Alberta High School Mathematics Competition

Solution to Part I – 2008

1. In order for m to be an integer, n must be a multiple of 5. Since there are 19 integer values of n within range, the number of corresponding values of m that are integers is also 19. The answer is **(b)**.

2. When $a+2b+3c$ is divided by 13, the remainder will be the same as when $9+2\times7+3\times10=53$ is divided by 13. The latter is clearly 1. The answer is **(a)**.

3. The total length of all but one side in a polygon with positive area must be greater than the length of that side. To minimize the perimeter, choose that side to be the longest. Hence the perimeter is greater than 20. Since all side lengths are integers, the minimum is at least 21. This minimum value may be achieved with a triangle of side lengths 5, 6 and 10. The answer is **(d)**.

4. Note that $f(x)=\frac{3}{x}$ and $g(x)=\frac{6}{x}$ satisfy the hypothesis. In any case, we have $f(\frac{1}{2})=g(1)=2f(1)=2g(2)=6$. The answer is **(e)**.

5. From the first condition, the desired remainder is one of 6, 13, 20 and 27. From the second condition, the desired remainder is one of 1, 5, 9, 13, 17, 21 and 25. The only common value is 13. The answer is **(d)**.

6. Consider a horizontal cross-section of the room as a clock, and let the ray start at the 12 o'clock position. It must travel directly at another point on the clock in order to return to its starting position after 11 bounces. However, some of these 11 paths take it back to the starting point before 11 bounces. The initial destination must be a point which is relatively prime to 12, namely, 1, 5, 7 or 11. The answer is **(a)**.

7. We have $|||x-1|-2|-3|-4|=5$. Now $|||x-1|-2|-3|=4\pm5$. Since the left side can never be equal to -1, we must have $|||x-1|-2|-3|=9$. Similarly, $||x-1|-2=12$, $|x-1|=14$ and $x=1\pm14=-13$ or 15. The answer is **(c)**.

8. Since $2008=2\times2\times2\times251$, the box must be one of $2\times4\times251$, $2\times2\times502$, $1\times8\times251$, $1\times4\times502$, $1\times2\times1004$ and $1\times1\times2008$, with respective surface area 3028, 4024, 4534, 5028, 6028 and 8034. The answer is **(a)**.

9. We have $b-a=11$ and $b^2-a^2=1001$, so that $b+a=91$. Hence $2b=102$ and $b=51$. The answer is **(b)**.

10. Let Linda's walking speed be v kilometres per hour. Then the length of the trains is given by $t_1(V-v)=t_2(V+v)$. It follows that $v=\frac{t_1-t_2}{t_1+t_2}V$. The answer is **(a)**.

11. The smallest value of a leads to the smallest value of b, and of c. The smallest value of a is 10, a 1 followed by $2-1$ 0s. The smallest value of b is 10^9, a 1 followed by $10-1$ 0s. The smallest value of c is 10^{10^9-1}, a 1 followed by 10^9-1 0s. The answer is **(b)**.

12. If we take both senior boys, we must complement them with both junior girls. If we take neither senior boy, we must take both senior girls and both junior boys. If we take only one senior boy, then we must take one senior girl, one junior girl and one junior boy. In each category, there are two choices. Hence the number of different teams is $1+1+2^4=18$. The answer is **(e)**.

13. The height of the circle is $2r$ where r is its radius. From $\pi r^2=1$, we have $2r=\frac{2}{\sqrt{\pi}}$. Since $2<\pi<4$, this is greater than $\frac{2}{\sqrt{4}}=1$, the height of the square, and less than $\frac{2}{\sqrt{2}}=\sqrt{2}$, the height of the right isosceles triangle. The height of the equilateral triangle is $\sqrt{3}s$ where $2s$ is its side length. From $\sqrt{3}s^2=1$, we have $\sqrt{3}s=\sqrt[4]{3}$. Since $3<4$, $\sqrt[4]{3}<\sqrt[4]{4}=\sqrt{2}$. The answer is **(b)**.

14. Let a, b and c be the roots. Then $a+b+c=0$ and $bc+ca+ab=-3$. It follows that $a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)=6$. Since $x^3=3x-1$, we have

$$x^5=x^2(3x-1)=3(3x-1)-x^2=-3+9x-x^2.$$

Hence $a^5+b^5+c^5=-9+9(a+b+c)-(a^2+b^2+c^2)=-15$. The answer is **(a)**.

15. In moving from city to city, we either pick up an extra prime or discard one. The numbers of prime divisors of $99=3\times3\times11$ and $100=2\times2\times5\times5$ are 3 and 4 respectively, and there are no common divisors. To change from one lot to the other, we need to make 7 moves. The answer is **(d)**.

16. We can guarantee that at least one ticket wins by buying the tickets (1,2), (1,3), (2,3) and (4,5). If neither 4 nor 5 is drawn, the last ticket wins. If at least one of them is drawn, then at most one of 1, 2 and 3 is drawn. Then one of the first three tickets wins. If we only buy three tickets, we have to choose 6 numbers. By the Pigeonhole Principle, one of the numbers 1, 2, 3, 4 and 5 will appear on at least two of our tickets. If it appears on all three, none of them wins if this number is drawn. Suppose it appears only on the first two tickets. If this number is drawn along with one of the numbers on our third ticket, we will not have a winning ticket. It follows that the minimum number of tickets to guarantee that at least one of them wins is 4. The answer is **(b)**.