Around the simplex mean width conjecture

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Abstract

In this note we discuss an old conjecture in Convex Geometry asserting that the regular simplex has the largest mean width among all simplices inscribed into the Euclidean ball and its relation to Information Theory. Equivalently, in the language of Gaussian processes, the conjecture states that the expectation of the maximum of $n+1$ standard Gaussian variables is maximal when the expectations of all pairwise products are $-1/n$, that is, when the Gaussian variables form a regular simplex in L_2. We mention other conjectures as well, in particular on the expectation of the smallest (in absolute value) order statistic of a sequence of standard Gaussian variables (not necessarily independent), where we expect the same answer.

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1 Introduction

By B_2^n and S^{n-1} we denote the standard unit Euclidean ball and the unit Euclidean sphere in $\mathbb{R}^n$. Then $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the corresponding Euclidean norm and inner product. By $\{e_i\}_{i=1}^n$ we denote the canonical basis of $\mathbb{R}^n$ and by Δ_n we denote the regular simplex inscribed into S^{n-1}. Given a convex body K in $\mathbb{R}^n$, its support function and mean width are defined by

$$h_K(u) = \max_{x \in K} \langle u, x \rangle \qquad \text{and} \qquad w(K) = 2 \int_{S^{n-1}} h_K(u)\, d\mu(u),$$

where μ is the normalized Lebesgue measure on S^{n-1}.

In this note we discuss the following long-standing open conjecture and some related results and conjectures.

Conjecture 1.1 *Among all simplices inscribed into B_2^n the regular simplex Δ_n has the maximal mean width. Moreover, Δ_n is the unique simplex maximizing mean width.*

This conjecture was briefly discussed in a survey of P. Gritzmann and V. Klee ([24], Section 9.10.2) and was mentioned several times by V. Klee in his talks. It was also mentioned in K. Böröczky's book [11] and in recent works by D. Hug and R. Schneider [25] and by K. Böröczky and R. Schneider [12]. We refer to [10] for related results on the mean width of a simplex and to [41] for the general theory of convex bodies and Brunn–Minkowski Theory. In the Information Theory community it was a general belief that the conjecture is known to be true (see e.g. [2, 3, 16, 45, 51]). We discuss the importance of Conjecture 1.1 to Information Theory in Section 2.

We would like to emphasize an important difference in the setting of this problem with a standard setting of problems in Asymptotic Geometric Analysis, where we usually identify bodies which can be obtained from each other by an affine transformation (or, in the centrally symmetric case, by a linear transformation), working thus with equivalence classes of bodies. In many problems we usually fix *a position* of a body, where by a position of a body we understand a certain affine (linear in the centrally symmetric case) image of it. In this context Conjecture 1.1 has to be compared with the following results by F. Barthe [8] (who developed the approach originated by K. Ball in [5, 6] to describe bodies with the maximal volume ratio and surface area) and by M. Schmuckenschläger [40] respectively, investigating maximizers and minimizers of mean width of bodies in John's and Löwner's positions.

Theorem 1.2 *Among all convex bodies in the John position, that is, bodies, whose maximal volume ellipsoid is the unit Euclidean ball B_2^n, the regular simplex $n\Delta_n$ has the largest mean width.*

Theorem 1.3 *Among all convex bodies in the Löwner position, that is, bodies, whose minimal volume ellipsoid is the unit Euclidean ball B_2^n, the regular simplex Δ_n has the smallest mean width.*

Corresponding statements for the class of centrally symmetric bodies were proved by G. Schechtman and M. Schmuckenschläger [39]. The maximizer of the mean width among all centrally symmetric bodies in the John position is the cube, while the minimizer of the mean width among all centrally symmetric bodies in the Löwner position is the cross-polytope (octahedron).

We now reformulate the conjecture in the language of Gaussian processes. By $g_1, g_2, g_3, \dots$ we always denote i.i.d. standard Gaussian random variables ($g_i \sim \mathcal{N}(0,1)$). $G = (g_1, \dots, g_n)$ denotes the standard Gaussian random vector in $\mathbb{R}^n$. The integration in polar coordinates leads to

$$w(K) = c_n \, \mathbb{E} \, h_K(G), \tag{1}$$

where c_n is a constant depending only on n (in fact, $c_n \geq \frac{2}{\sqrt{n}}$ and $c_n\sqrt{n} \to 2$).

When $K = \operatorname{conv}\{x_1, \dots, x_{n+1}\} \subset \mathbb{R}^n$, $|x_i| = 1$ for $i \leq n+1$, we have

$$\mathbb{E} \, h_K(G) = \mathbb{E} \max_{i \leq n+1} \langle G, x_i \rangle .$$

Denote $\xi_i = \langle G, x_i \rangle$, $i \leq n+1$. Then $\xi_i \sim \mathcal{N}(0,1)$ and

$$\sigma_{ij} = \sigma_{ij}(K) := \mathbb{E}\xi_i\xi_j = \langle x_i, x_j \rangle .$$

Recall, that the vertices $v_1, \dots, v_{n+1}$ of Δ_n satisfy

$$|v_i| = 1 \qquad \text{and} \qquad \langle v_i, v_j \rangle = -\frac{1}{n} \quad \text{for all} \quad i \neq j.$$

The $(n+1) \times (n+1)$ covariance matrix corresponding to the regular simplex, that is, $\sigma = \{\sigma_{ij}\}_{ij}$ with $\sigma_{ii} = 1$ and $\sigma_{ij} = -\frac{1}{n}$ for $i \neq j$, we denote by $\sigma(\Delta_n)$. Thus Conjecture 1.1 is equivalent to

Conjecture 1.4 *Among all Gaussian random vectors* $(\xi_1, \dots, \xi_{n+1})$ *with* $\xi_i \sim \mathcal{N}(0,1)$ *for all* $i \leq n+1$, *the expectation*

$$\mathbb{E} \max_{i \leq n+1} \xi_i$$

is maximal when the covariance matrix $\sigma = \sigma(\Delta_n)$. *The solution is unique.*

We would like to emphasize that if we add the absolute values to ξ_i's, that is, if we want to maximize $\mathbb{E} \max_{i \leq n+1} |\xi_i|$, then the answer is known – the maximum of such expectation attains when ξ_i's are independent as was proved by Z. Šidák [43] and E.D. Gluskin [21]. Geometrically it says the following.

Theorem 1.5 *Among all linear images of the cross-polytope contained in B_2^n the cross-polytope itself has the maximal mean width.*

Indeed, denote the cross-polytope by B_1^n and let T be a linear operator such that $TB_1^n \subset B_2^n$. Denote $x_i = Te_i$ for $i \geq 1$. Without loss of generality, $|x_i| = 1$. Let $\xi_i = \langle G, x_i \rangle$, $i \geq 1$. Then $\xi_i \sim \mathcal{N}(0,1)$. Therefore, by (1) and by the Šidák theorem,

$$w(TB_1^n) = c_n \mathbb{E} \max_{i \leq n} |\xi_i| \leq c_n \mathbb{E} \max_{i \leq n} |g_i| = w(B_1^n).$$

The other counterpart of this theorem follows from Proposition 4 in [35] in a similar way, namely we have

Theorem 1.6 *Among all linear images of the cross-polytope containing $\frac{1}{\sqrt{n}} B_2^n$ the cross-polytope itself has the minimal mean width.*

A. Balitskiy, R. Karasev, and A. Tsigler conjectured that for every $r > 0$ the Gaussian measure of a simplex S containing rB_2^n is minimized when S is regular (see Conjecture 3.3 in [4]). In the same way as Šidák's theorem and Proposition 4 in [35] imply Theorems 1.5 and 1.6, their conjecture would immediately imply Conjecture 1.1. Moreover, they showed that their conjecture implies Conjecture 2.4 formulated below.

This note is organized as follows. In Section 2, we discuss the *Simplex Code Conjecture* or the *Weak Simplex Conjecture*, which makes links between Conjecture 1.1 and Information theory. In particular, we mention a problem, Problem 2.2, on the behaviour of the maximum of a certain Gaussian process which is (if solution is a regular simplex) stronger than Conjecture 1.4 and which is needed to solve the corresponding problem in transmitting signals. In Section 3, we describe two other stronger conjectures related to the Steiner formula and to the intrinsic volumes. Then, in Section 4, we provide some asymptotic estimates, in particular we show that the mean width of a half-dimensional (flat) cross-polytope is surprisingly very close to the mean width of a regular simplex. We also show that the regular simplex is a solution in the asymptotic sense. Finally, in Section 5, we formulate another conjecture on the minimum of a Gaussian process, where, as we believe, the solution is also the regular simplex. It seems that Conjectures 3.1, 3.2, 5.1 have not appeared in the literature yet.

2 Simplex Code Conjecture

In this section we discuss the relation of Conjecture 1.1 to Information Theory. We first describe a problem in transmitting signals which goes back to works of V.A. Kotel'nikov [29], L.A. MacColl [37], and C.E. Shannon [42] published at the end of 40-s.

For positive integers n and N let $x_1, \ldots, x_N \in S^{n-1}$ be signal vectors to be transmitted. Let $Y = \lambda\, x_i + G$ be the observed (received) vector when x_i was transmitted. Here, G is the standard Gaussian random vector (corresponding to the white noise) and $\lambda > 0$ is the (fixed) signal-to-noise ratio. The problem is:

Problem 2.1 *Observed Y to reconstruct x_j which has been transmitted.*

To solve the problem one creates the *matched filter* – the vectors $y_i = \langle Y, x_i \rangle$, $i \le N$, and decides that x_j was transmitted if

$$y_j = \max_{i \le N} y_i.$$

We want to maximize the probability of the right decision, that is, to maximize the function

$$\psi_\lambda\left(\{x_i\}_{i=1}^N\right) = \frac{1}{N} \sum_{j=1}^{N} \mathbb{P}\left(y_j = \max_{i \le N} \langle Y, x_i \rangle \;\middle|\; Y = \lambda x_j + G\right).$$

In his works [1, 3] A.V. Balakrishnan essentially developed the theory (see also Chapter 14 of C.L. Weber's [51] book for self-contained presentation). In [1] Balakrishnan proved that the latter problem of maximizing ψ_λ is equivalent to the following problem.

Problem 2.2 *Given a Gaussian random vector $(\xi_1, \ldots, \xi_N)$ with $\xi_i \sim \mathcal{N}(0,1)$ for all $i \le N$ and a covariance matrix σ of rank n set*

$$\phi_\lambda(\sigma) := \mathbb{E} \exp\left(\lambda \max_{i \le N} \xi_i\right).$$

Maximize $\phi_\lambda(\sigma)$ over all choices of covariance matrices σ.

Differentiating with respect to λ, one gets the following [1].

Lemma 2.3 *If there exists a solution of Problem 2.2 which is independent of λ in an interval $(0, \lambda_0)$ for some $\lambda_0 > 0$, then this solution maximizes mean width of the convex hull of x_i's.*

We turn now to the case of the simplex, that is, we fix $N = n + 1$. The following is the Simplex Code Conjecture or the Weak Simplex Conjecture.

Conjecture 2.4 *Let $N = n + 1$. For every (fixed) $\lambda > 0$ the function $\phi_\lambda(\cdot)$ is maximal for the regular simplex, that is, when the covariance matrix of ξ_i's is $\sigma = \sigma(\Delta_n)$.*

The following two theorems were proved in [1] and [3] respectively.

Theorem 2.5 *Let $N = n + 1$. For every (fixed) $\lambda > 0$, the regular simplex is a local maximum in Problem 2.2. Furthermore, if there exists a solution of Problem 2.2 which is independent of λ on some interval (λ_1, λ_2), then this solution is given by the regular simplex.*

Theorem 2.6 *Given $\lambda > 0$ let σ_λ denote a point (or one of points) of maximum of $\phi_\lambda(\cdot)$. Then*

$$\lim_{\lambda \to \infty} \sigma_\lambda = \sigma(\Delta_n).$$

In [3] a similar statement about the limit at 0 is claimed but its proof essentially uses Conjecture 1.1.

In 1966 H.J. Landau and D. Slepian [30] published a proof of Conjecture 2.4 for arbitrary $n \geq 3$. The proof was based on the geometric technique developed by L. Fejes-Tóth ([18], pp. 137-138). However, as was noticed by S.M. Farber [17] and R.M. Tanner [48], the 3-dimensional proofs of corresponding geometric results in [18] do not extend to higher dimensions (see also [4] for more insight and related conjectures). Therefore the proof of Conjecture 2.4 in [30] holds only for $n = 3$. We refer to [49] for two more related conjectures, which are stronger than Conjecture 2.4. We would also like to mention that T.M. Cover [15] noticed that Conjecture 2.4 can be reduced to the following geometric problem.

Conjecture 2.7 *Let $A > 0$ and $B \subset S^{n-1}$ be a spherical cap with the center at $x \in S^{n-1}$. Let S be a spherical simplex of the area A, where by a spherical simplex we understand the intersection of the sphere S^{n-1} with n half-spaces. Then a regular spherical simplex with the center at x maximizes the area of the intersection $B \cap S$.*

Finally we would like to mention the Strong Simplex Conjecture, which asserts the same as the Weak Simplex Conjecture, but instead of constraints $x_i \in S^{n-1}$, $i \le n+1$, in the initial problem one uses the constraint

$$\sum_{i=1}^{n+1} |x_i|^2 = n+1,$$

i.e., instead of choosing x_i's on the sphere we fix the sum of their squared lengths. The Strong Simplex Conjecture was disproved by M. Steiner [45], who used essentially a one-dimensional example. For other counter-examples see [31, 47].

3 Two more geometric conjectures.

In this section we discuss connections of Conjecture 1.1 to the Steiner formula and to the intrinsic volumes and provide two more geometric conjectures which are stronger than Conjecture 1.1. Both conjectures were communicated to us after the first draft of this note was written. We refer to [41] and references therein for the general theory of convex bodies, Brunn–Minkowski Theory, in particular for information about and relations between the mean width functional, quermassintegrals, intrinsic volumes, etc. (see also [46] for relations between intrinsic volumes and Gaussian processes).

Let $t > 0$ and K be a convex body in $\mathbb{R}^n$. Consider the Minkowski sum

$$K_t := K + tB_2^n = \{x + ty \mid x \in K,\ y \in B_2^n\} = \{z \mid \operatorname{dist}(z, K) \le t\},$$

where dist denotes the Euclidean distance. The Steiner formula says that the (n-dimensional) volume $|K_t|$ of K_t is polynomial in t. It can be written as

$$|K_t| = \sum_{j=0}^{n} \kappa_{n-j} V_j(K) t^{n-j},$$

where $\kappa_0 = 1$, $\kappa_i = |B_2^i| = \pi^{i/2}/\Gamma(1 + i/2)$ for $i \ge 1$, and $V_i(K)$, $i \le n$, are coefficients which depend only on K. These coefficients are called intrinsic volumes of K. Analyzing the Steiner formula it is easy to see that $V_0(K) = 1$, $V_n(K) = |K|$, and $2V_{n-1}(K)$ is the surface area of K. Moreover, it is known that $V_1(K) = n\kappa_n/(2\kappa_{n-1})\, w(K)$ and $V_1 = \sqrt{2\pi}\mathbb{E} \max_{x\in K} \sup \langle G, x\rangle$ (see e.g. Proposition 2.4.14 in [46], cf. (1)).

R. van Handel [50] suggested the following natural extension of Conjecture 1.1.

Conjecture 3.1 *For every (fixed) $t > 0$, among all simplices $S \subset B_2^n$ the regular simplex Δ_n maximizes the volume $|S_t|$. Moreover, Δ_n is the unique simplex maximizing this volume.*

Conjecture 3.1 is stronger than Conjecture 1.1. Indeed, using the Steiner formula and that $V_0(K) = 1$ for every K, we have

$$|(\Delta_n)_t| - |S_t| = \sum_{j=0}^{n-1} \kappa_j \left(V_{n-j}(\Delta_n) - V_{n-j}(S)\right) t^j,$$

therefore, sending t to infinity, we observe $V_{n-1}(\Delta_n) \geq V_{n-1}(S)$.

Furthermore, Z. Kabluchko and D. Zaporozhets [28] suggested even stronger conjecture.

Conjecture 3.2 *Let $S \subset B_2^n$ be a simplex. Then for every $1 \leq i \leq n$ one has $V_i(S) \leq V_i(\Delta_n)$. Moreover, if S is not regular, then the inequality is strict for every i.*

We would like to note that some cases in Conjecture 3.2 are known. Indeed, the case $i = n$ corresponds to the volume. It follows from the John theorem ([26, 7], see also [49] for a geometric proof). The case $i = n - 1$, corresponding to the surface area, was proved by Tanner in [49].

4 Asymptotic results and comparison to cross-polytope

In this section we discuss asymptotic behaviour and compare mean width of the regular simplex with the mean width of corresponding half-dimensional cross-polytope, showing that surprisingly they are very close to each other. All results of this section with complete proofs can be found in [27].

As before, let g_i's denote i.i.d. standard Gaussian random variables. Let $(\eta_1, \dots, \eta_{n+1})$ denote a Gaussian random vector with the covariance matrix $\sigma(\Delta_n)$.

We believe that the following observation has been known for many years. A.V. Balakrishnan [1] tributes it to C.R. Chan (in the context of the function ψ_λ defined in the previous section). It can be obtained by direct calculations, since Δ_n can be realized as the (properly normalized) convex hull of the canonical basis $\{e_i\}_{i=1}^{n+1}$ in $\mathbb{R}^{n+1}$.

Claim 4.1

$$\mathbb{E} \max_{i\le n+1} \eta_i = \sqrt{\frac{n+1}{n}}\ \mathbb{E} \max_{i\le n+1} g_i = \left(1+\frac{1+o(1)}{2n}\right)\ \mathbb{E} \max_{i\le n+1} g_i.$$

The next statement claims that the regular simplex is the best asymptotically. Its proof is based on standard estimates for Gaussian processes.

Lemma 4.2 *If $S_n \subset B_2^n$ is a simplex with the maximal mean width then*

$$w(\Delta_n) \le w(S_n) \le \left(1+\frac{C\,\ln\ln n}{\ln n}\right) w(\Delta_n),$$

where C is a positive absolute constant.

Note here that in [19] the mean width of the regular simplex was calculated as

$$w(\Delta_n) = 2\sqrt{\frac{\ln n}{n}}\left(1-(1+o(1))\frac{\ln\ln n}{\ln n}\right).$$

We turn now to the comparison with the cross-polytope (octahedron). Recall that we consider convex hulls of $n+1$ points on the Euclidean sphere. Assume that $n = 2m-1$ and consider the m-dimensional cross-polytope $B_1^m = \operatorname{conv}\{\pm e_i\}_{i=1}^m$ in $\mathbb{R}^n$. Clearly, B_1^m is a (degenerated) simplex in $\mathbb{R}^n$. Surprisingly, the mean width of B_1^m is very close to the mean width of Δ_n as the next theorem shows (recall that the mean width can be computed via corresponding expectations of Gaussian processes and that Claim 4.1 relates dependent Gaussian random variables corresponding to the regular simplex with independent Gaussian random variables). The left hand side inequality in Theorem 4.3 is immediate by Slepian's Lemma ([44], see also [32, 34]).

Theorem 4.3 *Let $n = 2m-1$. Then*

$$\mathbb{E} \max_{i\le 2m} g_i \le \mathbb{E} \max_{i\le m} |g_i| = \left(1+\frac{1+o(1)}{4n\ln n}\right)\ \mathbb{E} \max_{i\le n+1} g_i.$$

In particular,

$$w(B_1^m) = \left(1 - \frac{1+o(1)}{2n}\right) w(\Delta_n).$$

Moreover, using the S. Chatterjee technique ([14]), a path σ_t, $t \in [0,1]$, in covariance matrices of Gaussian vectors can be constructed so that σ_0 corresponds to the vector

$$\sqrt{\frac{n+1}{n}}\,(g_1, \dots, g_{n+1})$$

(that is, to the regular simplex) and σ_1 corresponds to the vector

$$(g_1, -g_1, g_2, -g_2, \dots, g_m, -g_m)$$

(that is, to the m-dimensional cross-polytope) and such that the expectation of maximum is not-decreasing along this path. This gives the following estimate, which is slightly weaker for large n, but better for small n.

Theorem 4.4 *Let $n = 2m - 1$.*

$$\mathbb{E} \max_{i \le 2m} g_i \le \mathbb{E} \max_{i \le m} |g_i| \le \sqrt{\frac{n+1}{n}}\ \mathbb{E} \max_{i \le 2m} g_i.$$

In particular,

$$w(B_1^m) \le w(\Delta_n) \le \sqrt{\frac{n+1}{n}}\ w(B_1^m).$$

5 A conjecture on the smallest order statistic

In this section we formulate a conjecture on Gaussian processes, which, to the best of our knowledge, appears for the first time. Although it is not directly related to the mean width of convex bodies, we have decided to mention it here, since it also deals with an extreme order statistic of coordinates of the standard Gaussian vector (cf. Conjecture 1.4) and since we believe that it has the same solution as Conjecture 1.1.

Conjecture 5.1 *Let $n \ge 2$ and $p > 0$. Among all Gaussian random vectors $(\xi_1, \dots, \xi_{n+1})$ with $\xi_i \sim \mathcal{N}(0,1)$ for all $i \le n+1$, the expectation*

$$\mathbb{E} \min_{i \le n+1} |\xi_i|^p$$

is minimal when the covariance matrix $\sigma = \sigma(\Delta_n)$. The solution is unique.

The main motivation for this question comes from the Mallat-Zeitouni problem which is still open in full generality ([38], see also [36] for discussions, history, references, and a partial solution). In the original notes, published on O. Zeitouni's webpage in 2000, S. Mallat and O. Zeitouni suggested a way to solve it. Their method would have worked if a more general result in the spirit of Conjecture 5.1 had held with $p = 2$. Moreover, in [22, 23] the authors were able to prove that for every sequence of real numbers $\{a_i\}_{i=1}^{n+1}$ and every $p > 0$,

$$\mathbb{E} \min_{i \le n+1} |a_i g_i|^p \le \Gamma(2+p)\, \mathbb{E} \min_{i \le n+1} |a_i \xi_i|^p,$$

where $\Gamma(\cdot)$ is the Gamma-function. This result, together with the Šidák theorem also supported the intuition that the independent case gives the minimum (see also [33]). However later, R. van Handel checked that for $n = 2$ the arrangement corresponding for Δ_2 is better than three independent variables [38].

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