The Alberta High School Mathematics Competition
Solution to Part I, 2010

1. Since $10 \leq 4n \leq 99$, $3 \leq n \leq 24$. Hence there are $24 - 3 + 1 = 22$ such values. The answer is **(b)**.

2. There are 7 fences of length 4 and 5 fences of length 6. The total length is $7 \times 4 + 5 \times 6 = 58$. The answer is **(a)**.

3. Since the least common multiple is even, at least one number is even. Since the greatest common divisor is odd, exactly one number is even. We can show in a similar manner that exactly one of the two numbers is divisible by 5. Since neither is 10, one of them is 2 and the other is 5, yielding a sum of 7. The answer is **(c)**.

4. Note that x must be odd, and $x = 9 - \frac{2y}{3}$. Since $y \geq 0$, $x \leq 9$. Thus there are 5 triples $(x, y) = (9, 0)$, (7,3), (5,6), (3,9) and (1,12). The answer is **(b)**.

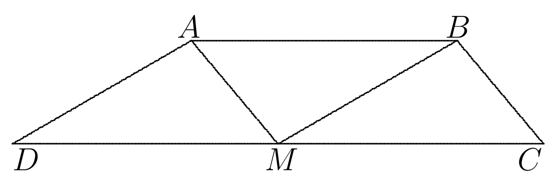
5. Note that $2010 \times \frac{5}{4} = 2512.5$. There are 502 multiples of 5 from 5 to 2510 inclusive. Hence 2512 is the $(2512 - 502)$-th or 2010-th number in the punctured sequence. The answer is **(b)**.

6. Alice and Ethel are 39 floors apart, and as long as the get-together floor is in between, the total number of floors they cover is 39. Similarly, the total number of floors Brian and Debra cover is 19. The minimum number of floors Colin covers is 0, when they get together on floor 3. The answer is **(e)**.

7. The pigeons can choose the pigeonholes in $9 \times 9 = 81$ ways. There are 12 pairs of rooms separated by an interior wall. Since the pigeons can choose these two rooms in 2 ways, the desired probability is $\frac{2\times12}{81} = \frac{8}{27}$. The answer is **(d)**.

8. Since $\frac{1}{x} \leq -3$, $x \leq 0$. Multiplying by $-\frac{x}{3}$, we have $-\frac{1}{3} \leq x$. Hence the set of all values of x is $\{-\frac{1}{3} \leq x < 0\}$. The answer is **(d)**.

9. Since $AB = DM$ and AB is parallel to DM, $ABMD$ is a parallelogram. Similarly, $ABCM$ is a parallelogram. Therefore, $\angle AMD = \angle BCD = 50°$ and $\angle BMC = \angle ADC = 30°$. Therefore, $\angle AMB = 180° - \angle AMD - \angle BMC = 180° - 50° - 30° = 100°$. The answer is **(c)**.



10. We first count the triangles in which the equal sides of length k are longer than the third side, which can be of length from 1 to $k-1$. Summing from $k = 1$ to 9, we have $0+1+2+\cdots+8 = 36$ such triangles. We now count the triangles in which the equal sides of length k are shorter than the third side, which can be of length from $k + 1$ to $2k - 1$. Summing from $k = 1$ to 9, we have $0 + 1 + 2 + 3 + 4 + 3 + 2 + 1 + 0 = 16$. The total is $36 + 16 = 52$. The answer is **(c)**.

11. The first three choices are equal respectively to $2^{2^{256}}$, $2^{2^{512}}$ and $2^{2^{81}}$. Clearly, the second one is the largest among them. The fourth number is equal to $2^{3^{16}}$. Since $2^{512} = 4^{256} > 3^{256} > 3^{16}$, the second number is larger than the fourth one. The fifth number is equal to $3^{2^{16}}$. Clearly, $2^{2^{256}} = 4^{2^{255}} > 3^{2^{255}} > 3^{2^{16}}$. Hence the second number is the largest among the five choices. The answer is **(b)**.

12. Note that $ab + a + b + 1 = (a + 1)(b + 1)$. Every composite number can be written in this form and no prime number can be written in this form. Therefore, the positive integers that are not gold numbers are those that are one less than a prime. By simple counting, we see there are 8 primes from 2 to 21. Therefore, the number of gold numbers between 1 and 20 inclusive is $20 - 8 = 12$. The answer is **(e)**.

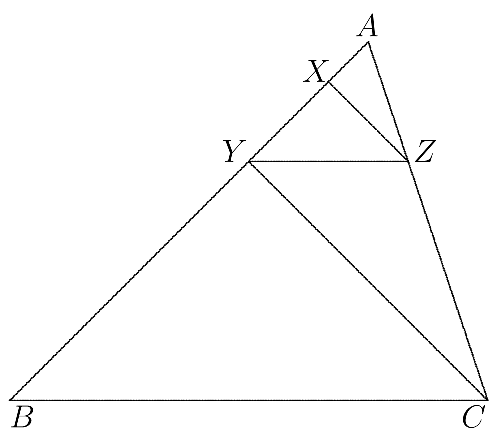
13. The sphere which passes through A, B, C and D also passes through the other four vertices of a $1 \times 2 \times 2$ block having A, B, C and D as four of its vertices. Since the space diagonal of this block is of length $\sqrt{2^2 + 2^2 + 1^2} = 3$, the radius of the sphere is $\frac{3}{2}$. The answer is **(a)**.

14. Each application of f doubles the exponent while each application of g quadruples the exponent. After 50 applications of f and 49 applications of g, the exponent has been doubled $50 + 2 \times 49 = 148$ times so that $n = 2^{148}$. The answer is **(c)**.

15. Denote the area of triangle T by $[T]$. Since triangles AXZ and AYC are similar, $ZC = 2AZ$. Since triangles AYZ and ABC are similar, $YB = 2AY$. It follows that

$$[XYZ] = \frac{2}{3}[AYZ] = \frac{2}{9}[AYC] = \frac{2}{27}[ABC] = \frac{2}{27}.$$

The answer is **(b)**.



16. When $n^3 - 3n + 2$ is divided by $2n + 1$, the quotient is $\frac{n^2}{2} - \frac{n}{4} - \frac{11}{8}$ and the remainder is $\frac{27}{8}$. Hence $8(n^3 - 3n + 2) = (2n + 1)(4n^2 - 2n - 11) + 27$, so that $2n + 1$ divides $n^3 - 3n + 2$ if and only if it divides 27. The set of all values of n for which $2n+1$ divides 27 is $\{-14, -5, -2, -1, 0, 1, 4, 13\}$, and there are 8 such values. The answer is **(e)**.