The Alberta High School Mathematics Competition
Solution to Part II, 2013.

Problem 1.
The given equation may be rewritten as $ab + \frac{36}{ab} + 12 = 25$. Therefore

$$(ab)^2 - 13ab + 36 = (ab-4)(ab-9) = 0.$$

Hence $ab = 4$ or $ab = 9$. Note that a and b are positive integers with $a \le b$. If $ab = 4$, we have $(a,b) = (1,4)$ or (2,2). If $ab = 9$, we have $(a,b) = (1,9)$ or (3,3). It is easy to verify that all four are indeed solutions.

Problem 2.

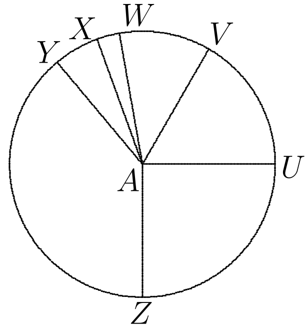
(a) Candy's perfect set may be $\{1,2,3,5,7,11,13,17,19\}$. We claim that this number is the highest possible. Now a maximal perfect set must contain the element 1, as otherwise we can add 1 and obtain a larger perfect set. Also, a maximal perfect set cannot contain an element which is divisible by two distinct primes, as otherwise we can replace that element by the two primes and obtain a larger perfect set. Hence each element pther than 1 is a positive power of a prime. Moreover, distinct elements are powers of distinct primes. Since there are only 8 primes less than 20, namely, 2, 3, 5, 7, 11, 13, 17 and 19, we claim is justified.

(b) Every maximal perfect set Candy can build must have the form

$$S = \{1, 2^{i_2}, 3^{i_3}, 5^{i_5}, 7^{i_7}, 11^{i_{11}}, 13^{i_{13}}, 17^{i_{17}}, 19^{i_{19}}\},$$

where each exponent is a positive integer. Since $5^2 > 20$, the exponent for all primes greater than or equal to 5 must be 1. Since $2^4 \le 20 \le 2^5$ and $3^2 \le 20 \le 3^3$, the exponent for 2 must be 1, 2, 3 or 4, and the exponent for 3 must be 1 or 2. This yields eight different maximal perfect sets.

Problem 3.
Let $n \ge 2$ be the number of rays drawn by Randy. Then there are $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of rays. Each pair determines two angles adding up to 360°. Hence the total number of angles between 2 of the n rays is exactly $n(n-1)$. The measure of such an angle is clearly less than 360°. Since it is supposed to be an integral multiple of 10°, there are at most 35 values for the measures of these angles. Since they are distinct, $n(n-1) \le 35$. Now $6 \times 5 = 30 < 35 < 42 = 7 \times 6$. Hence $n \le 6$. It is possible for Randy to draw 6 rays, determining 30 distinct angles. In the diagram below, $\angle UAV = 60°$, $\angle VAW = 40°$, $\angle WAX = 10°$, $\angle XAY = 20°$, $\angle YAA = 140°$ and $\angle ZAU = 90°$.

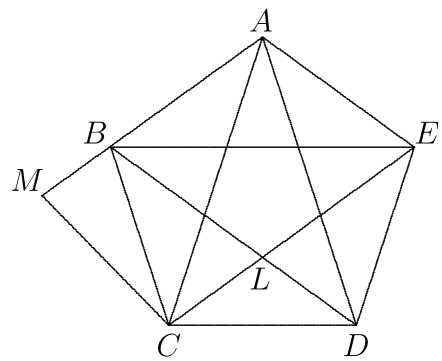


We now verify that the 30 angles between two rays are distinct. We have $\angle WAY = 30°$, $\angle VAX = 50°$, $\angle VAY = 70°$, $\angle UAW = 100°$, $\angle UAX = 110°$, $\angle UAY = 130°$, $\angle ZAV = 150°$, $\angle XAZ = 160°$ as well as $\angle WAZ = 170°$. These are 9 different angles distinct from the 6 between adjacent rays. All have measures less than 180°. Corresponding to these 15 angles, we have 15 other angles greater than 180°, yielding a total of 30 distinct angles.

Problem 4.
Let L be the point of intersection of EC and DB. Let M be the point on the extension of AB such that MC is parallel to AE. Then $ABLE$ and $AMCE$ are parallelograms. Note that triangles DLC and EAB are similar, as are triangles AMC and ELD. It follows that

$$\frac{EC}{AB} = \frac{EL + LC}{AB} = 1 + \frac{LC}{AB} = 1 + \frac{DL}{EA} = 1 + \frac{DL}{CM} = 1 + \frac{AB}{EC}.$$

Let $x = \frac{EC}{AB}$. Then $x = 1 + \frac{1}{x}$ so that $x^2 - x - 1 = 0$. Hence $x = \frac{1+\sqrt{5}}{2}$. Similarly, we have $\frac{DB}{AE} = \frac{AC}{ED} = \frac{AD}{BC} = \frac{EB}{DC} = \frac{1+\sqrt{5}}{2}$, so that $EC + DB + AC + AD + EB = 5(1 + \sqrt{5})$.



Remark:
The regular pentagon is used in the illustrative diagram. Many students may get the correct answer by treating only this special case, essentially proving that $\cos 36° = \frac{1+\sqrt{5}}{4}$.

Problem 5.
The conditions are:

$$\begin{aligned} r^2 + br + c &= 1, & (1)\\ s^2 + bs + c &= b, & (2)\\ t^2 + bt + c &= c, & (3)\\ r + t &= 2s. & (4) \end{aligned}$$

From (3), $t(t + b) = 0$ so that either $t = 0$ or $t = -b$. We consider these two cases separately.
Case 1: $t = 0$.
From (4), we have $r = 2s$. Substituting into (1), we have $4s^2 + 2bs + c = 1$. Subtracting (2) from this, we have $3s^2 + bs = 1 - b$ which may be rewritten as $(s + 1)(3s - 3 + b) = -2$. Hence 2 is divisible by $s + 1$, so that $s = -3, \ -2, \ 0$ or 1. However, since $s > t = 0$, we may only have $s = 1$. It follows that $b = -1$. Hence $f(x) = x^2 - x - 1$, with $r = 2, \ s = 1$ and $t = 0$.
Case 2: $t = -b$.
From (4), we have $r = 2s + b$. Substitoting into (1), we have $4s^2 + 6sb + 2b^2 + c = 1$. Subtracting (2) from this, we have $3s^2 + 5sb + 2b^2 = 1 - b$ which may be rewritten as $(3s + 2b + 3)(s + b - 1) = -2$. Hence -2 is divisible by $s + b - 1$. From $r > s > t = -b$, we have $s + b > 0$. Hence $s + b - 1 > -1$ so that $s + b - 1 = 1$ or 2. If $s + b - 1 = 1$, we have $3s + 2b + 3 = -2$ so that $s = -9$ and $b = 11$. Hence $f(x) = x^2 + 11x + 30$ with $r = -7, \ s = -9$ and $t = -11$. If $s + b - 1 = 2$, we have $3s + 2b + 3 = -1$ so that $s = -10$ and $b = 13$. Hence $f(x) = x^2 + 13x + 43$, with $r = -7, \ s = -10$ and $t = -13$.