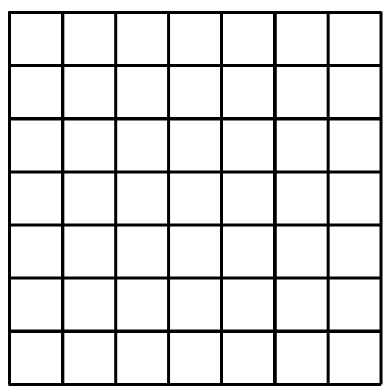
Alberta High School Mathematics Competition

Part II, 2009.

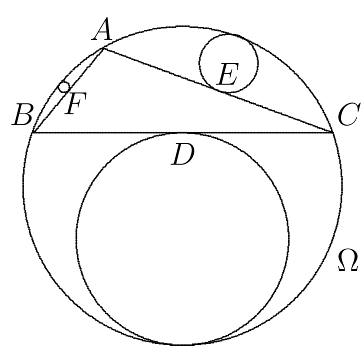
Problem 1.
Let w, x, y and z be non-negative numbers whose sum is 100. Determine the maximum possible value of $wx + xy + yz$.

Problem 2.
Determine all positive integers a and b, $a < b$, so that exactly $\frac{1}{100}$ of the consecutive integers $a^2,\ a^2+1,\ a^2+2,\ \ldots,\ b^2$ are the squares of integers.

Problem 3.
A game is played on a 7×7 board, initially blank. Betty Brown and Greta Green make alternate moves, with Betty going first. In each of her moves, Betty chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Greta chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Betty. Then Greta paints the remaining blank squares green. Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?



Problem 4.
A, B and C are points on a circle Ω with radius 1. Three circles are drawn outside triangle ABC and tangent to Ω internally. These three circles are also tangent to BC, CA and AB at their respective midpoints D, E and F. If the radii of two of these three circles are $\frac{2}{3}$ and $\frac{2}{11}$, what is the radius of the third circle?



Problem 5.
Prove that there are infinitely many positive integers k such that k^k can be expressed as the sum of the cubes of two positive integers.