frac{n}{k}$ is integer, $\theta = 1$ otherwise.

Proof: Let $m = n + k$, and let $K = \operatorname{conv}\{x_i\}_{i \leq m}$ be a non-degenerated polytope in $I\!R^n$. Without loss of generality we assume that $\delta(K)$ attains its value when center is chosen to be 0, i.e. $\delta(K) = \delta_0(K)$ (otherwise substitute the polytope K with $K - a = \operatorname{conv}\{x_i - a\}_{i \leq m}$, where a is the "right" center).

Consider the linear operator $T : I\!R^m \longrightarrow I\!R^n$ defined by $Te_i = x_i$, where $\{e_i\}_i$ is the canonical basis of $I\!R^m$. Denote the kernel of the operator T by L. Clearly, L is a k-dimensional subspace of $I\!R^m$.

Then we have

$$\begin{aligned}
\delta(K) &= \sup\left\{\sup_{i \leq m}\langle f, -x_i\rangle \mid f \in I\!R^n, \langle f, x_i\rangle \leq 1, i \leq m\right\} \\
&= \sup\left\{\sup_{i \leq m}\langle h, -e_i\rangle \mid h \in I\!R^m, h \in L^\perp, \langle h, e_i\rangle \leq 1, i \leq m\right\}.
\end{aligned}$$

Denote the set $\{h \in I\!R^m, \langle h, e_i\rangle \leq 1, i \leq m\}$ by S. Since $\delta(K) \geq 1$, we obtain

$$\delta(K) = \sup_{h \in S \cap L^\perp} \|h\|_\infty.$$

Passing to the dual problem we get

$$\delta(K) = \sup_{\|x\|_1} \inf_{y \in L} \sup_{h \in S}\langle h, x - y\rangle = \sup_{x = \pm e_j} \inf_{y \in L} \sup_{h \in S}\langle h, x + y\rangle.$$

Note, that $\sup_{h\in S}\langle h,z\rangle = \sum z_i$ if z_i are all positive and is infinity otherwise. Thus the last expression is equal

$$\sup_i \left\{ \inf \left\{ \sum_{j=1}^{m} y_j - 1 \mid y = \{y_j\} \in L, y_j \geq 0, j \leq m, y_i \geq 1 \right\} \right\}.$$

Let for every i the last infimum is attained on a vector $y_i = \{y_{ij}\}_j \in L$. Consider the matrix A with the entries y_{ij}. Then the rank of A is less or equal to the dimension of L, i.e. less or equal to k, and the trace of A is not less than m. By Lemma 1.1 there is i such that $\sum_j |y_{ij}| \geq m/k$. It implies

$$\delta(K) = \sup_{i\leq m} \sum_{j\leq m} |y_{ij}| - 1 \geq \frac{n}{k}.$$

To show the second part of theorem, let us note that it is enough to construct the matrix A of rank k with the entries y_{ij} satisfying $y_{ij} \geq 0$, $y_{ii} \geq 1$, and $\sum_j y_{ij} \leq \left[\frac{m}{k}\right] + \theta$, as can been seen from the proof above. To do this, one can consider a block-diagonal matrix with k blocks of rank one, consisting of entries of value one only. This proves the theorem. □

Remark 1. The example of the sharpness of the estimate can be described in geometric terms as follows. Consider the expansion $\mathbb{R}^n = \mathbb{R}^m \oplus ... \oplus \mathbb{R}^m \oplus \mathbb{R}^{m+1} \oplus ... \oplus \mathbb{R}^{m+1}$, where $m = [n/k]$. In each copy of $\mathbb{R}^l$ ($l = m,\ m+1$) take a non-degenerated simplex, i.e. convex hull of $l+1$ affine independent points. Take a convex hull K of those simplexes in $\mathbb{R}^n$. Clearly, the body K is the desired polytope.

Remark 2. It is possible to show that

$$\inf_K \delta(K) = \left[\frac{n+1}{2}\right],$$

where infimum is taken over all non-generated polytopes K in $\mathbb{R}^n$ with $n+2$ vertices.

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