The Alberta High School Mathematics Competition

Solution to Part II, 2015.

Problem 1.
Find the number of isosceles triangles of perimeter 2015 such that all three sides are **odd** integers.

Solution:
Let m, m and n be the lengths of the sides of the triangle. Then $2m + n = 2015$. We should have $2m > n$, hence $n \leq 1007$. Thus $(n, 2m) = (1, 2014), (3, 2012), (5, 2010), \cdots, (1005, 1010), (1007, 1008)$, or taking into account that m is odd, $(n, m) = (1, 1007), (5, 1005), \cdots, (1005, 505)$. The number of pairs is $(1007 - 505)/2 + 1 = 252$ and hence there are 252 triangles.

Problem 2.
Find all pairs (m, n) of positive integers such that $m^3 - n^3 = 5mn + 43$.

Solution:
Since $m > n$, $k = m - n$ is a positive integer. Then $(k + n)^3 - n^3 - 5(k + n)n = 43$ or

$$(3k - 5)n^2 + k(3k - 5)n + k^3 = 43 \tag{1}$$

We cannot have $k = 1$ because then (1) simplifies to $n^2+n+21 = 0$ which has no real solutions.Thus we assume $k \geq 2$, which means that $3k - 5 > 0$. From (1), we get $k^3 < 43$, and so we have $k \leq 3$. If $k = 2$, then (1) simplifies to $0 = n^2 + 2n - 35 = (n - 5)(n + 7)$, with the positive integral solution $n = 5$. If $k = 3$, from (1) we get $0 = n^2 + 3n - 4 = (n - 1)(n + 4)$, with the positive integral solution $n = 1$. Hence there are two such pairs, namely, $(m, n) = (7, 5)$ and (4,1).

Problem 3.
Let $f : [0, 4] \to [0, \infty)$ be such that $f(4) = 2$ and $f(x + y) \geq f(x) + f(y)$ for any real numbers x and y in the closed interval [0,4] such that $x + y \leq 4$.

(a) Suppose that $0 \leq x \leq y \leq 4$. Show that $f(y) \geq f(x)$.

(b) Show that $f(x) \leq x$ for any x in [0,4].

Solution:

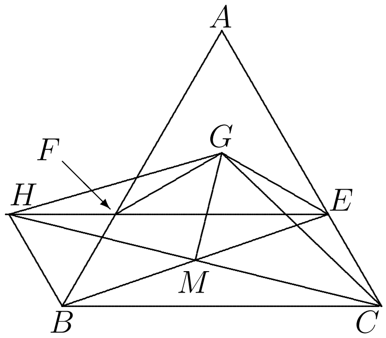
(a) If $0 \leq x \leq y \leq 4$ then $f(y) = f(x + (y - x)) \geq f(x) + f(y - x) \geq f(x)$.

(b) If x is in [2,4], then $f(x) \leq f(4) = 2 \leq x$. On the other hand, if x is in (0,2), then $\frac{4}{x} - \frac{2}{x} = \frac{2}{x} > 1$. Hence there is a positive integer n such that $\frac{2}{x} < n < \frac{4}{x}$ or $2 < nx < 4$, so that $f(nx) \leq nx$. Also, $f(nx) = f((n-1)x+x)) \geq f((n-1)x)+f(x) \geq f((n-2)x)+2f(x) \geq \cdots \geq nf(x)$ and hence $f(nx) \geq nf(x)$. Consequently $nf(x) \leq f(nx) \leq nx$, that is, $f(x) \leq x$ for any x in (0,2). Also, from $f(0) + f(0) \leq f(0)$ we obtain $f(0) \leq 0$, and since $f(0) \geq 0$, we get $f(0) = 0$. Therefore $f(x) \leq x$ for any x in [0,4].

Problem 4.
E and F are points on the sides CA and AB, respectively, of an equilateral triangle ABC such that EF is parallel to BC. G is the intersection point of medians in triangle AEF and M a point on the segment BE. Prove that $\angle MGC = 60^\circ$ if and only if M is the midpoint of BE.

Solution:

Let H be on the extension of EF such that $BCEH$ is a parallelogram. Then $\angle BHE = \angle BCE = 60^\circ$ and $\angle BFH = \angle AFE = 60^\circ$, so FBH is also an equilateral triangle, and in particular $HF = FB = EC$. Also $\angle HFG = 180^\circ - \angle GFE = 180^\circ - 30^\circ = 150^\circ$, and similarly $\angle CEG = 150^\circ$, thus $\angle HFG = \angle CEG$. Since $\angle GFE = 30^\circ = \angle GEF$, we know $FG = GE$. Hence triangles HFG and CEG are congruent (by SAS) so that $GH = GC$ and $\angle HGF = \angle CGE$. It follows that $\angle HGC = \angle HGF + \angle FGC = \angle CGE + \angle FGC = \angle EGF = 180^\circ - \angle GFE - \angle GEF = 180^\circ - 30^\circ - 30^\circ = 120^\circ$. Now M is the midpoint of the diagonal BE of the parallelogram if and only if it is the midpoint of the diagonal HC. Since triangle HGC is isosceles, this is equivalent to GM being the bisector of $\angle HGC$. In other words, $\angle MGC = 60^\circ$.



Problem 5.
Karys is helping her father move basketballs from his car to the gymnasium. She carries either 3 or 4 basketballs each trip, while her father carries 6 or 7 basketballs each trip. Altogether Karys makes 15 more trips and carries 15 fewer basketballs than her father.

(a) Determine the minimum number of basketballs that Karys carries.

(b) Determine the maximum number of basketballs that Karys carries.

Solution:
Suppose that Karys carries n basketballs in total. Then she will make at least $\frac{n}{4}$ and at most $\frac{n}{3}$ trips. Her father carries $n+15$ basketballs in total, so he makes at least $\frac{n+15}{7}$ and at most $\frac{n+15}{6}$ trips. Thus the difference between the number of trips Karys makes and the number of trips her father makes is at least $\frac{n}{4} - \frac{n+15}{6}$ and at most $\frac{n}{3} - \frac{n+15}{7}$. Thus we get

$$\frac{n}{4} - \frac{n+15}{6} \leq 15 \qquad \text{and} \qquad \frac{n}{3} - \frac{n+15}{7} \geq 15,$$

which simplify respectively to

$$2n - 60 \leq 360 \qquad \text{and} \qquad 4n - 45 \geq 315.$$

Thus $90 \leq n \leq 210$.

(a) It is possible for Karys to carry 90 basketballs in total and for her father to carry 90+15=105 basketballs, because Karys can carry 3 basketballs at a time for 30 trips, while her father carries 7 basketballs at a time for 15 trips, so Karys indeed makes $30 - 15 = 15$ more trips than her father.

(b) It is not possible for Karys to carry 210 basketballs in total. To do this she would need at least $\frac{210}{4} = 52.5$ trips, so at least 53 trips. Her father would take at most $\frac{225}{6} = 37.5$ trips, so at most 37 trips. So the difference in the number of trips is at least $53 - 37 = 16$, not 15. Similarly, 209 basketballs carried by Karys and 224 carried by her father is not possible either, because we get $\lceil \frac{209}{4} \rceil = 53$ and $\lfloor \frac{224}{6} \rfloor = 37$. However, 208 for Karys and 223 for her father is possible, because now $\lceil \frac{208}{4} \rceil = 52$ and $\lfloor \frac{223}{6} \rfloor = 37$, and $52 - 37 = 15$. Karys carries 4 basketballs at a time for 52 trips for a total of 208 basketballs, while her father carries 6 basketballs at a time for 36 trips and 7 basketballs at a time for 1 trip, for a total of $36 \cdot 6 + 7 = 223$ basketballs in 37 trips.

Remark:

Similarly we can check that Karys could carry 207 basketballs, but it turns out that 206 basketballs carried by Karys (and 221 by her father) is again not possible, because $\lceil \frac{206}{4} \rceil = 52$ and $\lfloor \frac{221}{6} \rfloor = 36$, and $52 - 36 = 16$.