Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-018-9717-4

On the Classification of Finite-Dimensional Linear Flows

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Received: 8 June 2018

Abstract
New elementary, self-contained proofs are presented for the topological and the smooth classification theorems of linear flows on finite-dimensional normed spaces. The arguments, and the examples that accompany them, highlight the fundamental roles of linearity and smoothness more clearly than does the existing literature.

Keywords Linear flow · Flow equivalence · Orbit equivalence · (uniform) core · Kronecker flow

Mathematics Subject Classification 34A30 · 34C41 · 37C15

1 Introduction

Let X be a finite-dimensional normed space over $\mathbb{R}$ and φ a flow on X, i.e., $\varphi : \mathbb{R} \times X \to X$ is continuous, with $\varphi\big(t, \varphi(s, x)\big) = \varphi(t + s, x)$ and $\varphi(0, x) = x$ for all $t, s \in \mathbb{R}$ and $x \in X$. A fundamental problem throughout dynamics is to decide precisely which flows are, in some sense, *essentially the same*. Formally, call two smooth flows φ, ψ on X, Y respectively C^ℓ**-orbit equivalent**, with $\ell \in \{0, 1, \dots, \infty\}$, if there exists a C^ℓ-diffeomorphism (or homeomorphism, in case $\ell = 0$) $h : X \to Y$ and a function $\tau : \mathbb{R} \times X \to \mathbb{R}$, with $\tau(\,\cdot\,, x)$ strictly increasing for each $x \in X$, such that

$$h\big(\varphi(t, x)\big) = \psi\big(\tau(t, x), h(x)\big) \quad \forall (t, x) \in \mathbb{R} \times X. \tag{1.1}$$

If τ in (1.1) can be chosen to be independent of x, and thus simply $\tau(t, x) = \alpha t$ with some $\alpha \in \mathbb{R}^+$, then φ, ψ are C^ℓ**-flow equivalent**; they are **linearly** (orbit or flow) **equivalent** if $h(x) = Hx$ with some linear isomorphism $H : X \to Y$. Notice that these definitions are tailor-made for the present article and differ somewhat from terminology in the literature which, however, is itself not completely unified. Usage herein of terminology pertaining to the equivalence of flows is informed by the magisterial text [21], as well as by [3,20]. Widely used alternative terms are (**topologically**) **conjugate** (for *flow equivalent*, often understood to include the additional requirement that $\alpha = 1$) and (**topologically**) **equivalent** (for *orbit equivalent*); see [5,6,12,16,18,22,23,30–34].

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Published online: 30 November 2018

$$\begin{array}{ccc}
\boxed{\text{linearly flow equivalent}} & \Longrightarrow & \boxed{\text{linearly orbit equivalent}} \\
\Downarrow & & \Downarrow \\
\boxed{C^{\infty}\text{-flow equivalent}} & \Longrightarrow & \boxed{C^{\infty}\text{-orbit equivalent}} \\
\Downarrow & & \Downarrow \\
\boxed{C^{\ell}\text{-flow equivalent}} & \Longrightarrow & \boxed{C^{\ell}\text{-orbit equivalent}} \\
\Downarrow & & \Downarrow \\
\boxed{C^{1}\text{-flow equivalent}} & \Longrightarrow & \boxed{C^{1}\text{-orbit equivalent}} \\
\Downarrow & & \Downarrow \\
\boxed{C^{0}\text{-flow equivalent}} & \begin{array}{c}\Longrightarrow \\ (\Longleftarrow \ \dim X = 1)\end{array} & \boxed{C^{0}\text{-orbit equivalent}}
\end{array}$$

Fig. 1 Notions of equivalence for flows on normed spaces over $\mathbb{R}$; no conceivable implication not shown in the diagram is valid in general

Clearly, linear equivalence implies C^ℓ-equivalence for any ℓ, which in turn implies C^0-equivalence; also, flow equivalence implies orbit equivalence. Simple examples show that none of these implications can be reversed in general, not even when $\dim X = 1$, though the latter case is somewhat special in that C^0-orbit equivalence does imply C^0-flow equivalence. In any case, however, it turns out that all such examples must involve non-linear flows. In fact, the main theme of this article is that for linear flows all the *infinitely many* different notions of equivalence do coalesce, rather amazingly, into just *two* notions; see Figs. 1 and 2.

A flow φ on X is **linear** if each homeomorphism (or time-t-map) $\varphi(t, \cdot) : X \to X$ is linear, or equivalently if $\varphi(t, \cdot) = e^{tA^\varphi}$ for every $t \in \mathbb{R}$, with a (unique) linear operator $A^\varphi : X \to X$, referred to as the **generator** of φ. Thus a linear flow simply encodes the totality of all solutions of the linear differential equation $\dot{x} = A^\varphi x$ on X, in that $\varphi(\cdot, x_0)$ is the unique solution of that equation satisfying $x(0) = x_0$. To emphasize the fundamental role played by linearity in all that follows, linear flows are henceforth denoted exclusively by upper-case Greek letters Φ, Ψ etc.

For linear flows, the weakest form of equivalence, C^0-orbit equivalence, implies the seemingly much stronger C^0-flow equivalence, and both properties can be characterized neatly in terms of linear algebra. To state the following **topological classification theorem**, the main topic of this article, recall that every linear flow Φ on X uniquely determines a Φ-invariant decomposition $X = X_S^\Phi \oplus X_C^\Phi \oplus X_U^\Phi$ into stable, central, and unstable subspaces, with a corresponding unique decomposition $\Phi \simeq \Phi_S \times \Phi_C \times \Phi_U$; see Sect. 5 for details.

Theorem 1.1 *Let Φ, Ψ be linear flows on X, Y, respectively. Then each of the following four statements implies the other three:*

(i) *Φ, Ψ are C^0-orbit equivalent;*
(ii) *Φ, Ψ are C^0-flow equivalent;*
(ii) *$\Phi_S \times \Phi_U$, $\Psi_S \times \Psi_U$ are C^0-flow equivalent, and Φ_C, Ψ_C are linearly flow equivalent;*
(iv) *$\dim X_S^\Phi = \dim Y_S^\Psi$, $\dim X_U^\Phi = \dim Y_U^\Psi$, and A^{Φ_C}, αA^{Ψ_C} are similar for some $\alpha \in \mathbb{R}^+$, i.e., $HA^{\Phi_C} = \alpha A^{\Psi_C} H$ with some linear isomorphism $H : X_C^\Phi \to Y_C^\Psi$.*

$$
\begin{array}{ccc}
\text{linearly flow equivalent} & \Longleftrightarrow & \text{linearly orbit equivalent} \\
\Updownarrow & & \Updownarrow \\
C^{\infty}\text{-flow equivalent} & \Longleftrightarrow & C^{\infty}\text{-orbit equivalent} \\
\Updownarrow & & \Updownarrow \\
C^{\ell}\text{-flow equivalent} & \Longleftrightarrow & C^{\ell}\text{-orbit equivalent} \\
\Updownarrow & & \Updownarrow \\
C^{1}\text{-flow equivalent} & \Longleftrightarrow & C^{1}\text{-orbit equivalent} \\
 & \Downarrow \ (\Uparrow\ X_{\mathrm{C}}^{\Phi} = X \text{ or } \dim X = 1) & \\
C^{0}\text{-flow equivalent} & \Longleftrightarrow & C^{0}\text{-orbit equivalent}
\end{array}
$$

“smooth”
(Theorem 1.2)

“topological”
(Theorem 1.1)

Fig. 2 By Theorems 1.1 and 1.2, for real linear flows all notions of equivalence coalesce into only two different notions, or even just one if $X_C^{\Phi} = X$ or $\dim X = 1$

In the presence of smoothness, i.e., for C^{ℓ}-equivalence with $\ell \geq 1$, the counterpart of Theorem 1.1 is the following **smooth classification theorem** which shows that in fact the weakest notion (C^1-orbit equivalence) implies the strongest (linear flow equivalence).

Theorem 1.2 *Let Φ, Ψ be linear flows. Then each of the following four statements implies the other three:*

(i) Φ, Ψ *are C^1-orbit equivalent;*
(ii) Φ, Ψ *are C^1-flow equivalent;*
(iii) Φ, Ψ *are linearly flow equivalent;*
(iv) $A^{\Phi}, \alpha A^{\Psi}$ *are similar for some $\alpha \in \mathbb{R}^+$.*

Taken together, Theorems 1.1 and 1.2 reveal a remarkable rigidity of finite-dimensional real linear flows: For such flows, there really are only two different notions of equivalence, informally referred to as **topological** and **smooth** equivalence; for central or one-dimensional flows, even these two notions coalesce. Moreover, the theorems characterize these equivalences in terms of elementary properties of the associated generators.

As far as the authors have been able to ascertain, variants of Theorem 1.1 were first proved, independently, in [24,26], though of course for hyperbolic linear flows the result dates back much further (see, e.g., [3,4,20]; a detailed discussion of the pertinent literature is deferred to Sect. 5 when all relevant technical terms will have been introduced). Given the clear, definitive nature of Theorem 1.1 and the fundamental importance of linear differential equations throughout science, it is striking that the details of [24,26] have not been disseminated more widely in over four decades [18]. A main objective of this article, then, is to provide an elementary, self-contained proof of Theorem 1.1 that hopefully will find its way into future textbooks on differential equations. In the process, several inaccuracies and gaps in the classical arguments are addressed as well. As presented here, Theorem 1.2 is a rather straightforward consequence of Theorem 1.1. Although the result itself seems to have long been part of dynamical systems folklore [3,4,6,12,30,34], the authors are not aware of any reference that would establish it in its full strength, that is, without imposing additional (and, as it turns out, unnecessary) assumptions on τ.

To appreciate the difference between Theorems 1.1 and 1.2, first note that for $\dim X = 1$, trivially *all* notions of equivalence coincide, yielding exactly three equivalence classes of real linear flows, represented by $\Phi(t, x) = e^{ta}x$ with $a \in \{-1, 0, 1\}$. However, already for $\dim X = 2$ the huge difference between the theorems becomes apparent: On the one hand, by Theorem 1.1, there are exactly eight topological (C^0) equivalence classes, represented by $\Phi(t, x) = e^{tA}x$ with A being precisely one of

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.2}$$

By Theorem 1.2, on the other hand, all smooth (C^1) equivalence classes are represented uniquely by the five left-most matrices in (1.2), together with $\pm\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and the five infinite families

$$\begin{bmatrix} -1 & 0 \\ 0 & a \end{bmatrix}, \pm\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \pm\begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix} \quad (a \in \mathbb{R}^+).$$

This article is organized as follows. Section 2 briefly reviews the notions of equivalence for flows, as well as a few basic dynamical concepts. It then introduces **cores**, a new family of invariant objects. Although these objects may well be useful in more general contexts, their properties are established here only as far as needed for the subsequent analysis of flows on finite-dimensional normed spaces. Section 3 specifically identifies cores for real linear flows, and shows how they can be iterated in a natural way. As it turns out, the proof of Theorem 1.1 also hinges on a careful analysis of bounded linear flows, and the latter is carried out in Sect. 4. With all required tools finally assembled, proofs of Theorems 1.1 and 1.2 are presented in Sect. 5, together with several comments on related results in the literature that prompted this work. While, for reasons that will become apparent in Sect. 6, the article focuses mostly on *real* spaces, the concluding section shows how the results carry over to *complex* spaces in a natural way. To keep the exposition focussed squarely on the main arguments, several elementary (and, presumably, known) facts of an auxiliary nature are stated without proof; for details regarding these facts, as well as others that are mentioned in passing but for which the authors were unable to identify a precise reference, the interested reader is referred to the accompanying document [37].

Throughout, the familiar symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}^+$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of all positive integers, non-negative integers, integers, rational, positive real, real, and complex numbers, respectively; for convenience, $c + \Omega = \{c + \omega : \omega \in \Omega\}$ and $c\Omega = \{c\omega : \omega \in \Omega\}$ for any $c \in \mathbb{C}$, $\Omega \subset \mathbb{C}$. Occasionally, for the purpose of coordinate-dependent arguments, elements of $\mathbb{Z}^m$, $\mathbb{R}^m$, or $\mathbb{C}^m$, with $m \in \mathbb{N} \setminus \{1\}$, are interpreted as $m \times 1$-column vectors.

2 Orbit Equivalence

Let X, Y be two finite-dimensional normed spaces over $\mathbb{R}$, and let φ, ψ, respectively, be flows on them; unless specified further, $\|\cdot\|$ denotes any norm on either space. Given two functions $h : X \to Y$ and $\tau : \mathbb{R} \times X \to \mathbb{R}$, say that φ is (h, τ)-**related** to ψ if h is a homeomorphism, $\tau(\cdot, x)$ is strictly increasing for each $x \in X$, and

$$h\big(\varphi(t, x)\big) = \psi\big(\tau(t, x), h(x)\big) \quad \forall (t, x) \in \mathbb{R} \times X. \tag{1.1}$$

In what follows, for each $t \in \mathbb{R}$ the homeomorphism $\varphi(t, \cdot) : X \to X$ usually is denoted φ_t, and for each $x \in \mathbb{R}$ the strictly increasing map $\tau(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is denoted τ_x. With this, (1.1) succinctly reads

$$h \circ \varphi_t(x) = \psi_{\tau_x(t)} \circ h(x) \quad \forall (t,x) \in \mathbb{R} \times X.$$

Thus φ is (h,τ)-related to ψ precisely if the homeomorphism h maps each φ-orbit into a ψ-orbit in an orientation-preserving way. Note that no assumption whatsoever is made regarding the x-dependence of τ_x. Still, utilizing the flow axioms of φ, ψ, and the continuity of h, h^{-1}, it is readily deduced from (1.1) that the function τ can be assumed to have several additional properties; cf. [5,6,31]. For convenience, these properties are understood to be part of what it means for φ to be (h,τ)-related to ψ throughout the remainder of this article.

Proposition 2.1 *Let φ, ψ be flows on X, Y, respectively, and assume that φ is (h,τ)-related to ψ. Then φ is $(h,\widetilde{\tau})$-related to ψ where $\widetilde{\tau}_x : \mathbb{R} \to \mathbb{R}$ is, for every $x \in X$, an (increasing) continuous bijection with $\widetilde{\tau}_x(0) = 0$.*

Recall from the Introduction that two flows φ, ψ are (C^0-)**orbit equivalent** if φ is (h,τ)-related to ψ for some h, τ; they are **flow equivalent** if, with the appropriate constant $\alpha \in \mathbb{R}^+$, the function τ can be chosen so that $\tau_x(t) = \alpha t$ for all $(t,x) \in \mathbb{R} \times X$. This terminology is justified.

Proposition 2.2 *Orbit equivalence and flow equivalence are equivalence relations in the class of all flows on finite-dimensional normed spaces.*

A simple, classical example of orbit equivalence, presented in essence (though not always in name) by many textbooks, is as follows [32, Sec. 3.1]: Assume that two flows φ, ψ on X are generated by the differential equations $\dot{x} = V(x)$, $\dot{x} = W(x)$, respectively, with C^∞-vector fields V, W. If $V = wW$ for some (measurable and locally bounded) function $w : X \to \mathbb{R}^+$ then φ is (id_X, τ)-related to ψ, with $\tau_x(t) = \int_0^t w\big(\varphi_s(x)\big)\,\mathrm{d}s$ for all $(t,x) \in \mathbb{R} \times X$.

For every $x \in X$, let $T_x^\varphi = \inf\{t \in \mathbb{R}^+ : \varphi_t(x) = x\}$, with the usual convention that $\inf \varnothing = +\infty$. Note that whenever the set $\{t \in \mathbb{R}^+ : \varphi_t(x) = x\}$ is non-empty, it equals either $\mathbb{R}^+$ or $\{nT_x^\varphi : n \in \mathbb{N}\}$. In the former case, $T_x^\varphi = 0$, and x is a **fixed point** of φ. In the latter case, $0 < T_x^\varphi < +\infty$, and x is T-**periodic**, i.e., $\varphi_T(x) = x$ with $T \in \mathbb{R}^+$, precisely for $T \in T_x^\varphi \mathbb{N}$; in particular, T_x^φ is the minimal φ-**period** of x. Denote by $\mathrm{Fix}\,\varphi$ and $\mathrm{Per}_T\varphi$ the sets of all fixed and T-periodic points respectively, and let $\mathrm{Per}\,\varphi = \bigcup_{T \in \mathbb{R}^+} \mathrm{Per}_T\varphi$. Note that $T_\cdot^\varphi$ is lower semi-continuous, with $T_x^\varphi = 0$ and $T_x^\varphi < +\infty$ if and only if $x \in \mathrm{Fix}\,\varphi$ and $x \in \mathrm{Per}\,\varphi$, respectively.

The φ-**orbit** of any $x \in X$ is $\varphi(\mathbb{R}, x) = \{\varphi_t(x) : t \in \mathbb{R}\}$. Recall that $C \subset X$ is φ-**invariant** if $\varphi_t(C) = C$ for all $t \in \mathbb{R}$, or equivalently if $\varphi(\mathbb{R}, x) \subset C$ for every $x \in C$. Clearly, $\mathrm{Fix}\,\varphi$ and $\mathrm{Per}\,\varphi$ are φ-invariant, and so is $\mathrm{Per}_T\varphi$ for every $T \in \mathbb{R}^+$. Another example of a φ-invariant set is $\mathrm{Bnd}\,\varphi := \big\{x \in X : \sup_{t\in\mathbb{R}} \|\varphi_t(x)\| < +\infty\big\}$, which simply is the union of all bounded φ-orbits. Plainly,

$$\mathrm{Fix}\,\varphi \subset \mathrm{Per}_T\varphi \subset \mathrm{Per}\,\varphi \subset \mathrm{Bnd}\,\varphi \quad \forall T \in \mathbb{R}^+.$$

Proposition 2.3 *Let φ, ψ be flows on X, Y, respectively, and assume that φ is (h,τ)-related to ψ. Then $C \subset X$ is φ-invariant if and only if $h(C) \subset Y$ is ψ-invariant. Moreover,*

$$h(Fix\,\varphi) = Fix\,\psi, \quad h(Per\,\varphi) = Per\,\psi, \quad h(Bnd\,\varphi) = Bnd\,\psi.$$

A simple observation with far-reaching consequences for the subsequent analysis is that, under the assumptions of Proposition 2.3, and for any $T \in \mathbb{R}^+$, the ψ-invariant set $h(\mathrm{Per}_T\varphi)$

may not be contained in $\mathrm{Per}_S\psi$ for any $S \in \mathbb{R}^+$. A numerical invariant that can be used to address this "scrambling" of $\mathrm{Per}\,\varphi \setminus \mathrm{Fix}\,\varphi$ by h is the φ-**height** of x, defined as

$$\langle x\rangle^{\varphi} = \limsup_{\widetilde{x}\in\mathrm{Per}\,\varphi, \widetilde{x}\to x} \frac{T^{\varphi}_{\widetilde{x}}}{T^{\varphi}_{x}} \quad \forall x \in \mathrm{Per}\,\varphi \setminus \mathrm{Fix}\,\varphi.$$

Note that $\langle x\rangle^{\varphi}$ equals either a positive integer or $+\infty$, and with $\langle x\rangle^{\varphi} := +\infty$ for every $x \in \mathrm{Fix}\,\varphi$, the function $\langle\cdot\rangle^{\varphi}$ is upper semi-continuous on $\mathrm{Per}\,\varphi$; cf. [26, Def. 5]. As is readily confirmed, minimal periods and heights are well-behaved under orbit equivalence.

Proposition 2.4 *Let φ, ψ be flows, and assume that φ is (h, τ)-related to ψ. Then, for every $x \in \mathrm{Per}\,\varphi$:*

(i) $T^{\psi}_{h(x)} = \tau_x(T^{\varphi}_x)$;
(ii) $\langle h(x)\rangle^{\psi} = \langle x\rangle^{\varphi}$.

The subsequent analysis relies heavily on the properties of certain invariant sets associated with the flows under consideration. Specifically, given a flow φ on X and any two points $x^-, x^+ \in X$, define the (x^-, x^+)-**core** $C_{x^-,x^+}(\varphi, X)$ as

$$C_{x^-,x^+}(\varphi, X) = \Big\{x \in X\colon \text{There exist sequences}\,(t_n^{\pm}) \text{ and } (x_n^{\pm}) \text{ with } t_n^{\pm} \to \pm\infty \\ \text{and } x_n^{\pm} \to x \text{ such that } \varphi_{t_n^{\pm}}(x_n^{\pm}) \to x^{\pm}\Big\};$$

here and throughout, expressions containing $\pm$ (or $\mp$) are to be read as two separate expressions containing only the upper and only the lower symbols, respectively. Note that $C_{x^-,x^+}(\varphi, X)$ is φ-invariant and closed, possibly empty. For linear flows, the $(0, 0)$-core $C_{0,0}(\varphi, X)$, henceforth simply denoted $C_0(\varphi, X)$, is naturally of particular relevance, and so is the **core**

$$C(\varphi, X) := \bigcup\nolimits_{x^-,x^+\in X} C_{x^-,x^+}(\varphi, X) \;\supset\; C_0(\varphi, X).$$

Clearly, $C(\varphi, X)$ also is φ-invariant and contains $\mathrm{Bnd}\,\varphi$ as well as all non-wandering points of φ. For instance, if X is one-dimensional then $C(\varphi, X)$ simply is the convex hull of $\mathrm{Fix}\,\varphi$, whereas $C_0(\varphi, X) = \{0\} \cap \mathrm{Fix}\,\varphi$. Most importantly, $C(\varphi, X)$ and $C_0(\varphi, X)$ both are well-behaved under orbit equivalence.

Lemma 2.5 *Let φ, ψ be flows on X, Y, respectively, and assume that φ is (h, τ)-related to ψ. Then*

$$h\big(C_{x^-,x^+}(\varphi, X)\big) = C_{h(x^-),h(x^+)}(\psi, Y) \quad \forall x^-, x^+ \in X. \tag{2.1}$$

Thus $h\big(C(\varphi, X)\big) = C(\psi, Y)$, and if $h(0) = 0$ then also $h\big(C_0(\varphi, X)\big) = C_0(\psi, Y)$.

The proof of Lemma 2.5 is facilitated by an elementary observation [37].

Proposition 2.6 *Let φ be a flow on X, and $x \in X$. Then the following are equivalent:*

(i) *For every $\varepsilon > 0$ there exists an $\widetilde{x} \in X$ such that $\|\varphi_t(\widetilde{x}) - x\| < \varepsilon$ for all $0 \le t \le \varepsilon^{-1}$;*
(ii) *$x \in \mathrm{Fix}\,\varphi$.*

Proof of Lemma 2.5 It suffices to prove (2.1), as all other assertions directly follow from it. To do this, given $x^-, x^+ \in X$, denote $C_{x^-,x^+}(\varphi, X)$ and $C_{h(x^-),h(x^+)}(\psi, Y)$ simply by C and D, respectively. From reversing the roles of (φ, X) and (ψ, Y), as well as h and h^{-1}, it is clear that all that needs to be shown is that $h(C) \subset D$.

Pick any $x \in C$, together with sequences $(t_n^\pm)$ and $(x_n^\pm)$ with $t_n^\pm \to \pm\infty$ and $x_n^\pm \to x$ such that $\varphi_{t_n^\pm}(x_n^\pm) \to x^\pm$; assume w.l.o.g. that $t_n^- < 0 < t_n^+$ for all n. Letting $s_n^\pm = \tau_{x_n^\pm}(t_n^\pm)$, note that $s_n^- < 0 < s_n^+$, and

$$h(x_n^\pm) \to h(x), \quad \psi_{s_n^\pm}\big(h(x_n^\pm)\big) \to h(x^\pm). \tag{2.2}$$

By considering appropriate subsequences, assume that $s_n^- \to s^- \in [-\infty, 0]$ and $s_n^+ \to s^+ \in [0, +\infty]$. Note that (2.2) immediately yields $h(x) \in D$ if $\{s^-, s^+\} = \{-\infty, +\infty\}$, so assume for instance that $s^+ < +\infty$. (The case of $s^- > -\infty$ is completely analogous.) Then $\psi_{s^+}\big(h(x)\big) = h(x^+)$ by (2.2), and, as will be shown below, in fact

$$h(x) \in \operatorname{Per} \psi. \tag{2.3}$$

Assuming (2.3), let $T \in \mathbb{R}^+$ be any ψ-period of $h(x)$, and $y_n^+ = h(x)$, $r_n^+ = s^+ + nT$ for every $n \in \mathbb{N}$. With this clearly $r_n^+ \to +\infty$, and $\psi_{r_n^+}\big(h(x)\big) = h(x^+)$ for all n. Thus to complete the proof it only remains to verify (2.3).

Assume first that $s^+ = 0$, and hence $x = x^+$. For each $n \in \mathbb{N}$, define a non-negative continuous function $f_n : \mathbb{R} \to \mathbb{R}$ as $f_n(s) = \|\varphi_{st_n^+}(x_n^+) - x\|$, and note that $f_n(0) = \|x_n^+ - x\| \to 0$, but also $f_n(1) = \|\varphi_{t_n^+}(x_n^+) - x\| \to 0$. In fact, more is true:

$$\lim_{n\to\infty} f_n(s) = 0 \quad \text{uniformly on } [0, 1]\,. \tag{2.4}$$

To prove (2.4), suppose by way of contradiction that

$$\varepsilon_0 \le f_{n_k}(s_k) = \|\varphi_{s_k t_{n_k}^+}(x_{n_k}^+) - x\| \quad \forall k \in \mathbb{N}, \tag{2.5}$$

with appropriate $\varepsilon_0 > 0$, $s_k \in [0, 1]$, and integers $n_k \ge k$. Since $0 \le r_k := \tau_{x_{n_k}^+}(s_k t_{n_k}^+) \le \tau_{x_{n_k}^+}(t_{n_k}^+) = s_{n_k}^+ \to 0$, clearly $h\big(\varphi_{s_k t_{n_k}^+}(x_{n_k}^+)\big) = \psi_{r_k}\big(h(x_{n_k}^+)\big) \to h(x)$, which, together with (2.5), contradicts the continuity of h^{-1} at $h(x)$, and hence establishes (2.4). Deduce that, given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ with $\max_{s\in[0,1]} f_N(s) < \varepsilon$ as well as $t_N^+ > \varepsilon^{-1}$. But then $\|\varphi_t(x_N^+) - x\| < \varepsilon$ for all $0 \le t \le \varepsilon^{-1}$, and Proposition 2.6 yields $x \in \operatorname{Fix} \varphi$. By Proposition 2.3, $h(x) \in \operatorname{Fix} \psi$, which proves (2.3) when $s^+ = 0$.

Finally, assume that $s^+ \in \mathbb{R}^+$, and let $t^+ = \tau_x^{-1}(s^+) > 0$. Then $h\big(\varphi_{t^+}(x)\big) = \psi_{s^+}\big(h(x)\big) = h(x^+)$, and consequently $\varphi_{t^+}(x) = x^+$, as well as

$$\psi_{\tau_{x_n^+}(t_n^+ - t^+)}\big(h(x_n^+)\big) = h \circ \varphi_{-t^+}\big(\varphi_{t_n^+}(x_n^+)\big) \;\to\; h \circ \varphi_{-t^+}(x^+) = h(x).$$

Since $0 \le \tau_{x_n^+}(t_n^+ - t^+) \le s_n^+$ for all large n, assume w.l.o.g. that $\tau_{x_n^+}(t_n^+ - t^+) \to r \in [0, s^+]$, and hence $\psi_r\big(h(x)\big) = h(x)$. On the one hand, if $r \in \mathbb{R}^+$ then clearly $h(x) \in \operatorname{Per} \psi$. On the other hand, if $r = 0$ then (2.4) holds with $f_n(s) = \|\varphi_{s(t_n^+ - t^+)}(x_n^+) - x\|$, and the same argument as above shows that $x \in \operatorname{Fix} \varphi$. Thus (2.3) also holds when $s^+ \in \mathbb{R}^+$. □

A crucial step in the subsequent analysis is the decomposition of flows into simpler, well-understood parts. To prepare for this, recall that two flows φ, ψ on X, Y, respectively, together induce the product flow $\varphi \times \psi$ on $X \times Y$, by letting $(\varphi \times \psi)_t = \varphi_t \times \psi_t$ for all $t \in \mathbb{R}$. Endow $X \times Y$ with any norm. It is readily seen that

$$C_{(x^-,y^-),(x^+,y^+)}(\varphi \times \psi, X \times Y) \subset C_{x^-,x^+}(\varphi, X) \times C_{y^-,y^+}(\psi, Y) \quad \forall x^-, x^+ \in X, y^-, y^+ \in Y,$$

and therefore also

$$C(\varphi \times \psi, X \times Y) \subset C(\varphi, X) \times C(\psi, Y); \tag{2.6}$$

the same inclusion is valid with C_0 instead of C. Quite trivially, equality holds in (2.6) and its analogue for C_0 if one factor is at most one-dimensional. As the following example shows, however, equality does not hold in general if $\min\{\dim X, \dim Y\} \geq 2$.

Example 2.7 Let $X = \mathbb{R}^2$, and write $X^+ = (\mathbb{R}^+)^2$ and $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for convenience. Consider the flow φ on X generated by $\dot{x} = V(x)$, with the C^∞-vector field

$$V(x) = \begin{cases} \dfrac{1}{s}\begin{bmatrix} f(s)x_1 \log x_1 - sf(s)x_1 \log x_2 \\ sf(s)x_2 \log x_1 + f(s)x_2 \log x_2 \end{bmatrix} & \text{if } x \in X^+ \setminus \{\mathbf{1}\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $s = s(x) = \sqrt{(\log x_1)^2 + (\log x_2)^2}$, and $f(s) = e^{-s-1/s}$ for all $s \in \mathbb{R}^+$. Clearly, $(X \setminus X^+) \cup \{\mathbf{1}\} \subset \operatorname{Fix} \varphi$. Introducing (exponential) polar coordinates $x_1 = e^{r\cos\theta}$, $x_2 = e^{r\sin\theta}$ in X^+ transforms $\dot{x} = V(x)$ into

$$\dot{r} = \dot{\theta} = f(r). \tag{2.7}$$

Deduce from (2.7) that $\lim_{t\to-\infty} r(t) = 0$, $\lim_{t\to+\infty} r(t) = +\infty$, and $r - \theta$ is constant. Consequently, $\lim_{t\to-\infty} \varphi_t(x) = \mathbf{1}$ for every $x \in X^+$, but also, given any $x \in X^+ \setminus \{\mathbf{1}\}$, there exists a sequence (t_n^+) with $t_n^+ \to +\infty$ such that $\theta(t_n^+) + \frac{3}{4}\pi \in 2\pi\mathbb{Z}$ for all n, and hence $\lim_{n\to\infty} \varphi_{t_n^+}(x) = 0$. Thus $x \in C_{\mathbf{1},0}(\varphi, X)$ for every $x \in X^+ \setminus \{\mathbf{1}\}$, and $C(\varphi, X) = X$; see also Fig. 3.

Next, note that f is decreasing on $[1, +\infty[$, and hence any two solutions (r, θ), $(\widetilde{r}, \widetilde{\theta})$ of (2.7) with $r(0), \widetilde{r}(0) \geq 1$ satisfy

$$\left|r(t)-\widetilde{r}(t)\right| \leq \left|r(0)-\widetilde{r}(0)\right|, \quad \left|\theta(t)-\widetilde{\theta}(t)\right| \leq \left|r(0)-\widetilde{r}(0)\right|+\left|\theta(0)-\widetilde{\theta}(0)\right| \quad \forall t \geq 0; \tag{2.8}$$

moreover, $\theta - \widetilde{\theta}$ is constant whenever $r(0) = \widetilde{r}(0)$. Pick any $a \geq e^{1/\sqrt{2}}$, and consider

$$u = \begin{bmatrix} a \\ a^{-1} \end{bmatrix}, \quad \widetilde{u} = \begin{bmatrix} a^{-1} \\ a \end{bmatrix}. \tag{2.9}$$

Then $r(t) = s\big(\varphi_t(u)\big) = s\big(\varphi_t(\widetilde{u})\big) = \widetilde{r}(t) \geq 1$ and $\theta(t) - \widetilde{\theta}(t) \in \pi + 2\pi\mathbb{Z}$ for all $t \geq 0$. Also, let

$$U = \left\{x \in X^+ : x_2^{\sqrt{3}} > \max\{x_1, x_1^3\}^{-1}\right\} = \left\{x \in X^+ \setminus \{\mathbf{1}\} : \theta \in \left]-\tfrac{1}{3}\pi, \tfrac{5}{6}\pi\right[+ 2\pi\mathbb{Z}\right\}.$$

For any $\varepsilon > 0$ sufficiently small, it is clear from (2.8) that for every $t \geq 0$ at least one of the two open sets $\varphi_t\big(B_\varepsilon(u)\big)$ and $\varphi_t\big(B_\varepsilon(\widetilde{u})\big)$ is entirely contained in U. Note that $B_\varepsilon(u) \times B_\varepsilon(\widetilde{u})$ is a neighbourhood of $(u, \widetilde{u})$ in $X \times X$. Consequently, $\big((\varphi \times \varphi)_{t_n^+}(x_n, \widetilde{x}_n)\big)$ is unbounded whenever $t_n^+ \to +\infty$ and $(x_n, \widetilde{x}_n) \to (u, \widetilde{u})$. Thus, $(u, \widetilde{u}) \notin C(\varphi \times \varphi, X \times X)$, whereas clearly $(u, \widetilde{u}) \in C(\varphi, X) \times C(\varphi, X)$, and so the inclusion (2.6) is strict in this example.

Good behaviour of certain invariant objects under products is indispensable for the analysis in later sections. Negative examples such as Example 2.7 therefore suggest that the cores $C(\varphi, X)$ and $C_0(\varphi, X)$ be supplanted, or at least supplemented with similar objects that are well-behaved under products. To this end, note that

$$C(\varphi, X) = \big\{x \in X : \text{There exist sequences } (t_n^\pm) \text{ and } (x_n^\pm) \text{ with } t_n^\pm \to \pm\infty \\ \text{and } x_n^\pm \to x \text{ such that } \big(\varphi_{t_n^\pm}(x_n^\pm)\big) \text{ both are bounded}\big\}.$$

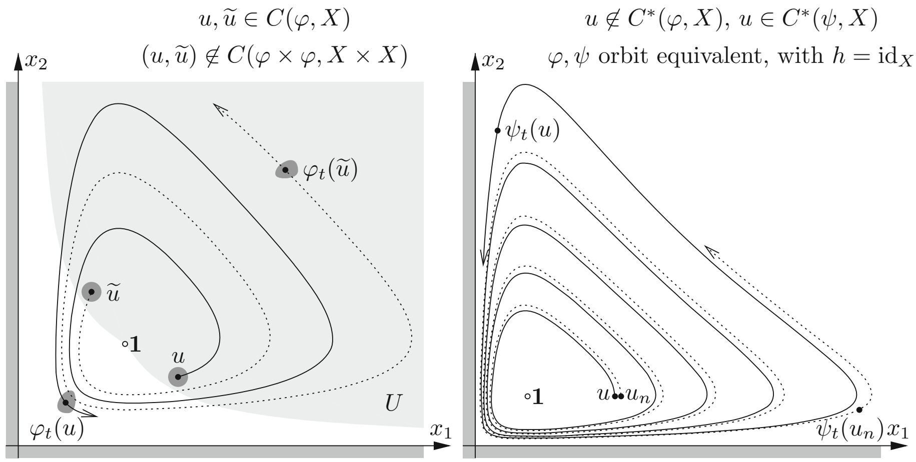

Fig. 3 In general, (non-uniform) cores are well-behaved under orbit equivalence but not under products (left; see Example 2.7), whereas for uniform cores the situation is the exact opposite (right; see Example 2.9)

In light of this, define the **uniform core** $C^*(\varphi, X)$ as

$$C^*(\varphi, X) = \big\{x \in X : \text{For } \textit{every} \text{ sequence } (t_n) \text{ with } |t_n| \to +\infty \text{ there exists a sequence } (x_n) \text{ with } x_n \to x \text{ such that } \big(\varphi_{t_n}(x_n)\big) \text{ is bounded}\big\};$$

analogously, define the **uniform** (0, 0)-**core** $C_0^*(\varphi, X)$ as

$$C_0^*(\varphi, X) = \big\{x \in X : \text{For } \textit{every} \text{ sequence } (t_n) \text{ with } |t_n| \to +\infty \text{ there exists a sequence } (x_n) \text{ with } x_n \to x \text{ such that } \varphi_{t_n}(x_n) \to 0\big\} \subset C^*(\varphi, X).$$

Again, $C^*(\varphi, X)$ and $C_0^*(\varphi, X)$ are φ-invariant, and they obviously are contained in their non-uniform counterparts, i.e.,

$$C^*(\varphi, X) \subset C(\varphi, X), \quad C_0^*(\varphi, X) \subset C_0(\varphi, X). \tag{2.10}$$

Moreover, $C^*(\varphi, X) \supset \mathrm{Bnd}\,\varphi$, just as for (non-uniform) cores. For the flow φ in Example 2.7, it is clear that $C^*(\varphi, X) = \mathrm{Fix}\,\varphi \neq X = C(\varphi, X)$; see also Example 2.9 below. Thus the left inclusion in (2.10) is strict in general, and so is the right inclusion.

As alluded to earlier, $C^*(\varphi, X)$ and $C_0^*(\varphi, X)$ are useful for the purpose of this article because, unlike their non-uniform counterparts, they are well-behaved under products.

Lemma 2.8 *Let φ, ψ be flows on X, Y, respectively. Then*

$$C^*(\varphi \times \psi, X \times Y) = C^*(\varphi, X) \times C^*(\psi, Y)$$

as well as

$$C_0^*(\varphi \times \psi, X \times Y) = C_0^*(\varphi, X) \times C_0^*(\psi, Y).$$

Proof The asserted equality for C^* (respectively, C_0^*) is an immediate consequence of the fact that $\big((\varphi \times \psi)_{t_n}(x_n, y_n)\big)$ is bounded (converges to 0) if and only if $\big(\varphi_{t_n}(x_n)\big)$ and $\big(\psi_{t_n}(y_n)\big)$ both are bounded (converge to 0). □

Regarding the behaviour of uniform cores under equivalence, it is readily checked that if φ, ψ are *flow* equivalent then $h\big(C^*(\varphi, X)\big) = C^*(\psi, Y)$; moreover, $h\big(C_0^*(\varphi, X)\big) = C_0^*(\psi, Y)$ if $h(0) = 0$. These equalities may fail under mere *orbit* equivalence, however, so the analogue of Lemma 2.5 for uniform cores does not hold. The following example demonstrates this.

Example 2.9 With the identical objects as in Example 2.7, first deduce from (2.8) that, given any $x \in X^+ \setminus \{\mathbf{1}\}$ and sufficiently small $\varepsilon > 0$, one may chose (t_n) with $t_n \to +\infty$ such that $\varphi_{t_n}\big(B_\varepsilon(x)\big) \subset U$ for all n. But then clearly $\big(\varphi_{t_n}(x_n)\big)$ is unbounded whenever $x_n \to x$, and hence $x \notin C^*(\varphi, X)$. Thus, $C^*(\varphi, X) = (X \setminus X^+) \cup \{\mathbf{1}\} = \mathrm{Fix}\,\varphi \neq C(\varphi, X)$; see also Fig. 3.

Next, fix a decreasing C^∞-function $g : \mathbb{R} \to \mathbb{R}$ with $g(s) = 1$ for all $s \le 1$ and $g(s) = 0$ for all $s \ge 2$. Let ψ be the flow on X generated by $\dot{x} = v(x)V(x)$, where $v : X \to \mathbb{R}$ is given by

$$v(x) = \begin{cases} 1 + e^{4\pi} g\big((s - \log x_1 \cos s - \log x_2 \sin s)s^{-1}e^{s-1/s}\big) & \text{if } x \in X^+ \setminus \{\mathbf{1}\}, \\ 1 & \text{otherwise}; \end{cases} \tag{2.11}$$

note that the vector field vV is C^∞. Similarly to Example 2.7, $(X \setminus X^+) \cup \{\mathbf{1}\} = \mathrm{Fix}\,\psi$, and (exponential) polar coordinates in X^+ transform $\dot{x} = v(x)V(x)$ into

$$\dot{r} = \dot{\theta} = f(r) + e^{4\pi} f(r) g\Big(\big(1 - \cos(\theta - r)\big)e^{r-1/r}\Big). \tag{2.12}$$

Note that $r - \theta$ again is constant for every solution of (2.12). Specifically, given any $0 \le a \le \frac{1}{2}$, let (r_a, θ_a) be the solution of (2.12) with $r(0) = 2\pi(1+a)$ and $\theta(0) = 0$. Then $r_a(t) - \theta_a(t) = 2\pi(1 + a)$ and $r_a(t) - r_0(t) \le 2\pi a$ for all $t \ge 0$. Notice that $\lim_{t\to+\infty} r_a(t) = +\infty$. Consequently, for every $0 < a \le \frac{1}{2}$ there exists a $t_a \in \mathbb{R}^+$ such that $\dot{r}_a = f(r_a)$ for all $t \ge t_a$, but also $e^{-1/r_0}(1 + e^{4\pi}) > 1 + e^{3\pi}$. Clearly, $\lim_{a\downarrow 0} t_a = +\infty$; assume w.l.o.g. that $a \mapsto t_a$ is decreasing on $]0, \frac{1}{2}]$. It follows that $\dot{r}_0 \ge e^{-r_0}(1 + e^{3\pi})$ as well as $\dot{r}_a \le e^{-r_a}$ on $[t_a, +\infty[$, and therefore also, with $\widetilde{t}_a := t_a + e^{4\pi + r_0(t_a)}$,

$$\theta_0(t) - \theta_a(t) = r_0(t) - r_a(t) + 2\pi a \ge 2\pi a + \log \frac{e^{r_0(t_a)} + (t - t_a)(1 + e^{3\pi})}{e^{r_a(t_a)} + t - t_a} > 3\pi \quad \forall t \ge \widetilde{t}_a.$$

Deduce from this and the continuity of $a \mapsto \theta_a(t)$, that, given any integer $j \ge 2$ and $t \ge \widetilde{t}_{1/j}$, there exists $0 < a_j(t) \le j^{-1}$ such that $\theta_{a_j(t)}(t) + \frac{3}{4}\pi \in 2\pi\mathbb{Z}$.

With these preparations, consider the point $u = \begin{bmatrix} e^{2\pi} \\ 1 \end{bmatrix} \notin C^*(\varphi, X)$, and let (t_n) be any sequence with $|t_n| \to +\infty$. If $t_n \to -\infty$ then $\big(\psi_{t_n}(u)\big)$ is bounded, in fact $\psi_{t_n}(u) \to \mathbf{1}$, so it suffices to assume that (t_n) is increasing, and $t_1 \ge \widetilde{t}_{1/2}$. Pick a sequence (j_n) with $\widetilde{t}_{1/j_n} \le t_n < \widetilde{t}_{1/j_{n+1}}$ for all n. Note that $j_n \to \infty$, and hence $0 < a_{j_n}(t_n) < j_n^{-1} \to 0$. Writing $b_n := a_{j_n}(t_n)$ for convenience, consider

$$u_n = \begin{bmatrix} e^{2\pi(1+b_n)} \\ 1 \end{bmatrix} \quad \forall n \in \mathbb{N}.$$

With this, not only $u_n \to u$, but also $\psi_{t_n}(u_n) = e^{-r_{b_n}(t_n)/\sqrt{2}}\mathbf{1} \to 0$, showing that $\big(\psi_{t_n}(u_n)\big)$ is bounded. In other words, $u \in C^*(\psi, X)$. Recall that φ and ψ are generated by $\dot{x} = V(x)$ and $\dot{x} = v(x)V(x)$, respectively, with v given by (2.11), and $1 \le v \le 1 + e^{4\pi}$. As pointed out right after Proposition 2.2, the flows φ, ψ are orbit equivalent with $h = \mathrm{id}_X$, and yet $h\big(C^*(\varphi, X)\big) \neq C^*(\psi, X)$.

3 Cores of Linear Flows

In a linear flow, naturally an invariant set is of particular interest if it also is a (linear) subspace. For instance, Fix Φ and Bnd Φ (but not, in general, Per Φ) are Φ-invariant subspaces for any linear flow Φ, and so are all uniform cores. As seen in the previous section, uniform cores are well-behaved under products (Lemma 2.8) but not under orbit equivalence (Example 2.9), whereas for (non-uniform) cores the situation is the exact opposite (Lemma 2.5 and Example 2.7). This discrepancy is consistent with a lack of equality in (2.10) in general. One main result of this section, Theorem 3.5 below, shows that both inclusions in (2.10) are in fact equalities—provided that φ is linear. As an important consequence, all cores of linear flows are invariant subspaces that are well-behaved under orbit equivalence *and* under products. With regard to the last assertion in Lemma 2.5, the following additional property of orbit equivalences is useful when dealing with linear flows; again, for convenience this property is hereafter assumed to be part of what it means for Φ to be (h, τ)-related to Ψ.

Proposition 3.1 *Let Φ, Ψ be linear flows, and assume that Φ is (h, τ)-related to Ψ. Then Φ is $(\widetilde{h}, \tau)$-related to Ψ where $\widetilde{h}(0) = 0$.*

In a first step towards Theorem 3.5, cores of irreducible linear flows are considered. Recall that Φ is **irreducible** if $X = Z \oplus \widetilde{Z}$, with Φ-invariant subspaces $Z, \widetilde{Z}$, implies that $Z = \{0\}$ or $\widetilde{Z} = \{0\}$. Plainly, Φ is irreducible if and only if, relative to the appropriate basis, A^Φ is a single real Jordan block. In particular, for irreducible Φ the spectrum $\sigma(\Phi) := \sigma(A^\Phi)$ is either a real singleton or a non-real complex conjugate pair. In order to clarify the structure of cores of irreducible linear flows, for every $s \in \mathbb{R}$ denote by $\lceil s \rceil$ and $\lfloor s \rfloor$ the smallest integer $\geq s$ and the largest integer $\leq s$, respectively.

Lemma 3.2 *Let Φ be an irreducible linear flow on X. Then $C^*(\Phi, X) = C(\Phi, X)$, and*

$$\dim C^*(\Phi, X) = \dim C(\Phi, X) = \begin{cases} 0 & \text{if } \sigma(\Phi) \cap \imath\mathbb{R} = \varnothing, \\ \lceil \frac{1}{2} \dim X \rceil & \text{if } \sigma(\Phi) = \{0\}, \\ 2\lceil \frac{1}{4} \dim X \rceil & \text{if } \sigma(\Phi) \subset \imath\mathbb{R} \setminus \{0\}. \end{cases}$$

Similarly, $C_0^*(\Phi, X) = C_0(\Phi, X)$, and

$$\dim C_0^*(\Phi, X) = \dim C_0(\Phi, X) = \begin{cases} 0 & \text{if } \sigma(\Phi) \cap \imath\mathbb{R} = \varnothing, \\ \lfloor \frac{1}{2} \dim X \rfloor & \text{if } \sigma(\Phi) = \{0\}, \\ 2\lfloor \frac{1}{4} \dim X \rfloor & \text{if } \sigma(\Phi) \subset \imath\mathbb{R} \setminus \{0\}. \end{cases}$$

The proof of Lemma 3.2 utilizes explicit calculations involving several families of special matrices. These matrices are reviewed beforehand for the reader's convenience. First, given any $m \in \mathbb{N}$ and $\omega \in \mathbb{C}$, consider the diagonal matrix

$$D_m(\omega) = \operatorname{diag}[1, \omega, \ldots, \omega^{m-1}] \in \mathbb{C}^{m \times m},$$

for which $D_m(\omega) \in \mathbb{R}^{m \times m}$ whenever $\omega \in \mathbb{R}$, as well as the nilpotent Jordan block of size m,

$$J_m = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ & & & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{m \times m}.$$

Clearly, $D_m(1) = \mathrm{id}_{\mathbb{R}^m} =: I_m$, and $D_m(\omega)^{-1} = D_m(\omega^{-1})$ whenever $\omega \neq 0$, but also

$$D_m(\omega)^{-1} \text{ and } \omega^{1-m} D_m(\omega) \text{ are bounded (in fact, converge) as } |\omega| \to +\infty. \tag{3.1}$$

Moreover, recall that $J_m^m = 0$, and hence

$$e^{tJ_m} = I_m + tJ_m + \cdots + \frac{t^{m-1}}{(m-1)!} J_m^{m-1} \quad \forall t \in \mathbb{R}.$$

A simple lower bound for the size of $e^{tJ_m}x$ is as follows.

Proposition 3.3 *For every $m \in \mathbb{N}$ and norm $\|\cdot\|$ on $\mathbb{R}^m$ there exists a $\nu \in \mathbb{R}^+$ such that*

$$\left\| e^{tJ_m}x \right\| \geq \frac{\nu \|x\|}{\sqrt{1+t^{2m-2}}} \quad \forall t \in \mathbb{R}, x \in \mathbb{R}^m.$$

Next, recall that the function $1/\Gamma$, the reciprocal of the Euler Gamma function, is entire [1, Ch. 6]. In particular, given any $m, n \in \mathbb{N}$ and $\omega \in \mathbb{C}$, the Toeplitz-type matrix

$$\Delta_{m,n}^{[\omega]} := \begin{bmatrix} 1/\Gamma(\omega+1) & 1/\Gamma(\omega+2) & \cdots & 1/\Gamma(\omega+n) \\ 1/\Gamma(\omega) & 1/\Gamma(\omega+1) & \cdots & 1/\Gamma(\omega-1+n) \\ \vdots & \vdots & & \vdots \\ 1/\Gamma(\omega-m+2) & 1/\Gamma(\omega-m+3) & \cdots & 1/\Gamma(\omega-m+n+1) \end{bmatrix} \in \mathbb{C}^{m\times n}$$

is well-defined, each of its entries depending analytically on ω. Note that $\Delta_{m,n}^{[\omega]} \in \mathbb{R}^{m\times n}$ whenever $\omega \in \mathbb{R}$, and $\Delta_{m,n}^{[\omega]}$ is upper triangular (respectively, the zero matrix) if and only if ω is an integer ≤ 0 (an integer $\leq -n$). Also, in the case of a square matrix, the function $\det \Delta_{m,m}^{[\cdot]}$ is entire and not constant, and hence $\Delta_{m,m}^{[\omega]}$ is invertible for most ω.

Proposition 3.4 *Let $m \in \mathbb{N}$ and $\omega \in \mathbb{C}$. Then*

$$\det \Delta_{m,m}^{[\omega]} = \prod\nolimits_{j=1}^{m} \frac{\Gamma(j)}{\Gamma(\omega+j)};$$

in particular, $\Delta_{m,m}^{[\omega]}$ is invertible unless ω is a negative integer.

To appreciate the usefulness of the matrices D_m and $\Delta_{m,n}^{[\omega]}$ in the study of linear flows, note that

$$e^{tJ_m} = \begin{bmatrix} 1 & t & & \cdots & \dfrac{t^{m-1}}{(m-1)!} \\ 0 & \ddots & \ddots & & \vdots \\ & & & \ddots & \\ \vdots & & & \ddots & t \\ 0 & \cdots & & 0 & 1 \end{bmatrix} = D_m(t)^{-1}\Delta_{m,m}^{[0]} D_m(t) \quad \forall t \in \mathbb{R}\setminus\{0\}.$$

More generally, for any $1 \leq j \leq m$ and $t \neq 0$, the $m \times m$-matrix e^{tJ_m} can be partitioned as

$$e^{tJ_m} = \left[\begin{array}{c|c} D_j(t)^{-1}\Delta_{j,m-j}^{[0]} D_{m-j}(t) & t^{m-j} D_j(t)^{-1} \Delta_{j,j}^{[m-j]} D_j(t) \\ \hline t^{-j} D_{m-j}(t)^{-1} \Delta_{m-j,m-j}^{[-j]} D_{m-j}(t) & t^{m-2j} D_{m-j}(t)^{-1} \Delta_{m-j,j}^{[m-2j]} D_j(t) \end{array}\right]. \tag{3.2}$$

Proof of Lemma 3.2 For simplicity, suppress the symbols (Φ, X) in all cores, i.e., write C instead of $C(\Phi, X)$ etc. Note that if $\dim X \le 1$ then $C_0^* = C_0 = \{0\}$, whereas $C^* = C$ equals $\{0\}$ or X, depending on whether $\Phi \ne 0$ or $\Phi = 0$. Thus the lemma holds if $\dim X \le 1$. Henceforth assume $\dim X \ge 2$, and let $(b_1, \ldots, b_{\dim X})$ be an ordered basis of X relative to which A^Φ is a single real Jordan block. Throughout, no notational distinction is made between linear operators on (respectively, elements of) X on the one hand, and their coordinate matrices (column vectors) relative to (b_j) on the other hand.

Assume for the time being that $\sigma(\Phi) = \{a\}$ with $a \in \mathbb{R}$, and hence $A^\Phi = aI_{\dim X} + J_{\dim X}$. In this case,

$$\|\Phi_t x\| = \|e^{ta} e^{tJ_{\dim X}} x\| \ge e^{ta} \frac{\nu \|x\|}{\sqrt{1 + t^{2\dim X - 2}}} \quad \forall t \in \mathbb{R}, x \in X, \tag{3.3}$$

by Proposition 3.3. Pick any $x \in C$. If $a \ne 0$ and $(\Phi_{t_n} x_n)$ is bounded for appropriate sequences (t_n) and (x_n) with $at_n \to +\infty$ and $x_n \to x$, then (3.3) implies that $x = 0$. Thus $C = \{0\}$ whenever $a \ne 0$, and only the case of $a = 0$ has to be considered further.

Assume first that $\dim X$ is *odd*, say $\dim X = 2d + 1$ with $d \in \mathbb{N}$. Letting $m = 2d + 1$, deduce from (3.2) with $j = d + 1$ that for all $t \ne 0$,

$$\Phi_t = \left[\begin{array}{c|c} D_{d+1}(t)^{-1} \Delta^{[0]}_{d+1,d} D_d(t) & t^d D_{d+1}(t)^{-1} \Delta^{[d]}_{d+1,d+1} D_{d+1}(t) \\ \hline 0 & t^{-1} D_d(t)^{-1} \Delta^{[-1]}_{d,d+1} D_{d+1}(t) \end{array}\right], \tag{3.4}$$

because $\Delta^{[-d-1]}_{d,d} = 0$, whereas with $j = d$,

$$\Phi_t = \left[\begin{array}{c|c} D_d(t)^{-1} \Delta^{[0]}_{d,d+1} D_{d+1}(t) & t^{d+1} D_d(t)^{-1} \Delta^{[d+1]}_{d,d} D_d(t) \\ \hline t^{-d} D_{d+1}(t)^{-1} \Delta^{[-d]}_{d+1,d+1} D_{d+1}(t) & t D_{d+1}(t)^{-1} \Delta^{[1]}_{d+1,d} D_d(t) \end{array}\right]. \tag{3.5}$$

Let $V = \operatorname{span}\{b_1, \ldots, b_d\}$, pick any $x = \left[\begin{array}{c} v \\ \hline 0 \end{array}\right] \in V$ with $v \in \mathbb{R}^d$, and consider

$$x_t := \left[\begin{array}{c} v \\ \hline -t^{-d} D_{d+1}(t)^{-1} \big(\Delta^{[d]}_{d+1,d+1}\big)^{-1} \Delta^{[0]}_{d+1,d} D_d(t) v \end{array}\right] \quad \forall t \in \mathbb{R} \setminus \{0\}.$$

(Recall that $\Delta^{[d]}_{d+1,d+1}$ is invertible by Proposition 3.4.) From (3.1), it is clear that $\lim_{|t|\to+\infty} x_t = x$, and together with the expression for Φ_t in (3.4) also

$$\Phi_t x_t = \left[\begin{array}{c} 0 \\ \hline -t^{-(d+1)} D_d(t)^{-1} \Delta^{[-1]}_{d,d+1} \big(\Delta^{[d]}_{d+1,d+1}\big)^{-1} \Delta^{[0]}_{d+1,d} D_d(t) v \end{array}\right] \overset{|t|\to+\infty}{\longrightarrow} 0.$$

Thus $x \in C_0^*$. Since $x \in V$ was arbitrary, $V \subset C_0^*$. Conversely, given any $x = \left[\begin{array}{c} v \\ \hline w \end{array}\right] \in C_0$, with $v \in \mathbb{R}^d$, $w \in \mathbb{R}^{d+1}$, there exist sequences (t_n), (v_n), and (w_n) with $t_n \to +\infty$, $v_n \to v$, and $w_n \to w$ such that

$$\Phi_{t_n} \left[\begin{array}{c} v_n \\ \hline w_n \end{array}\right] = \left[\begin{array}{c} D_{d+1}(t_n)^{-1} \big(\Delta^{[0]}_{d+1,d} D_d(t_n) v_n + t_n^d \Delta^{[d]}_{d+1,d+1} D_{d+1}(t_n) w_n\big) \\ \hline \cdots \end{array}\right] \to 0. \tag{3.6}$$

Recall from (3.1) that $\left(t_n^{-d} D_{d+1}(t_n)\right)$ converges, and apply these matrices to the first component of (3.6) to obtain

$$t_n^{-d} \Delta_{d+1,d}^{[0]} D_d(t_n) v_n + \Delta_{d+1,d+1}^{[d]} D_{d+1}(t_n) w_n \to 0.$$

With (3.1) also $t_n^{-d} \Delta_{d+1,d}^{[0]} D_d(t_n) v_n \to 0$, and hence $\Delta_{d+1,d+1}^{[d]} D_{d+1}(t_n) w_n \to 0$. Since $\Delta_{d+1,d+1}^{[d]}$ is invertible and $\left(D_{d+1}(t_n)^{-1}\right)$ converges, $w_n \to 0 = w$, i.e., $x \in V$. As $x \in C_0$ was arbitrary, $C_0 \subset V$, and hence $C_0^* = C_0 = V$; note that $\dim V = d = \lfloor \frac{1}{2} \dim X \rfloor$.

Next, given any $x = \left[\begin{array}{c} w \\ \hline 0 \end{array}\right] \in V \oplus \operatorname{span}\{b_{d+1}\}$, with $w \in \mathbb{R}^{d+1}$, consider

$$x_t := \left[\begin{array}{c} w \\ \hline -t^{-(d+1)} D_d(t)^{-1} \left(\Delta_{d,d}^{[d+1]}\right)^{-1} \Delta_{d,d+1}^{[0]} D_{d+1}(t) w \end{array}\right] \quad \forall t \in \mathbb{R} \setminus \{0\},$$

which again is well-defined as $\Delta_{d,d}^{[d+1]}$ is invertible. As before, (3.1) implies $\lim_{|t| \to +\infty} x_t = x$, and together with the expression for Φ_t in (3.5) also shows that

$$\Phi_t x_t = \left[\begin{array}{c} 0 \\ \hline t^{-d} D_{d+1}(t)^{-1} \left(\Delta_{d+1,d+1}^{[-d]} - \Delta_{d+1,d}^{[1]} \left(\Delta_{d,d}^{[d+1]}\right)^{-1} \Delta_{d,d+1}^{[0]}\right) D_{d+1}(t) w \end{array}\right]$$

converges as $|t| \to +\infty$, and hence $x \in C^*$. Thus $V \oplus \operatorname{span}\{b_{d+1}\} \subset C^*$. Conversely, given any $x = \left[\begin{array}{c} w \\ \hline v \end{array}\right] \in C$, there exist sequences (t_n), (w_n), and (v_n) with $t_n \to +\infty$, $w_n \to w$, and $v_n \to v$ such that

$$\Phi_{t_n} \left[\begin{array}{c} w_n \\ \hline v_n \end{array}\right] = \left[\begin{array}{c} D_d(t_n)^{-1} \left(\Delta_{d,d+1}^{[0]} D_{d+1}(t_n) w_n + t_n^{d+1} \Delta_{d,d}^{[d+1]} D_d(t_n) v_n\right) \\ \hline \cdots \end{array}\right] \tag{3.7}$$

is bounded as $n \to \infty$. Since $t_n^{-(d+1)} D_d(t_n) \to 0$, applying these matrices to the first component of (3.7) yields

$$t_n^{-(d+1)} \Delta_{d,d+1}^{[0]} D_{d+1}(t_n) w_n + \Delta_{d,d}^{[d+1]} D_d(t_n) v_n \to 0.$$

As before, also $\Delta_{d,d}^{[d+1]} D_d(t_n) v_n \to 0$, and hence $v_n \to 0 = v$, i.e., $x \in V \oplus \operatorname{span}\{b_{d+1}\}$. In summary, $C^* = C = V \oplus \operatorname{span}\{b_{d+1}\}$. This establishes the lemma when $\sigma(\Phi) \subset \mathbb{R}$ and $\dim X$ is odd, as $\dim V \oplus \operatorname{span}\{b_{d+1}\} = d + 1 = \lceil \frac{1}{2} \dim X \rceil$.

The case of $\dim X$ *even*, say $\dim X = 2d$, is similar but simpler: In this case, (3.2) with $m = 2d$, $j = d$ yields

$$\Phi_t = \left[\begin{array}{c|c} D_d(t)^{-1} \Delta_{d,d}^{[0]} D_d(t) & t^d D_d(t)^{-1} \Delta_{d,d}^{[d]} D_d(t) \\ \hline 0 & D_d(t)^{-1} \Delta_{d,d}^{[0]} D_d(t) \end{array}\right] \quad \forall t \in \mathbb{R} \setminus \{0\}.$$

On the one hand, if $x = \left[\begin{array}{c} v \\ \hline 0 \end{array}\right] \in V$ with $v \in \mathbb{R}^d$, then

$$x_t := \left[\begin{array}{c} v \\ \hline -t^{-d} D_d(t)^{-1} \left(\Delta_{d,d}^{[d]}\right)^{-1} \Delta_{d,d}^{[0]} D_d(t) v \end{array}\right] \overset{|t| \to +\infty}{\longrightarrow} x,$$

by (3.1), but also

$$\Phi_t x_t = \left[\begin{array}{c} 0 \\ \hline -t^{-d} D_d(t)^{-1}\Delta_{d,d}^{[0]}\big(\Delta_{d,d}^{[d]}\big)^{-1}\Delta_{d,d}^{[0]} D_d(t)v \end{array}\right] \overset{|t|\to+\infty}{\longrightarrow} 0,$$

showing that $V \subset C_0^*$. On the other hand, if $x = \left[\begin{array}{c} u \\ \hline v \end{array}\right] \in C$ with $u, v \in \mathbb{R}^d$, then there exist sequences (t_n), (u_n), and (v_n) with $t_n \to +\infty$, $u_n \to u$, and $v_n \to v$, such that

$$\Phi_{t_n}\left[\begin{array}{c} u_n \\ \hline v_n \end{array}\right] = \left[\begin{array}{c} D_d(t_n)^{-1}\big(\Delta_{d,d}^{[0]} D_d(t_n)u_n + t_n^d \Delta_{d,d}^{[d]} D_d(t_n)v_n\big) \\ \hline \ldots \end{array}\right]$$

is bounded as $n \to \infty$. Applying $t_n^{-d} D_d(t_n) \to 0$ to the first component yields $v_n \to 0 = v$, as before, and hence $x \in V$. In summary, $C_0^* = C^* = C_0 = C = V$. Noting that $\dim V = d = \frac{1}{2}\dim X$ establishes the lemma when $\sigma(\Phi) \subset \mathbb{R}$ and $\dim X$ is even.

Finally, it remains to consider the case of $\sigma(\Phi) = \{a \pm \imath b\}$ with $a \in \mathbb{R}$, $b \in \mathbb{R}^+$. Since $\dim X$ is even in this case, let $m = \frac{1}{2}\dim X$. Then $A^\Phi = aI_{2m} + \left[\begin{array}{c|c} J_m & -bI_m \\ \hline bI_m & J_m \end{array}\right]$, which in turn yields

$$\Phi_t = e^{ta}\left[\begin{array}{c|c} \cos(bt)I_m & -\sin(bt)I_m \\ \hline \sin(bt)I_m & \cos(bt)I_m \end{array}\right]\left[\begin{array}{c|c} e^{tJ_m} & 0 \\ \hline 0 & e^{tJ_m} \end{array}\right] \quad \forall t \in \mathbb{R}. \tag{3.8}$$

From (3.8) and Proposition 3.3, it is clear that, with an appropriate $\widetilde{\nu} \in \mathbb{R}^+$,

$$\|\Phi_t x\| \ge e^{ta}\frac{\widetilde{\nu}\|x\|}{\sqrt{1+t^{2m-2}}} \quad \forall t \in \mathbb{R}, x \in X.$$

As before, it follows that $C = \{0\}$ unless $a = 0$, so only that case has to be analyzed further. This analysis is virtually identical to the one above, simply because the left matrix on the right-hand side of (3.8) does not in any way affect boundedness or convergence to 0 of $\Phi_t x$: On the one hand, if $m = 2d + 1$ then, with $W = \operatorname{span}\{b_1, \ldots, b_d, b_{m+1}, \ldots, b_{m+d}\}$,

$$C_0^* = C_0 = W, \quad C^* = C = W \oplus \operatorname{span}\{b_{d+1}, b_{m+d+1}\}.$$

On the other hand, if $m = 2d$ then $C_0^* = C_0 = C^* = C = W$. In either case, $\dim W = 2d = 2\lfloor \frac{1}{4}\dim X\rfloor$ and $\dim W \oplus \operatorname{span}\{b_{d+1}, b_{m+d+1}\} = 2d + 2 = 2\lceil \frac{1}{4}\dim X\rceil$. □

Given any Φ-invariant subspace Z of X, denote by Φ_Z the linear flow induced by Φ on Z, that is, $\Phi_Z(t, x) = \Phi_t x$ for all $(t, x) \in \mathbb{R} \times Z$. Note that if $X = \bigoplus_{j=1}^{\ell} Z_j$ with Φ-invariant subspaces $Z_1, \ldots, Z_\ell$, then Φ is flow equivalent to the linear flow $\times_{j=1}^{\ell}\Phi_{Z_j}$ on $\times_{j=1}^{\ell} Z_j$, via the linear isomorphism $h(x) = (P_1 x, \ldots, P_\ell x)$ and $\tau_x = \mathrm{id}_{\mathbb{R}}$ for all $x \in X$; here P_j denotes the linear projection of X onto Z_j along $\bigoplus_{k\ne j} Z_k$. With this, an immediate consequence of Lemma 3.2 announced earlier is

Theorem 3.5 *Let Φ be a linear flow on X. Then $C^*(\Phi, X) = C(\Phi, X)$, $C_0^*(\Phi, X) = C_0(\Phi, X)$, and both sets are Φ-invariant subspaces of X_C^Φ.*

Proof Let $X = \bigoplus_{j=1}^{\ell} Z_j$ be such that each flow Φ_{Z_j} is irreducible. With h as above,

$$\begin{aligned} C(\Phi, X) &= h^{-1}C\left(\times_{j=1}^{\ell}\Phi_{Z_j}, \times_{j=1}^{\ell} Z_j\right) \subset h^{-1}\left(\times_{j=1}^{\ell} C(\Phi_{Z_j}, Z_j)\right) \\ &= h^{-1}\left(\times_{j=1}^{\ell} C^*(\Phi_{Z_j}, Z_j)\right) = h^{-1}C^*\left(\times_{j=1}^{\ell}\Phi_{Z_j}, \times_{j=1}^{\ell} Z_j\right) = C^*(\Phi, X), \end{aligned}$$

where, from left to right, the equalities are due to Lemmas 2.5, 3.2, and 2.8, and the fact that Φ and $\times_{j=1}^{\ell} \Phi_{Z_j}$ are flow equivalent via h, respectively, whereas the inclusion is the ℓ-factor analogue of (2.6). With (2.10), therefore, $C^*(\Phi, X) = C(\Phi, X)$, and recalling that $h(0) = 0$, also $C_0^*(\Phi, X) = C_0(\Phi, X)$. Let $J = \{1 \le j \le \ell : \sigma(\Phi_{Z_j}) \subset \imath\mathbb{R}\}$. By Lemma 3.2, $C(\Phi_{Z_j}, Z_j) = \{0\}$ whenever $j \notin J$, and consequently

$$C(\Phi, X) = h^{-1}\left(\times_{j=1}^{\ell} C(\Phi_{Z_j}, Z_j)\right) = \bigoplus\nolimits_{j \in J} C(\Phi_{Z_j}, Z_j) \subset \bigoplus\nolimits_{j \in J} Z_j = X_C^{\Phi}.$$

□

In light of Theorem 3.5, when dealing with linear flows only the symbols C and C_0 are used henceforth. Note that if Z is a Φ-invariant subspace of X then one may also consider cores of the flow Φ_Z, and this idea of restriction can be iterated. To do so in a systematic way, given any binary sequence $\epsilon = (\epsilon_k)_{k \in \mathbb{N}_0}$, that is, $\epsilon_k \in \{0, 1\}$ for all k, let $C^{\epsilon, -1}(\Phi, X) = X$ and, for every $k \in \mathbb{N}_0$, let

$$C^{\epsilon,k}(\Phi, X) = \begin{cases} C\big(\Phi_{C^{\epsilon,k-1}(\Phi,X)}, C^{\epsilon,k-1}(\Phi, X)\big) & \text{if } \epsilon_k = 0, \\ C_0\big(\Phi_{C^{\epsilon,k-1}(\Phi,X)}, C^{\epsilon,k-1}(\Phi, X)\big) & \text{if } \epsilon_k = 1. \end{cases}$$

Clearly $X \supset C^{\epsilon,0}(\Phi, X) \supset C^{\epsilon,1}(\Phi, X) \supset \cdots$, and hence the **iterated core**

$$C^{\epsilon}(\Phi, X) := \lim\nolimits_{k \to \infty} C^{\epsilon,k}(\Phi, X) = \bigcap\nolimits_{k \in \mathbb{N}_0} C^{\epsilon,k}(\Phi, X)$$

is a Φ-invariant subspace naturally inheriting basic properties from $C(\Phi, X)$ and $C_0(\Phi, X)$.

Lemma 3.6 *Let Φ, Ψ be linear flows on X, Y, respectively, and ϵ a binary sequence.*

(i) *If Φ is (h, τ)-related to Ψ then $h\big(C^{\epsilon}(\Phi, X)\big) = C^{\epsilon}(\Psi, Y)$.*
(ii) *$C^{\epsilon}(\Phi \times \Psi, X \times Y) = C^{\epsilon}(\Phi, X) \times C^{\epsilon}(\Psi, Y)$.*

Proof With Lemma 2.5 and Proposition 3.1, $h\big(C^{\epsilon,0}(\Phi, X)\big) = C^{\epsilon,0}(\Psi, Y)$. By induction, for every $k \ge 1$, $\Phi_{C^{\epsilon,k-1}(\Phi,X)}$ is (h_k, τ_k)-related to $\Psi_{C^{\epsilon,k-1}(\Psi,Y)}$, with h_k and τ_k denoting the restrictions of h and τ to $C^{\epsilon,k-1}(\Phi, X)$ and $\mathbb{R} \times C^{\epsilon,k-1}(\Phi, X)$ respectively. Hence $h\big(C^{\epsilon,k}(\Phi, X)\big) = C^{\epsilon,k}(\Psi, Y)$, which proves (i). Similarly, with Lemma 2.8 and Theorem 3.5, induction yields $C^{\epsilon,k}(\Phi \times \Psi, X \times Y) = C^{\epsilon,k}(\Phi, X) \times C^{\epsilon,k}(\Psi, Y)$ for every $k \ge 1$, which establishes (ii). □

It is not hard to see that $C^{\epsilon}(\Phi, X) = \{0\}$ whenever $\epsilon_k = 1$ for infinitely many k. In what follows, therefore, only terminating binary sequences (i.e., $\epsilon_k = 0$ for all large k) are of interest. Any such sequence (uniquely) represents a non-negative integer. More precisely, given any $n \in \mathbb{N}_0$, let $\epsilon(n)$ be the binary sequence of base-2 digits of n in reversed (i.e., ascending) order, that is,

$$n = \sum\nolimits_{k=0}^{\infty} 2^k \epsilon(n)_k \quad \forall n \in \mathbb{N}_0 \,;$$

thus, for instance, $\epsilon(4) = (0, 0, 1, 0, 0, \ldots)$ and $\epsilon(13) = (1, 0, 1, 1, 0, 0, \ldots)$. To understand the structure of $C^{\epsilon(n)}(\Phi, X)$, first consider the case of an irreducible flow.

Lemma 3.7 *Let Φ be an irreducible linear flow on X.*

(i) *If $\sigma(\Phi) \cap \imath\mathbb{R} = \varnothing$ then $C^{\epsilon(n)}(\Phi, X) = \{0\}$ for all $n \in \mathbb{N}_0$.*
(ii) *If $\sigma(\Phi) = \{0\}$ then* $C^{\epsilon(n)}(\Phi, X) = \begin{cases} \mathrm{Fix}\, \Phi & \text{if } n < \dim X, \\ \{0\} & \text{if } n \ge \dim X. \end{cases}$

(iii) *If* $\sigma(\Phi) \subset \imath\mathbb{R} \setminus \{0\}$ *then* $C^{\epsilon(n)}(\Phi, X) = \begin{cases} \mathit{Per}\,\Phi & \mathit{if}\, n < \frac{1}{2}\,\mathit{dim}\, X, \\ \{0\} & \mathit{if}\, n \geq \frac{1}{2}\,\mathit{dim}\, X. \end{cases}$

Proof Recall from Lemma 3.2 that $C(\Phi, X) = \{0\}$ whenever $\sigma(\Phi) \cap \imath\mathbb{R} = \varnothing$, and in this case $C^{\epsilon(n)}(\Phi, X) = \{0\}$ for every $n \in \mathbb{N}_0$, proving (i).

To establish (ii) and (iii), let $(b_1, \dots, b_{\dim X})$ be an ordered basis of X, relative to which A^Φ is a single real Jordan block. If $\sigma(\Phi) = \{0\}$ consider the two increasing functions $f_0, f_1 : \mathbb{R} \to \mathbb{R}$, given by $f_0(s) = \lceil \frac{1}{2}s \rceil$ and $f_1(s) = \lfloor \frac{1}{2}s \rfloor$, respectively. Let $m_k = f_{\epsilon(n)_k} \circ \cdots \circ f_{\epsilon(n)_0}(\dim X)$. As seen in the proof of Lemma 3.2, $C^{\epsilon(n),k}(\Phi, X) = \mathrm{span}\,\{b_1, \dots, b_{m_k}\}$ for every $k \in \mathbb{N}_0$, provided that $m_k \geq 1$, and $C^{\epsilon(n),k}(\Phi, X) = \{0\}$ otherwise. Note that

$$\lim_{k\to\infty} \underbrace{f_0 \circ \cdots \circ f_0}_{k\text{times}}(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0. \end{cases} \tag{3.9}$$

Consequently, $\epsilon(0) = (0, 0, \dots)$, and $\lim_{k\to\infty} m_k = 1$, so $C^{\epsilon(0)}(\Phi, X) = \mathrm{span}\,\{b_1\}$. Henceforth, assume $n = \epsilon(n)_0 + 2\epsilon(n)_1 + \cdots + 2^\ell \epsilon(n)_\ell \geq 1$, with $\ell \in \mathbb{N}_0$ and $\epsilon(n)_\ell = 1$. Notice that

$$f_{\epsilon(n)_k} \circ \cdots \circ f_{\epsilon(n)_0}(n) = \epsilon(n)_{k+1} + 2\epsilon(n)_{k+2} + \cdots + 2^{\ell-k-1}\epsilon(n)_\ell \quad \forall k = 0, \dots, \ell - 1,$$

hence in particular $f_{\epsilon(n)_{\ell-1}} \circ \cdots \circ f_{\epsilon(n)_0}(n) = \epsilon(n)_\ell = 1$, which implies $f_{\epsilon(n)_\ell} \circ \cdots \circ f_{\epsilon(n)_0}(n) = 0$. Since f_0, f_1 are increasing, $f_{\epsilon(n)_\ell} \circ \cdots \circ f_{\epsilon(n)_0}(i) \leq 0$ for all $i \leq n$, and since $\epsilon(n)_k = 0$ for all $k > \ell$, it follows from (3.9) that $\lim_{k\to\infty} f_{\epsilon(n)_k} \circ \cdots \circ f_{\epsilon(n)_0}(i) = 0$. In particular, $C^{\epsilon(n)}(\Phi, X) = \{0\}$ whenever $\dim X \leq n$. Next, notice that

$$f_{\epsilon(n)_k} \circ \cdots \circ f_{\epsilon(n)_0}(n + 1) = 1 + f_{\epsilon(n)_k} \circ \cdots \circ f_{\epsilon(n)_0}(n) \quad \forall k = 0, \dots, \ell,$$

hence in particular $f_{\epsilon(n)_\ell} \circ \cdots \circ f_{\epsilon(n)_0}(n + 1) = 1$. Again by monotonicity, $f_{\epsilon(n)_\ell} \circ \cdots \circ f_{\epsilon(n)_0}(i) \geq 1$ for all $i \geq n + 1$, and with (3.9) $\lim_{k\to\infty} f_{\epsilon(n)_k} \circ \cdots \circ f_{\epsilon(n)_0}(i) = 1$. This shows that $C^{\epsilon(n)}(\Phi, X) = \mathrm{span}\,\{b_1\} = \mathrm{Fix}\,\Phi$ whenever $\dim X \geq n + 1$, proving (ii).

Finally, to prove (iii) recall from the proof of Lemma 3.2 that

$$C^{\epsilon(n),k}(\Phi, X) = \mathrm{span}\,\{b_1, \dots, b_{\widetilde{m}_k}, b_{\frac{1}{2}\dim X+1}, \dots, b_{\frac{1}{2}\dim X+\widetilde{m}_k}\},$$

provided that $\widetilde{m}_k = f_{\epsilon(n)_k} \circ \cdots \circ f_{\epsilon(n)_0}(\frac{1}{2}\dim X) \geq 1$, and $C^{\epsilon(n),k}(\Phi, X) = \{0\}$ otherwise. Again, $\lim_{k\to\infty} \widetilde{m}_k$ equals 1 if $\frac{1}{2}\dim X \geq n + 1$, and equals 0 if $\frac{1}{2}\dim X \leq n$. This proves (iii) since $\mathrm{span}\,\{b_1, b_{\frac{1}{2}\dim X+1}\} = \mathrm{Per}\,\Phi$. □

Given an arbitrary linear flow Φ on X, let $X = \bigoplus_{j=1}^\ell Z_j$ be such that Φ_{Z_j} is irreducible for every $j = 1, \dots, \ell$. By combining Lemmas 3.6 and 3.7, it is clear that $C^{\epsilon(0)}(\Phi, X) = \mathrm{Bnd}\,\Phi$, and that $\left(C^{\epsilon(n)}(\Phi, X)\right)_{n\in\mathbb{N}_0}$ is a decreasing sequence of nested spaces, with $C^{\epsilon(n)}(\Phi, X) = \{0\}$ for all $n \geq \max_{j=1}^\ell \dim Z_j$. Moreover, for every $n \in \mathbb{N}_0$,

$$\begin{aligned} \dim C^{\epsilon(n)}(\Phi, X) \;=\; &\#\Big\{1 \leq j \leq \ell : \sigma(\Phi_{Z_j}) = \{0\}, \dim Z_j > n\Big\} \\ &+ 2\#\Big\{1 \leq j \leq \ell : \sigma(\Phi_{Z_j}) \subset \imath\mathbb{R} \setminus \{0\}, \dim Z_j > 2n\Big\}. \end{aligned} \tag{3.10}$$

By Lemma 3.6, these numbers are preserved under orbit equivalence. Thus, iterated cores, and especially their dimensions, provide crucial information regarding the numbers and sizes of blocks in the real Jordan normal form of A^Φ. However, these cores do not per se distinguish between different eigenvalues of $A^{\Phi_\mathbb{C}}$. To distinguish blocks corresponding to different elements of $\sigma(\Phi) \cap \imath\mathbb{R}$, ideally in a way that is preserved under orbit equivalence, a finer analysis of $\mathrm{Bnd}\,\Phi$ is needed.

4 Bounded Linear Flows

Call a linear flow Φ on X **bounded** if $\mathrm{Bnd}\,\Phi = X$. (Recall that X is a finite-dimensional normed space over $\mathbb{R}$.) Clearly, every bounded linear flow is central, i.e., $X_C^\Phi = X$; see also Sect. 5. Unless explicitly stated otherwise, every linear flow considered in this section is bounded. Note that Φ is bounded precisely if $\sigma(\Phi) \subset \imath\mathbb{R}$ and A^Φ is diagonalisable (over $\mathbb{C}$), in which case Theorem 1.1 takes a particularly simple form.

Theorem 4.1 *Two bounded linear flows Φ, Ψ are C^0-orbit equivalent if and only if $A^\Phi, \alpha A^\Psi$ are similar for some $\alpha \in \mathbb{R}^+$.*

The main purpose of this section is to provide a proof of Theorem 4.1, divided into several steps for the reader's convenience. Given a non-empty set $\Omega \subset \mathbb{C}$, refer to any element of $\Omega_\mathbb{Q} := \{\omega\mathbb{Q} : \omega \in \Omega\}$ as a **rational class** generated by Ω. Note that for every $\omega, \widetilde{\omega} \in \mathbb{C}$ either $\omega\mathbb{Q} = \widetilde{\omega}\mathbb{Q}$ or $\omega\mathbb{Q} \cap \widetilde{\omega}\mathbb{Q} = \{0\}$. Given $\omega \in \mathbb{C}$ and a bounded linear flow Φ on X, associate with $\omega\mathbb{Q}$ the Φ-invariant subspace

$$X_{\omega\mathbb{Q}}^\Phi := \ker A^\Phi \oplus \bigoplus\nolimits_{s \in \mathbb{R}^+ : \imath s \in \omega\mathbb{Q}} \ker\left((A^\Phi)^2 + s^2\,\mathrm{id}_X\right) \supset \mathrm{Fix}\,\Phi.$$

A few basic properties of such spaces follow immediately from this definition.

Proposition 4.2 *Let Φ be a bounded linear flow on X, and $\omega, \widetilde{\omega} \in \mathbb{C}$. Then:*

(i) *$X_{\omega\mathbb{Q}}^\Phi \cap X_{\widetilde{\omega}\mathbb{Q}}^\Phi \neq \mathrm{Fix}\,\Phi$ if and only if $\omega\mathbb{Q} = \widetilde{\omega}\mathbb{Q} = \lambda\mathbb{Q}$ for some $\lambda \in \sigma(\Phi) \setminus \{0\}$, and hence $X_{\omega\mathbb{Q}}^\Phi = \mathrm{Fix}\,\Phi$ precisely if $\omega\mathbb{Q} \cap \sigma(\Phi) \subset \{0\}$;*
(ii) *For $\lambda, \widetilde{\lambda} \in \sigma(\Phi)$, $X_{\lambda\mathbb{Q}}^\Phi = X_{\widetilde{\lambda}\mathbb{Q}}^\Phi$ if and only if $\lambda\mathbb{Q} = \widetilde{\lambda}\mathbb{Q}$;*
(iii) *$\sum_{\lambda \in \sigma(\Phi)} X_{\lambda\mathbb{Q}}^\Phi = X$;*
(iv) *$X_{\{0\}}^\Phi = \mathrm{Fix}\,\Phi$, and $\bigcup_{\lambda \in \sigma(\Phi)} X_{\lambda\mathbb{Q}}^\Phi = \mathrm{Per}\,\Phi$;*
(v) *For every $\lambda \in \sigma(\Phi) \setminus \{0\}$, $X_{\lambda\mathbb{Q}}^\Phi = \mathrm{Per}_T\,\Phi$, with*

$$T = T_{\lambda\mathbb{Q}}^\Phi := \min \bigcap\nolimits_{s \in \mathbb{R}^+ : \{-\imath s, \imath s\} \cap \lambda\mathbb{Q} \cap \sigma(\Phi) \neq \varnothing} \frac{2\pi}{s}\mathbb{N},$$

and $\{x \in X_{\lambda\mathbb{Q}}^\Phi : T_x^\Phi = T_{\lambda\mathbb{Q}}^\Phi\}$ is open and dense in $X_{\lambda\mathbb{Q}}^\Phi$.

Recall from Sect. 2 that if φ is (h, τ)-related to ψ then $h(\mathrm{Per}\,\varphi) = \mathrm{Per}\,\psi$, and yet $h(\mathrm{Per}_T\,\varphi)$ may not be contained in $\mathrm{Per}_S\,\psi$ for any $S \in \mathbb{R}^+$. Taken together, the following two lemmas show that such a situation cannot occur for linear flows.

Lemma 4.3 *Let Φ, Ψ be bounded linear flows on X, Y, respectively, and assume that $\mathrm{Per}\,\Phi = X$. If Φ is (h, τ)-related to Ψ then there exists an $\alpha \in \mathbb{R}^+$ with the following properties:*

(i) *$T_{h(x)}^\Psi = \alpha T_x^\Phi$ for every $x \in X$;*
(ii) *$h(\mathrm{Per}_T\,\Phi) = \mathrm{Per}_{\alpha T}\,\Psi$ for every $T \in \mathbb{R}^+$;*
(iii) *$A^\Phi, \alpha A^\Psi$ are similar.*

Proof By Proposition 4.2(iv), $\operatorname{Per}\Phi = X$ if and only if $X^{\Phi}_{\lambda\mathbb{Q}} = X$ for some $\lambda \in \sigma(\Phi)$, and since $\operatorname{Per}\Psi = h(\operatorname{Per}\Phi) = h(X) = Y$, also $Y^{\Psi}_{\mu\mathbb{Q}} = Y$ for some $\mu \in \sigma(\Psi)$. Clearly, if $\lambda = 0$ then $\mu = 0$, in which case every $\alpha \in \mathbb{R}^+$ has all the desired properties. Henceforth assume that $\lambda \neq 0$, or equivalently that $\operatorname{Fix}\Phi \neq X$, and hence $\mu \neq 0$ as well.

To prove (i), pick any $x \in X \setminus \operatorname{Fix}\Phi$. By Proposition 4.2(v), there exists a sequence (x_n) with $x_n \to x$ and $T^{\Phi}_{x_n} = T^{\Phi}_{\lambda\mathbb{Q}}$ for all n, so $\langle x\rangle^{\Phi} = T^{\Phi}_{\lambda\mathbb{Q}}/T^{\Phi}_x$, and similarly $\langle h(x)\rangle^{\Psi} = T^{\Psi}_{\mu\mathbb{Q}}/T^{\Psi}_{h(x)}$. By Proposition 2.4(ii), therefore, $T^{\Psi}_{h(x)}/T^{\Phi}_x = T^{\Psi}_{\mu\mathbb{Q}}/T^{\Phi}_{\lambda\mathbb{Q}}$, that is, (i) holds with $\alpha = T^{\Psi}_{\mu\mathbb{Q}}/T^{\Phi}_{\lambda\mathbb{Q}}$.

To prove (ii), pick any $T \in \mathbb{R}^+$ and $x \in \operatorname{Per}_T\Phi \setminus \operatorname{Fix}\Phi$. Then $T/T^{\Phi}_x = m$ for some $m \in \mathbb{N}$, and (i) yields $\alpha T/T^{\Psi}_{h(x)} = m$, that is, $h(x) \in \operatorname{Per}_{\alpha T}\Psi$. Thus $h(\operatorname{Per}_T\Phi) \subset \operatorname{Per}_{\alpha T}\Psi$, and reversing the roles of Φ and Ψ yields $h(\operatorname{Per}_T\Phi) = \operatorname{Per}_{\alpha T}\Psi$.

To prove (iii), denote A^{Φ}, A^{Ψ} simply by A, B, respectively, and let $\sigma(\Phi) \setminus \{0\} = \{\pm\imath a_1, \ldots, \pm\imath a_m\}$ and $\sigma(\Psi) \setminus \{0\} = \{\pm\imath b_1, \ldots, \pm\imath b_n\}$ with appropriate $m, n \in \mathbb{N}$ and real numbers $a_1 > \cdots > a_m > 0$ and $b_1 > \cdots > b_n > 0$; for convenience, $a_0 := b_0 := 0$. Also, let $X_0 = \ker A$, $Y_0 = \ker B$, as well as $X_s = \ker(A^2 + s^2\,\mathrm{id}_X)$, $Y_s = \ker(B^2 + s^2\,\mathrm{id}_Y)$ for every $s \in \mathbb{R}^+$. Since A, B are diagonalisable (over $\mathbb{C}$), to establish (iii) it suffices to show that in fact $m = n$, and that moreover

$$a_k = \alpha b_k \quad \text{and} \quad \dim X_{a_k} = \dim Y_{b_k} \quad \forall k = 0, 1, \ldots, m. \tag{4.1}$$

To this end, notice first that $\operatorname{Per}_{2\pi/s}\Phi = \bigoplus_{k\in\mathbb{N}_0} X_{ks}$, and similarly $\operatorname{Per}_{2\pi/s}\Psi = \bigoplus_{k\in\mathbb{N}_0} Y_{ks}$. For the purpose of induction, assume that, for some integer $0 \le \ell < \min\{m, n\}$,

$$a_k = \alpha b_k \quad \text{and} \quad \dim X_{a_k} = \dim Y_{b_k} \quad \forall k = 0, 1, \ldots, \ell. \tag{4.2}$$

Now, recall that $h(X_0) = h(\operatorname{Fix}\Phi) = \operatorname{Fix}\Psi = Y_0$, and hence $\dim X_0 = \dim Y_0$ by the topological invariance of dimension [17, ch. 2]. In other words, (4.2) holds for $\ell = 0$. Next, let $K_\ell = \bigl\{k \in \mathbb{N}_0 : ka_{\ell+1} \in \{a_0, a_1, \ldots a_\ell\}\bigr\}$, and note that $K_\ell \subset \mathbb{N}_0$ is finite with $0 \in K_\ell$ and $1 \notin K_\ell$. Moreover, since $a_{\ell+1} > 0$,

$$\operatorname{Per}_{2\pi/a_{\ell+1}}\Phi = \bigoplus\nolimits_{k\in\mathbb{N}_0} X_{ka_{\ell+1}} = \bigoplus\nolimits_{k\in\mathbb{N}\setminus K_\ell} X_{ka_{\ell+1}} \oplus \bigoplus\nolimits_{k\in K_\ell} X_{ka_{\ell+1}} = X_{a_{\ell+1}} \oplus \bigoplus\nolimits_{k\in K_\ell} X_{ka_{\ell+1}},$$

whereas by (ii),

$$h(\operatorname{Per}_{2\pi/a_{\ell+1}}\Phi) = \operatorname{Per}_{2\pi\alpha/a_{\ell+1}}\Psi = \bigoplus\nolimits_{k\in\mathbb{N}_0} Y_{ka_{\ell+1}/\alpha} = \bigoplus\nolimits_{k\in\mathbb{N}\setminus K_\ell} Y_{ka_{\ell+1}/\alpha} \oplus \bigoplus\nolimits_{k\in K_\ell} Y_{ka_{\ell+1}/\alpha}.$$

By assumption (4.2), $\dim X_{ka_{\ell+1}} = \dim Y_{ka_{\ell+1}/\alpha}$ for every $k \in K_\ell$. Since $\dim \operatorname{Per}_{2\pi/a_{\ell+1}}\Phi = \dim \operatorname{Per}_{2\pi\alpha/a_{\ell+1}}\Psi$, again by the topological invariance of dimension, clearly $\dim X_{a_{\ell+1}} = \sum_{k\in\mathbb{N}\setminus K_\ell} \dim Y_{ka_{\ell+1}/\alpha} > 0$. This shows that $\imath k a_{\ell+1}/\alpha \in \sigma(\Psi)$ for some $k \in \mathbb{N} \setminus K_\ell$, and also $\dim X_{a_{\ell+1}} \ge \dim Y_{a_{\ell+1}/\alpha}$ because $1 \in \mathbb{N} \setminus K_\ell$. Note that $ka_{\ell+1}/\alpha \notin \{b_0, b_1, \ldots, b_\ell\}$ whenever $k \in \mathbb{N} \setminus K_\ell$. Thus $ka_{\ell+1}/\alpha \le b_{\ell+1}$, and in particular $a_{\ell+1} \le \alpha b_{\ell+1}$. The same argument with the roles of Φ and Ψ reversed yields $a_{\ell+1} \ge \alpha b_{\ell+1}$ and $\dim X_{b_{\ell+1}/\alpha} \le \dim Y_{b_{\ell+1}}$. Consequently, (4.2) holds with $\ell + 1$ instead of ℓ, and in fact for all $\ell \le \min\{m, n\}$ by induction. Since Φ, Ψ are bounded, $X = \bigoplus_{\ell=0}^m X_{a_\ell}$, $Y = \bigoplus_{\ell=0}^n Y_{b_\ell}$, from which it is clear that $m = n$, showing in turn that (4.1) holds. As observed earlier, this proves that $A^{\Phi} = A$ and $\alpha A^{\Psi} = \alpha B$ are similar. □

As seen in the above proof, the assumption $\operatorname{Per}\Phi = X$ in Lemma 4.3 simply means that $X^{\Phi}_{\lambda\mathbb{Q}} = X$ for some $\lambda \in \sigma(\Phi)$. Thus $\sigma(\Phi)$ generates at most one rational class other than $\{0\}$. Even when $\sigma(\Phi)$ does generate several rational classes, however, it turns out that if Φ is (h, τ)-related to Ψ then $h(X^{\Phi}_{\lambda\mathbb{Q}})$ always equals $Y^{\Psi}_{\mu\mathbb{Q}}$ with an appropriate μ. This way the homeomorphism h induces a bijection between the rational classes generated by $\sigma(\Phi)$ and $\sigma(\Psi)$.

Lemma 4.4 *Let Φ, Ψ be bounded linear flows on X, Y, respectively. If Φ is (h,τ)-related to Ψ then there exists a (unique) bijection $h_{\mathbb{Q}} : \sigma(\Phi)_{\mathbb{Q}} \to \sigma(\Psi)_{\mathbb{Q}}$ with $h(X^{\Phi}_{\lambda\mathbb{Q}}) = Y^{\Psi}_{h_{\mathbb{Q}}(\lambda\mathbb{Q})}$ for every $\lambda \in \sigma(\Phi)$; in particular, $\sigma(\Phi)$ and $\sigma(\Psi)$ generate the same number of rational classes.*

The proof of Lemma 4.4 is facilitated by a simple topological observation [37].

Proposition 4.5 *Let $Z_1, \ldots, Z_\ell$ be subspaces of X, with $\ell \in \mathbb{N}$. If $\dim X/Z_j \ge 2$ for every $j = 1, \ldots, \ell$ then $X \setminus \bigcup_{j=1}^{\ell} Z_j$ is connected.*

Remark 4.6 Proposition 4.5 remains valid when $\dim X = \infty$, provided that each Z_j is closed. It also holds when X is a normed space over $\mathbb{C}$, in which case it suffices to require that $Z_j \ne X$ for every j.

Proof of Lemma 4.4 Assume that $\sigma(\Phi)$ and $\sigma(\Psi)$ both generate at least two different rational classes other than $\{0\}$. (Otherwise, the lemma trivially is correct.) Fix any $\lambda \in \sigma(\Phi) \setminus \{0\}$. Given $x \in X^{\Phi}_{\lambda\mathbb{Q}} \subset \operatorname{Per}\Phi$, Propositions 2.3 and 4.2(iv) guarantee that $h(x) \in Y^{\Psi}_{\mu\mathbb{Q}}$ for an appropriate, possibly x-dependent $\mu \in \sigma(\Psi)$. Thus the family of closed, connected sets $\left\{h^{-1}(Y^{\Psi}_{\mu\mathbb{Q}}) : \mu \in \sigma(\Psi) \setminus \{0\}\right\}$ constitutes a finite cover of $X^{\Phi}_{\lambda\mathbb{Q}} \setminus \operatorname{Fix}\Phi$; by Proposition 4.5, the latter set is connected. If $X^{\Phi}_{\lambda\mathbb{Q}} \setminus \operatorname{Fix}\Phi$ was not entirely contained in $h^{-1}(Y^{\Psi}_{\mu\mathbb{Q}})$ for some μ, then one could choose $\mu_1, \mu_2 \in \sigma(\Psi) \setminus \{0\}$ with $\mu_1\mathbb{Q} \ne \mu_2\mathbb{Q}$ such that

$$\begin{aligned}\varnothing \ne h^{-1}(Y^{\Psi}_{\mu_1\mathbb{Q}}) \cap h^{-1}(Y^{\Psi}_{\mu_2\mathbb{Q}}) \cap (X^{\Phi}_{\lambda\mathbb{Q}} \setminus \operatorname{Fix}\Phi) &= h^{-1}(Y^{\Psi}_{\mu_1\mathbb{Q}} \cap Y^{\Psi}_{\mu_2\mathbb{Q}}) \cap X^{\Phi}_{\lambda\mathbb{Q}} \setminus \operatorname{Fix}\Phi \\ &\subset h^{-1}(\operatorname{Fix}\Psi) \setminus \operatorname{Fix}\Phi = \varnothing,\end{aligned}$$

an obvious contradiction. Hence indeed $h(X^{\Phi}_{\lambda\mathbb{Q}}) \subset Y^{\Psi}_{\mu\mathbb{Q}}$ for some $\mu \in \sigma(\Psi)$, and reversing the roles of Φ and Ψ yields $h(X^{\Phi}_{\lambda\mathbb{Q}}) = Y^{\Psi}_{\mu\mathbb{Q}}$. Note that the rational class $\mu\mathbb{Q}$ is uniquely determined by $\lambda\mathbb{Q}$, due to Proposition 4.2(ii). Letting $h_{\mathbb{Q}}(\lambda\mathbb{Q}) = \mu\mathbb{Q}$ precisely when $h(X^{\Phi}_{\lambda\mathbb{Q}}) = Y^{\Psi}_{\mu\mathbb{Q}}$ therefore (uniquely) defines a map $h_{\mathbb{Q}} : \sigma(\Phi)_{\mathbb{Q}} \to \sigma(\Psi)_{\mathbb{Q}}$. Since h is one-to-one, so is $h_{\mathbb{Q}}$, and hence $\#\sigma(\Phi)_{\mathbb{Q}} \le \#\sigma(\Psi)_{\mathbb{Q}}$. Again, reversing the roles of Φ and Ψ yields $\#\sigma(\Phi)_{\mathbb{Q}} = \#\sigma(\Psi)_{\mathbb{Q}}$, and $h_{\mathbb{Q}}$ is a bijection. $\square$

Combining Lemmas 4.3 and 4.4, notice that if $\lambda \in \sigma(\Phi) \setminus \{0\}$ and Φ, Ψ are C^0-orbit equivalent, then the respective (linear) flows induced on $X^{\Phi}_{\lambda\mathbb{Q}}$ and $Y^{\Psi}_{h_{\mathbb{Q}}(\lambda\mathbb{Q})}$ are linearly flow equivalent with $\tau_x = \alpha_{\lambda\mathbb{Q}}\mathrm{id}_{\mathbb{R}}$ for every $x \in X^{\Phi}_{\lambda\mathbb{Q}}$, where $\alpha_{\lambda\mathbb{Q}} = T^{\Psi}_{h_{\mathbb{Q}}(\lambda\mathbb{Q})}/T^{\Phi}_{\lambda\mathbb{Q}}$. As it turns out, Theorem 4.1 is but a direct consequence of the fact that $\alpha_{\lambda\mathbb{Q}}$ does not actually depend on $\lambda\mathbb{Q}$.

Lemma 4.7 *Let Φ, Ψ be bounded linear flows. If Φ is (h,τ)-related to Ψ then*

$$\frac{T^{\Psi}_{h_{\mathbb{Q}}(\lambda\mathbb{Q})}}{T^{\Phi}_{\lambda\mathbb{Q}}} = \frac{T^{\Psi}_{h_{\mathbb{Q}}(\widetilde{\lambda}\mathbb{Q})}}{T^{\Phi}_{\widetilde{\lambda}\mathbb{Q}}} \quad \forall \lambda, \widetilde{\lambda} \in \sigma(\Phi) \setminus \{0\};$$

here $h_{\mathbb{Q}}$ denotes the bijection of Lemma 4.4.

The proof of Lemma 4.7 given below is somewhat subtle. It makes use of a few elementary facts regarding maps of the 2-torus $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$. Specifically, recall that with every continuous map $f : \mathbb{T} \to \mathbb{T}$ one can associate a continuous function $F_f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x + \mathbb{Z}^2) = F_f(x) + \mathbb{Z}^2$ for all $x \in \mathbb{R}^2$, as well as $\sup_{x\in\mathbb{R}^2} \|F_f(x) - L_f x\| < +\infty$ for a unique $L_f \in \mathbb{Z}^{2\times2}$. Two continuous maps $f, \widetilde{f} : \mathbb{T} \to \mathbb{T}$ are homotopic if and only if $L_f = L_{\widetilde{f}}$; moreover, $L_{f\circ\widetilde{f}} = L_f L_{\widetilde{f}}$. Also, if $f_n \to f$ uniformly on $\mathbb{T}$, then $L_{f_n} = L_f$ for all sufficiently large n.

Given any $u \in \mathbb{R}^2$, let $\kappa_u(t, z) = z + ut$ for all $(t, z) \in \mathbb{R} \times \mathbb{T}$. Thus κ_u simply is the Kronecker (or parallel) flow on $\mathbb{T}$ generated by the differential equation $\dot{z} = u$. Recall that for every $z \in \mathbb{T}$, the κ_u-orbit $\kappa_u(\mathbb{R}, z)$ is either a singleton (if $u = 0$), homeomorphic to a circle (if $au \in \mathbb{Z}^2 \setminus \{0\}$ for some $a \in \mathbb{R}$), or dense in $\mathbb{T}$. Variants of the following simple rigidity property of Kronecker flows appear to have long been part of dynamical systems folklore; cf. [2, Thm. 2] and [26, Lem. 6].

Proposition 4.8 *Let $u, \widetilde{u} \in \mathbb{R}^2$. If $f : \mathbb{T} \to \mathbb{T}$ is continuous and maps some κ_u-orbit into a $\kappa_{\widetilde{u}}$-orbit, i.e., $f \circ \kappa_u(\mathbb{R}, z) \subset \kappa_{\widetilde{u}}(\mathbb{R}, \widetilde{z})$ for some $z, \widetilde{z} \in \mathbb{T}$, then $L_f u, \widetilde{u}$ are linearly dependent.*

Remark 4.9 All concepts regarding $\mathbb{T}$ recalled above have precise analogues on the m-torus $\mathbb{R}^m/\mathbb{Z}^m$ for all $m \in \mathbb{N}$, and Proposition 4.8 carries over verbatim with $u, \widetilde{u} \in \mathbb{R}^m$ and their associated m-dimensional Kronecker flows. Only the special case of $m = 2$, however, plays a role in what follows.

Proof of Lemma 4.7 As in the proof of Lemma 4.3, denote A^Φ, A^Ψ simply by A, B. Also, let $\lambda_1\mathbb{Q}, \ldots, \lambda_\ell\mathbb{Q}$ and $\mu_1\mathbb{Q}, \ldots, \mu_\ell\mathbb{Q}$, with $\ell \in \mathbb{N}_0$, be the distinct rational classes other than $\{0\}$ generated by $\sigma(\Phi)$ and $\sigma(\Psi)$ respectively, and $h_\mathbb{Q}(\lambda_j\mathbb{Q}) = \mu_j\mathbb{Q}$ for $j = 1, \ldots, \ell$. As there is nothing to prove otherwise, assume $\ell \geq 2$, and let $\lambda_1 = \lambda$, $\lambda_2 = \widetilde{\lambda}$. For the reader's convenience, the proof is carried out in several separate steps.

Step I—Topological preliminaries Let $X_{j,k} = \sum_{\lambda\in\sigma(\Phi)\cap(\lambda_j\mathbb{Q}+\lambda_k\mathbb{Q})} X^\Phi_{\lambda\mathbb{Q}}$ for every $1 \leq j \leq k \leq \ell$, and similarly let $Y_{j,k} = \sum_{\mu\in\sigma(\Psi)\cap(\mu_j\mathbb{Q}+\mu_k\mathbb{Q})} Y^\Psi_{\mu\mathbb{Q}}$. Clearly, $X_{j,k}$ is Φ-invariant and contains both $X^\Phi_{\lambda_j\mathbb{Q}}$ $(= X_{j,j})$ and $X^\Phi_{\lambda_k\mathbb{Q}}$. Moreover, if $\{j_1, k_1\} \neq \{j_2, k_2\}$ then $X_{j_1,k_1} \cap X_{j_2,k_2} \subset X^\Phi_{\lambda\mathbb{Q}} \subset \operatorname{Per}\Phi$, with an appropriate $\lambda \in \sigma(\Phi)$. Also, note that $x \in X_{j,k} \setminus \operatorname{Per}\Phi$ for some j, k if and only if $\overline{\Phi(\mathbb{R}, x)}$ is homeomorphic to $\mathbb{T}$. Since this property is preserved under orbit equivalence, given any $x \in X_{1,2}$, there exist j, k, possibly depending on x, such that $h(x) \in Y_{j,k}$. Thus the closed, connected sets $\{h^{-1}(Y_{j,k}) : 1 \leq j \leq k \leq \ell\}$ cover $X_{1,2} \setminus \operatorname{Per}\Phi$. Since the latter set is connected, and $h^{-1}(Y_{j_1,k_1} \cap Y_{j_2,k_2}) \subset h^{-1}(\operatorname{Per}\Psi) = \operatorname{Per}\Phi$, the same argument as in the proof of Lemma 4.4 demonstrates that $h(X_{1,2}) \subset Y_{j,k}$ for some j, k, and since $Y^\Psi_{\mu_i\mathbb{Q}} = h(X^\Phi_{\lambda_i\mathbb{Q}})$ for $i = 1, 2$, it is clear that in fact $h(X_{1,2}) \subset Y_{1,2}$. Reversing the roles of Φ and Ψ yields $h(X_{1,2}) = Y_{1,2}$. Henceforth, assume w.l.o.g. that $X_{1,2} = X$ and $Y_{1,2} = Y$. (Otherwise, all topological notions employed in Steps III to V below have to be interpreted relative to $X_{1,2}$ and $Y_{1,2}$, respectively.)

Step II—Arithmetical preliminaries For convenience, let $Z_0 = \ker A = \operatorname{Fix}\Phi$, and for every $j = 1, \ldots, \ell$ let $Z_j = \bigoplus_{s\in\mathbb{R}^+:\imath s\in\lambda_j\mathbb{Q}} \ker(A^2 + s^2 \operatorname{id}_X)$, and also let $T_j = T^\Phi_{\lambda_j\mathbb{Q}}$. With this, $X = \bigoplus_{j=0}^\ell Z_j$, and for each $j = 1, \ldots, \ell$ the eigenvalue λ_j is a rational multiple of $2\pi\imath/T_j$. Since $X = X_{1,2}$ by assumption, there exist unique $k_{j,1}, k_{j,2} \in \mathbb{Z}$, $k_j \in \mathbb{N}$ with $\gcd(k_{j,1}, k_{j,2}, k_j) = 1$ and

$$k_j/T_j = k_{j,1}/T_1 + k_{j,2}/T_2 \quad \forall j = 1, \ldots, \ell.$$

Let $\mathcal{L}_\mathbb{R}$ be the subspace of $\mathbb{R}^\ell$ given by

$$\mathcal{L}_\mathbb{R} = \{x \in \mathbb{R}^\ell : k_{j,1}x_{,1} + k_{j,2}x_{,2} - k_j x_{,j} = 0 \ \forall j = 1, \ldots, \ell\}.$$

Note that $\mathcal{L}_{\mathbb{R}}$ is two-dimensional and contains two linearly independent integer vectors. (If $\ell = 2$ then simply $\mathcal{L}_{\mathbb{R}} = \mathbb{R}^2$.) Hence $\mathcal{L}_{\mathbb{Z}} := \mathcal{L}_{\mathbb{R}} \cap \mathbb{Z}^\ell$ is a two-dimensional lattice, that is, a discrete additive subgroup of $\mathcal{L}_{\mathbb{R}}$. Let $b_1, b_2 \in \mathbb{Z}^\ell$ be a basis of this lattice, i.e., $\mathcal{L}_{\mathbb{Z}} = b_1\mathbb{Z} + b_2\mathbb{Z}$. Though not unique per se, the basis b_1, b_2 is uniquely determined under the additional assumption that

$$b_{1,1} > 0, \quad b_{2,1} = 0, \quad 0 \leq b_{1,2} < b_{2,2}. \tag{4.3}$$

(Note that if $\ell = 2$ then simply $b_{1,1} = b_{2,2} = 1$, $b_{2,1} = b_{1,2} = 0$.) Since clearly $[\,1/T_1, \dots, 1/T_\ell\,]^\top \in \mathcal{L}_{\mathbb{R}} \setminus \{0\}$, there exists a unique $u \in \mathbb{R}^2 \setminus \{0\}$ such that

$$[\,b_1 \,|\, b_2\,]\,u = [\,1/T_1, \dots, 1/T_\ell\,]^\top. \tag{4.4}$$

Notice in particular that $u_{,1}\mathbb{Q} + u_{,2}\mathbb{Q} = 1/T_1\mathbb{Q} + 1/T_2\mathbb{Q}$, and hence $u_{,1}, u_{,2}$ are rationally independent because $1/T_1, 1/T_2$ are.

A completely analogous construction can be carried out in Y: Let $W_0 = \ker B = \operatorname{Fix}\Psi$, and let $W_j = \bigoplus_{s\in\mathbb{R}^+ : \imath s \in \mu_j \mathbb{Q}} \ker (B^2 + s^2\,\mathrm{id}_Y)$ for $j = 1, \dots, \ell$, as well as $S_j = T^\Psi_{\mu_j\mathbb{Q}}$. Then $Y = \bigoplus_{j=0}^{\ell} W_j$, and the same procedure as above yields unique $c_1, c_2 \in \mathbb{Z}^\ell$ with

$$c_{1,1} > 0, \quad c_{2,1} = 0, \quad 0 \leq c_{1,2} < c_{2,2}, \tag{4.5}$$

(and in fact $c_{1,1} = c_{2,2} = 1$, $c_{2,1} = c_{1,2} = 0$ in case $\ell = 2$), together with a unique $\widetilde{u} \in \mathbb{R}^2 \setminus \{0\}$ such that

$$[\,c_1 \,|\, c_2\,]\,\widetilde{u} = [\,1/S_1, \dots, 1/S_\ell\,]^\top ; \tag{4.6}$$

again, $\widetilde{u}_{,1}, \widetilde{u}_{,2}$ are rationally independent.

Step III—Construction of maps on $\mathbb{T}$ Denote by $P_0, \dots, P_\ell$ the complementary linear projections associated with the decomposition $X = \bigoplus_{j=0}^{\ell} Z_j$, i.e., P_0 is the projection of X onto Z_0 along $\bigoplus_{j=1}^{\ell} Z_j$ etc. Note that $P_j\Phi_t = \Phi_t P_j$ for all $j = 0, 1, \dots, \ell$ and $t \in \mathbb{R}$, due to the Φ-invariance of Z_j. Given any $x \in X$, define $p_x : \mathbb{T} \to X$ as

$$p_x(z) = P_0 x + \sum\nolimits_{j=1}^{\ell} \Phi_{(b_{1,j}z_{,1} + b_{2,j}z_{,2})T_j} P_j x \quad \forall z \in \mathbb{T},$$

with $b_1, b_2 \in \mathbb{Z}^\ell$ as in Step II. Clearly, p_x is continuous, $p_x(0 + \mathbb{Z}^2) = x$, and with an appropriate constant $\nu \in \mathbb{R}^+$,

$$\|p_x(z) - p_{\widetilde{x}}(z)\| \leq \nu \|x - \widetilde{x}\| \quad \forall x, \widetilde{x} \in X, z \in \mathbb{T}. \tag{4.7}$$

Thus $p_{x_n} \to p_x$ uniformly on $\mathbb{T}$ whenever $x_n \to x$. Also, with the unique u from (4.4)

$$p_x(ut + \mathbb{Z}^2) = P_0 x + \sum\nolimits_{j=1}^{\ell} \Phi_{t(b_{1,j}u_{,1} + b_{2,j}u_{,2})T_j} P_j x = P_0 x + \sum\nolimits_{j=1}^{\ell} \Phi_t P_j x = \Phi_t x \quad \forall t \in \mathbb{R}.$$

In terms of the Kronecker flow κ_u on $\mathbb{T}$, this simply means that

$$p_x \circ \kappa_u(t, 0 + \mathbb{Z}^2) = \Phi_t x \quad \forall t \in \mathbb{R}. \tag{4.8}$$

Since $u_{,1}, u_{,2}$ are rationally independent, the κ_u-orbit $\kappa_u(\mathbb{R}, 0 + \mathbb{Z}^2)$ is dense in $\mathbb{T}$, and hence $p_x(\mathbb{T}) = \overline{\Phi(\mathbb{R}, x)}$. Thus, p_x maps $\mathbb{T}$ continuously onto the closure of the Φ-orbit of x, for every $x \in X$.

Next, consider $U := \{x \in X : T^\Phi_{P_j x} = T_j \ \forall j = 1, \dots, \ell\}$, an open, dense, and connected subset of X by Propositions 4.2 and 4.5. Whenever $x \in U$, note that $p_x(z) = p_x(\widetilde{z})$ implies $z - \widetilde{z} \in \mathbb{Z}^2$, i.e., p_x is one-to-one and hence a homeomorphism from $\mathbb{T}$ onto $\overline{\Phi(\mathbb{R}, x)}$. Moreover, p_x^{-1} depends continuously on $x \in U$ in the following sense: If $x_n \to x$ in

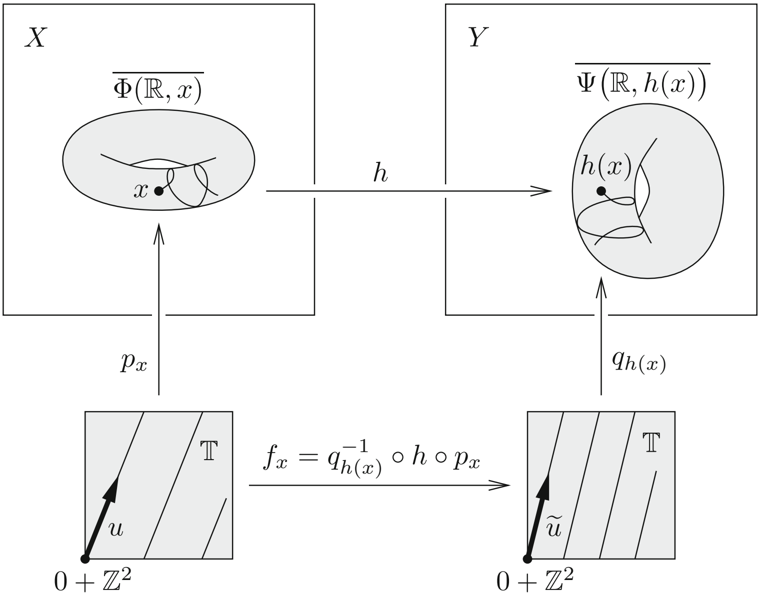

Fig. 4 The map $f_x : \mathbb{T} \to \mathbb{T}$ is well-defined and continuous provided that $x \in h^{-1}(V) \subset X$ and is a homeomorphism whenever $x \in U \cap h^{-1}(V)$

U, and if $(\widetilde{x}_n)$ converges to some $\widetilde{x}$ with $\widetilde{x}_n \in p_{x_n}(\mathbb{T})$ for every n, then $\widetilde{x} \in p_x(\mathbb{T})$ and $p_{x_n}^{-1}(\widetilde{x}_n) \to p_x^{-1}(\widetilde{x})$ in $\mathbb{T}$. To see this, let $\widetilde{x}_n = p_{x_n}(z_n)$ with the appropriate $z_n \in \mathbb{T}$, and note that every subsequence (z_{n_k}) contains a subsequence that converges in $\mathbb{T}$ to some z with $\widetilde{x} = p_x(z)$. Since p_x is one-to-one, z is uniquely determined by this property, and so $(z_n) = \big(p_{x_n}^{-1}(\widetilde{x}_n)\big)$ converges to $z = p_x^{-1}(\widetilde{x})$.

Again, a completely analogous construction can be carried out in Y: Denote by $Q_0, \dots, Q_\ell$ the projections associated with the decomposition $Y = \bigoplus_{j=0}^{\ell} W_j$ and, given any $y \in Y$, define $q_y : \mathbb{T} \to Y$ as

$$q_y(z) = Q_0 y + \sum\nolimits_{j=1}^{\ell} \Psi_{(c_{1,j} z_{,1} + c_{2,j} z_{,2}) S_j} Q_j y \quad \forall z \in \mathbb{T},$$

with $c_1, c_2 \in \mathbb{Z}^\ell$ as in Step II. As before, q_y is continuous, $q_y(0+\mathbb{Z}^2) = y$, and $q_{y_n} \to q_y$ uniformly on $\mathbb{T}$ whenever $y_n \to y$. In analogy to (4.8), with the unique $\widetilde{u}$ from (4.6),

$$q_y \circ \kappa_{\widetilde{u}}(t, 0+\mathbb{Z}^2) = \Psi_t y \quad \forall t \in \mathbb{R}, \tag{4.9}$$

and q_y maps $\mathbb{T}$ continuously onto $\overline{\Psi(\mathbb{R}, y)}$. With the open, dense, and connected subset $V := \{y \in Y : T^{\Psi}_{Q_j y} = S_j \ \forall j = 1, \dots, \ell\}$ of Y, the map q_y is one-to-one whenever $y \in V$, and q_y^{-1} depends continuously on $y \in V$, in the sense made precise earlier.

Combining the homeomorphism h with the maps introduced so far yields a continuous map $f_x : \mathbb{T} \to \mathbb{T}$, given by

$$f_x(z) := q_{h(x)}^{-1} \circ h \circ p_x(z) \quad \forall z \in \mathbb{T},$$

with $f_x(0+\mathbb{Z}^2) = 0+\mathbb{Z}^2$, provided that $x \in h^{-1}(V)$; see also Fig. 4. Notice that $h^{-1}(V) \subset X$ is open, dense, and connected. As seen earlier, if $x_n \to x$ in $h^{-1}(V)$ then $f_{x_n} \to f_x$ pointwise.

In fact, using the analogue for $q_{h(x)}$ of (4.7), it is readily seen that $f_{x_n} \to f_x$ uniformly on $\mathbb{T}$. Thus $x \mapsto L_{f_x}$ is continuous on $h^{-1}(V)$, and indeed constant because $h^{-1}(V)$ is connected. In other words, $L_{f_x} = L$ for a unique $L \in \mathbb{Z}^{2\times 2}$ and every $x \in h^{-1}(V)$. Recall that p_x, and hence also f_x, is a homeomorphism whenever $x \in U \cap h^{-1}(V)$. This set, though perhaps not connected, is open and dense in X, so certainly not empty. Thus L is invertible over $\mathbb{Z}$, or equivalently $|\det L| = 1$.

Step IV—Properties of L and $[\, b_1 \,|\, b_2 \,], [\, c_1 \,|\, c_2 \,]$ The scene is now set for recognizing some finer properties of the matrices $L \in \mathbb{Z}^{2\times 2}$ and $[\, b_1 \,|\, b_2 \,], [\, c_1 \,|\, c_2 \,] \in \mathbb{Z}^{\ell\times 2}$, which truly is the crux of this proof. Concretely, it will be shown both that $L = I_2$ and that the first two rows of $[\, c_1 \,|\, c_2 \,]$ are positive integer multiples of the corresponding rows of $[\, b_1 \,|\, b_2 \,]$. To this end, for $i = 1, 2$ fix $x_i \in Z_i$ so that $T^{\Phi}_{x_i} = T_i$. Then $y_i := h(x_i) \in W_0 \oplus W_i$ and $T^{\Psi}_{y_i} = S_i$. Also, by (4.7) and its analogue for q_y, picking $\nu \in \mathbb{R}^+$ large enough ensures that

$$\|p_x(z) - p_{\widetilde{x}}(z)\| \le \nu\|x - \widetilde{x}\|, \quad \|q_y(z) - q_{\widetilde{y}}(z)\| \le \nu\|y - \widetilde{y}\| \quad \forall x, \widetilde{x} \in X, y, \widetilde{y} \in Y, z \in \mathbb{T}.$$

Since S_i is the minimal Ψ-period of y_i, given any $\varepsilon > 0$, there exists a $\delta_1(\varepsilon) > 0$ such that

$$\|\Psi_t y_i - \Psi_{\widetilde{t}} y_i\| < \delta_1(\varepsilon) \text{ for some } t, \widetilde{t} \in \mathbb{R} \quad \Longrightarrow \quad \min_{k\in\mathbb{Z}} |(t - \widetilde{t})/S_i - k| < \varepsilon. \tag{4.10}$$

By the continuity of h and the periodicity of x_i, there also exists a $\delta_2(\varepsilon) > 0$ such that

$$\|x - \Phi_t x_i\| < \delta_2(\varepsilon) \text{ for some } t \in \mathbb{R} \quad \Longrightarrow \quad \|h(x) - h(\Phi_t x_i)\| < \frac{\delta_1(\varepsilon)}{2(1+\nu)}. \tag{4.11}$$

Moreover, notice the simple estimate, valid for $x \in h^{-1}(V)$ and $i = 1, 2$,

$$\|q_{y_i} \circ f_x(z) - h \circ p_{x_i}(z)\| \le \nu\|h(x) - y_i\| + \|h \circ p_x(z) - h \circ p_{x_i}(z)\| \quad \forall z \in \mathbb{T}. \tag{4.12}$$

Finally, let $z_s, \widetilde{z}_s \in \mathbb{T}$ be given by

$$z_s = \begin{bmatrix} 0 \\ s \end{bmatrix} + \mathbb{Z}^2, \quad \widetilde{z}_s = \begin{bmatrix} -b_{2,2}s \\ b_{1,2}s \end{bmatrix} + \mathbb{Z}^2 \quad \forall s \in \mathbb{R},$$

and observe that, for $i = 1, 2$,

$$p_{x_i}(z_s) = \Phi_{b_{2,i}sT_i} x_i, \quad q_{y_i} \circ f_x(z_s) = \Psi_{\gamma_i(s)S_i} y_i \quad \forall s \in \mathbb{R}, \tag{4.13}$$

with $\gamma_i(s) = [\, c_{1,i}, c_{2,i} \,] F_{f_x}(z_s)$. Similarly

$$p_{x_i}(\widetilde{z}_s) = \Phi_{(-b_{1,i}b_{2,2}+b_{1,2}b_{2,i})sT_i} x_i, \quad q_{y_i} \circ f_x(\widetilde{z}_s) = \Psi_{\widetilde{\gamma}_i(s)S_i} y_i \quad \forall s \in \mathbb{R}, \tag{4.14}$$

with $\widetilde{\gamma}_i(s) = [\, c_{1,i}, c_{2,i} \,] F_{f_x}(\widetilde{z}_s)$. With these preparations, it is possible to analyze f_x for x close to x_1 or x_2. For the reader's convenience, the analysis is carried out in two separate sub-steps.

Sub-step IVa—Analysis of f_x for x close to x_1 Given any $0 < \varepsilon < \frac{1}{4}$, let $\delta = \delta_2(\varepsilon)/(1+\nu)$ for convenience, and assume that $x \in h^{-1}(V)$ with $\|x - x_1\| < \delta$. Then $\|h(x) - y_1\| < \frac{1}{2}\delta_1(\varepsilon)/(1+\nu)$ by (4.11), and using (4.13) with $i = 1$, recalling that $b_{2,1} = 0$,

$$\|p_x(z_s) - p_{x_1}(z_s)\| = \|p_x(z_s) - x_1\| < \frac{\nu}{1+\nu}\delta_2(\varepsilon) < \delta_2(\varepsilon),$$

and hence $\|h \circ p_x(z_s) - h \circ p_{x_1}(z_s)\| < \frac{1}{2}\delta_1(\varepsilon)/(1+\nu)$ as well. With (4.12), therefore, $\|\Psi_{\gamma_1(s)S_1} y_1 - y_1\| < \frac{1}{2}\delta_1(\varepsilon)$, and (4.10) yields $\min_{k\in\mathbb{Z}} |\gamma_1(s) - k| < \varepsilon$ for all $s \in \mathbb{R}$. Since γ_1 is continuous, there exists a unique $k \in \mathbb{Z}$ such that $|\gamma_1(s) - k| < \varepsilon$ for all s. Recall that $\gamma_1(s) = [\, c_{1,1}, 0 \,] F_{f_x}(z_s)$ and that $\sup_{s\in\mathbb{R}} |F_{f_x}(z_s) - L_{f_x} z_s| < +\infty$. Consequently,

$$\sup_{s\in\mathbb{R}} |c_{1,1} L_{1,2} s| = \sup_{s\in\mathbb{R}} \left|[\, c_{1,1}, 0 \,] L z_s\right| < +\infty,$$

and since $c_{1,1} > 0$, it follows that $L_{1,2} = 0$, which in turn implies $|L_{1,1}| = |L_{2,2}| = 1$, because $|\det L| = 1$.

Similarly, using (4.14) with $i = 1$,

$$\| p_x(\widetilde{z}_s) - \Phi_{-b_{1,1}b_{2,2}sT_1} x_1 \| = \| p_x(\widetilde{z}_s) - p_{x_1}(\widetilde{z}_s) \| < \frac{\nu}{1+\nu} \delta_2(\varepsilon),$$

and again $\|h \circ p_x(\widetilde{z}_s) - h \circ p_{x_1}(\widetilde{z}_s)\| < \frac{1}{2}\delta_1(\varepsilon)/(1+\nu)$, so that (4.12) now yields

$$\| \Psi_{\widetilde{\gamma}_1(s)S_1} y_1 - \Psi_{\tau_{x_1}(-b_{1,1}b_{2,2}sT_1)} y_1 \| = \| q_{y_1} \circ f_x(\widetilde{z}_s) - h \circ p_{x_1}(\widetilde{z}_s) \| < \tfrac{1}{2}\delta_1(\varepsilon).$$

Hence $\min_{k\in\mathbb{Z}} |\widetilde{\gamma}_1(s) - \tau_{x_1}(-b_{1,1}b_{2,2}sT_1)/S_1 - k| < \varepsilon$ for all $s \in \mathbb{R}$. Similarly to before, and since $L_{1,2} = 0$, this implies that

$$\sup_{s\in\mathbb{R}} |b_{2,2}c_{1,1}L_{1,1}s - \tau_{x_1}(b_{1,1}b_{2,2}sT_1)/S_1| < +\infty.$$

As $b_{1,1}, b_{2,2}, c_{1,1}$ all are positive, and τ_{x_1} is increasing, $L_{1,1} \geq 0$, and so in fact $L_{1,1} = 1$.

Finally, let $r = 1/(b_{1,1}b_{2,2})$ and note that $p_{x_1}(\widetilde{z}_{s+r}) = p_{x_1}(\widetilde{z}_s)$ for all s, but also

$$\begin{aligned}
\| \Psi_{\widetilde{\gamma}_1(s+r)S_1} y_1 - \Psi_{\widetilde{\gamma}_1(s)S_1} y_1 \| &= \| q_{y_1} \circ f_x(\widetilde{z}_{s+r}) - q_{y_1} \circ f_x(\widetilde{z}_s) \| \\
&\leq 2\nu \|h(x) - y_1\| + \|h \circ p_x(\widetilde{z}_{s+r}) - h \circ p_{x_1}(\widetilde{z}_{s+r})\| + \|h \circ p_x(\widetilde{z}_s) - h \circ p_{x_1}(\widetilde{z}_s)\| \\
&< 2\nu \frac{\delta_1(\varepsilon)}{2(1+\nu)} + \frac{\delta_1(\varepsilon)}{2(1+\nu)} + \frac{\delta_1(\varepsilon)}{2(1+\nu)} \\
&= \delta_1(\varepsilon) \quad \forall s \in \mathbb{R}.
\end{aligned}$$

Deduce from (4.10) that, with a unique $k \in \mathbb{Z}$,

$$|\widetilde{\gamma}_1(s+r) - \widetilde{\gamma}_1(s) + k| < \varepsilon \quad \forall s \in \mathbb{R}. \tag{4.15}$$

Adding (4.15) with $s = 0, r, \ldots, (n-1)r$ yields $|\widetilde{\gamma}_1(nr) - \widetilde{\gamma}_1(0) + nk| < n\varepsilon$ for every $n \in \mathbb{N}$. Since the difference between $\widetilde{\gamma}_1(nr) = [\, c_{1,1}, 0\,] F_{f_x}(\widetilde{z}_{nr})$ and $[\, c_{1,1}, 0\,] L\widetilde{z}_{nr} = -c_{1,1}n/b_{1,1}$ remains bounded as $n \to \infty$, it follows that $|c_{1,1}/b_{1,1} - k| \leq \varepsilon$. Moreover, since $\varepsilon > 0$ was arbitrary and $b_{1,1}, c_{1,1}$ are positive, in fact $c_{1,1}/b_{1,1} = k \in \mathbb{N}$. In summary, the analysis for x being sufficiently close to x_1 shows that $L_{1,1} = 1$, $L_{1,2} = 0$, and $c_{1,1}/b_{1,1} \in \mathbb{N}$.

Sub-step IVb—Analysis of f_x for x close to x_2 A completely analogous analysis can be carried out for x being close to x_2. Specifically, given any $0 < \varepsilon < \frac{1}{4}$, assume that $x \in h^{-1}(V)$ with $\|x - x_2\| < \delta$. Similarly to before, (4.12) and (4.13) now yield

$$\| \Psi_{\gamma_2(s)S_2} y_2 - \Psi_{\tau_{x_2}(b_{2,2}sT_2)} y_2 \| = \| q_{y_2} \circ f_x(z_s) - h \circ p_{x_2}(z_s) \| < \tfrac{1}{2}\delta_1(\varepsilon) \quad \forall s \in \mathbb{R},$$

and consequently $\min_{k\in\mathbb{Z}} |\gamma_2(s) - \tau_{x_2}(b_{2,2}sT_2)/S_2 - k| < \varepsilon$. As $\gamma_2(s) = [\, c_{1,2}, c_{2,2}\,] F_{f_x}(z_s)$, this implies that

$$\sup_{s\in\mathbb{R}} |c_{2,2}L_{2,2}s - \tau_{x_2}(b_{2,2}sT_2)/S_2| < +\infty,$$

and hence $L_{2,2} \geq 0$, so in fact $L_{2,2} = 1$. As well, $p_{x_2}(z_{s+1/b_{2,2}}) = p_{x_2}(z_s)$ for all s, but also

$$\begin{aligned}
&\| \Psi_{\gamma_2(s+1/b_{2,2})S_2} y_2 - \Psi_{\gamma_2(s)S_2} y_2 \| = \| q_{y_2} \circ f_x(z_{s+1/b_{2,2}}) - q_{y_2} \circ f_x(z_s) \| \\
&\quad \leq 2\nu\|h(x) - y_2\| + \|h \circ p_x(z_{s+1/b_{2,2}}) - h \circ p_{x_2}(z_{s+1/b_{2,2}})\| + \|h \circ p_x(z_s) - h \circ p_{x_2}(z_s)\| \\
&\quad < \delta_1(\varepsilon) \quad \forall s \in \mathbb{R},
\end{aligned}$$

implying that $|\gamma_2(s + 1/b_{2,2}) - \gamma_2(s) - k| < \varepsilon$ for a unique $k \in \mathbb{Z}$ and all $s \in \mathbb{R}$. By adding these inequalities for $s = 0, 1/b_{2,2}, \ldots, (n-1)/b_{2,2}$, similarly to before, it follows that

$$|c_{2,2}/b_{2,2} - k| = \limsup_{n\to\infty} |\gamma_2(n/b_{2,2})/n - k| \leq \varepsilon,$$

and since $\varepsilon > 0$ was arbitrary, $c_{2,2}/b_{2,2} \in \mathbb{N}$. Finally, utilizing (4.12) and (4.14) with $i = 2$,

$$\|\Psi_{\widetilde{\gamma}_2(s)S_2} y_2 - y_2\| = \|q_{y_2} \circ f_x(\widetilde{z}_s) - h \circ p_{x_2}(\widetilde{z}_s)\| < \tfrac{1}{2}\delta_1(\varepsilon) \quad \forall s \in \mathbb{R},$$

yields $\min_{k\in\mathbb{Z}} |\widetilde{\gamma}_2(s) - k| < \varepsilon$ for all s. Consequently, as $L_{1,1} = L_{2,2} = 1$ and $L_{1,2} = 0$,

$$\sup_{s\in\mathbb{R}} |s(b_{1,2}c_{2,2} - b_{2,2}c_{1,2} - L_{2,1}b_{2,2}c_{2,2})| = \sup_{s\in\mathbb{R}} \left|[\, c_{1,2}, c_{2,2}\,] L\widetilde{z}_s\right| < +\infty.$$

Thus necessarily $L_{2,1} = b_{1,2}/b_{2,2} - c_{1,2}/c_{2,2}$. By (4.3) and (4.5), both ratios $b_{1,2}/b_{2,2}$ and $c_{1,2}/c_{2,2}$ are non-negative and strictly less than 1. Thus $L_{2,1} = 0$ and $b_{1,2}/b_{2,2} = c_{1,2}/c_{2,2}$. In summary, the analysis for x being sufficiently close to x_2 shows that $L_{2,1} = 0$, $L_{2,2} = 1$, and hence $L = I_2$, as well as $c_{2,2}/b_{2,2} \in \mathbb{N}$ and $b_{1,2}/b_{2,2} = c_{1,2}/c_{2,2}$.

Step V—Concluding the proof For every $x \in U \cap h^{-1}(V)$ the map $g_x : \mathbb{T} \to \mathbb{T}$ given by $g_x = p_x^{-1} \circ h^{-1} \circ q_{h(x)}$ is a homeomorphism of $\mathbb{T}$, with $g_x = f_x^{-1}$, and carrying out Step IV with the roles of Φ and Ψ reversed yields $L_{g_x} = L^{-1} = I_2$, as well as $b_{1,1}/c_{1,1}, b_{2,2}/c_{2,2} \in \mathbb{N}$. This shows that in fact $b_{1,1} = c_{1,1}$, $b_{2,2} = c_{2,2}$, and hence also $b_{1,2} = c_{1,2}$. With this, the proof is readily completed: Combine (4.8), (4.9), the definition of f_x, and the fact that Φ is (h, τ)-related to Ψ, to deduce that for every $x \in h^{-1}(V)$,

$$f_x \circ \kappa_u(t, 0 + \mathbb{Z}^2) = q_{h(x)}^{-1} \circ h(\Phi_t x) = q_{h(x)}^{-1}\big(\Psi_{\tau_x(t)} h(x)\big) = \kappa_{\widetilde{u}}(\tau_x(t), 0 + \mathbb{Z}^2) \quad \forall t \in \mathbb{R},$$

where $u, \widetilde{u} \in \mathbb{R}^2 \setminus \{0\}$ are determined by (4.4) and (4.6) respectively. In particular

$$\begin{bmatrix} b_{1,1} & 0 \\ b_{1,2} & b_{2,2} \end{bmatrix} u = \begin{bmatrix} 1/T_1 \\ 1/T_2 \end{bmatrix}, \quad \begin{bmatrix} c_{1,1} & 0 \\ c_{1,2} & c_{2,2} \end{bmatrix} \widetilde{u} = \begin{bmatrix} 1/S_1 \\ 1/S_2 \end{bmatrix}. \tag{4.16}$$

By Proposition 4.8, the vectors $L_{f_x} u, \widetilde{u}$ are linearly dependent. Since $L_{f_x} = I_2$ and the two matrices in (4.16) are identical, linear dependence of $L_{f_x} u, \widetilde{u}$ implies linear dependence of $[\, 1/T_1, 1/T_2\,]^\top, [\, 1/S_1, 1/S_2\,]^\top$, that is,

$$0 = \begin{vmatrix} 1/T_1 & 1/S_1 \\ 1/T_2 & 1/S_2 \end{vmatrix} = \frac{1}{S_1 S_2}\left(\frac{S_1}{T_1} - \frac{S_2}{T_2}\right).$$

Thus, $T^{\Psi}_{\mu_1\mathbb{Q}}/T^{\Phi}_{\lambda_1\mathbb{Q}} = T^{\Psi}_{\mu_2\mathbb{Q}}/T^{\Phi}_{\lambda_2\mathbb{Q}}$, as claimed. □

As alluded to earlier, by combining Lemmas 4.3, 4.4, and 4.7 it is now easy to establish the "only if" part of Theorem 4.1. (The "if" part is obvious.)

Proof of Theorem 4.1 As in the proof of Lemma 4.7, let $\lambda_1\mathbb{Q}, \ldots, \lambda_\ell\mathbb{Q}$, with $\ell \in \mathbb{N}_0$, be the distinct rational classes other than $\{0\}$ generated by $\sigma(\Phi)$; again there is nothing to prove unless $\ell \geq 2$. For convenience, denote the generators of the linear flows induced on $X^{\Phi}_{\lambda_j\mathbb{Q}}$ and $Y^{\Psi}_{h_\mathbb{Q}(\lambda_j\mathbb{Q})}$ by A_j and B_j respectively, and let $X^{\Phi}_{\lambda_j\mathbb{Q}} = \mathrm{Fix}\,\Phi \oplus \bigoplus_{k=1}^{m_j} X_{a_{j,k}}$, $Y^{\Psi}_{h_\mathbb{Q}(\lambda_j\mathbb{Q})} = \mathrm{Fix}\,\Psi \oplus \bigoplus_{k=1}^{m_j} Y_{a_{j,k}/\alpha_j}$, in accordance with the proof of Lemma 4.3. As seen in that proof, $H_j A_j = \alpha_j B_j H_j$, with $\alpha_j = T^{\Psi}_{h_\mathbb{Q}(\lambda_j\mathbb{Q})}/T^{\Phi}_{\lambda_j\mathbb{Q}}$ and an isomorphism $H_j : X^{\Phi}_{\lambda_j\mathbb{Q}} \to Y^{\Psi}_{h_\mathbb{Q}(\lambda_j\mathbb{Q})}$ satisfying $H_j \mathrm{Fix}\,\Phi = \mathrm{Fix}\,\Psi$ as well as $H_j X_{a_{j,k}} = Y_{a_{j,k}/\alpha_j}$ for $k = 1, \ldots, m_j$. By Lemma 4.7, $\alpha_j = \alpha_1$ for all $j = 1, \ldots, \ell$. Since $X = \sum_{j=1}^{\ell} X^{\Phi}_{\lambda_j\mathbb{Q}}$ and $Y = \sum_{j=1}^{\ell} Y^{\Psi}_{h_\mathbb{Q}(\lambda_j\mathbb{Q})}$, letting $Hx = H_j x$ for $x \in \bigoplus_{k=1}^{m_j} X_{a_{j,k}}$ and $Hx = H_1 x$ for $x \in \mathrm{Fix}\,\Phi$, yields a linear isomorphism $H : X \to Y$ with $HA^{\Phi} = \alpha_1 A^{\Psi} H$. □

5 Proof of the Classification Theorems

Let Φ be a linear flow on X, a finite-dimensional normed space over $\mathbb{R}$. The subspaces

$$\begin{aligned} X_S^\Phi &:= \{x \in X : \lim_{t\to+\infty} \Phi_t x = 0\}, \\ X_C^\Phi &:= \{x \in X : \lim_{|t|\to+\infty} e^{-\varepsilon|t|}\Phi_t x = 0 \ \forall \varepsilon > 0\}, \\ X_U^\Phi &:= \{x \in X : \lim_{t\to-\infty} \Phi_t x = 0\}, \end{aligned}$$

referred to as the **stable**, **central**, and **unstable** space of Φ, respectively, are Φ-invariant, and $X = X_S^\Phi \oplus X_C^\Phi \oplus X_U^\Phi$; see, e.g., [12] for an authoritative account on linear dynamical systems. Call Φ **hyperbolic** if $X_C^\Phi = \{0\}$, and **central** if $X_C^\Phi = X$. For $\bullet = S, C, U$, let $P_\bullet^\Phi$ be the linear projection onto $X_\bullet^\Phi$ along $\bigoplus_{\circ\neq\bullet} X_\circ^\Phi$. With this and $\Phi_\bullet := \Phi_{X_\bullet^\Phi}$, clearly Φ is linearly flow equivalent to the product flow $\times_\bullet \Phi_\bullet$, via the isomorphism $\times_\bullet P_\bullet^\Phi$ and with $\tau_x = \mathrm{id}_\mathbb{R}$ for all $x \in X$. By invariance, $P_\bullet^\Phi \Phi_t = \Phi_t P_\bullet^\Phi$ for all $t \in \mathbb{R}$, and hence also $P_\bullet^\Phi A^\Phi = A^\Phi P_\bullet^\Phi$. Notice that if Φ is (h, τ)-related to Ψ then $h(X_S^\Phi) = Y_S^\Psi$, $h(X_U^\Phi) = Y_U^\Psi$, whereas it is possible that $h(X_C^\Phi) \neq Y_C^\Psi$.

Proof of Theorem 1.1 To establish that (i)$\Rightarrow$(iv), assume that Φ is (h, τ)-related to Ψ. Then $h(X_S^\Phi) = Y_S^\Psi$, $h(X_U^\Phi) = Y_U^\Psi$, hence $\dim X_S^\Phi = \dim Y_S^\Psi$, $\dim X_U^\Phi = \dim Y_U^\Psi$, and it only remains to prove the assertion regarding Φ_C, Ψ_C. To this end, in analogy to the proofs in Sect. 4, denote A^{Φ_C}, A^{Ψ_C} by A, B respectively, and let $X_0 = \ker A$, $Y_0 = \ker B$, as well as $X_s = \ker(A^2 + s^2 \mathrm{id}_{X_C^\Phi})$, $Y_s = \ker(B^2 + s^2 \mathrm{id}_{Y_C^\Psi})$ for every $s \in \mathbb{R}^+$. For $s \geq 0$ and $n \in \mathbb{N}_0$, let $c_n^\Phi(s) = \dim\left(X_s \cap C^{\epsilon(n)}(\Phi, X)\right)$. Recall from Sect. 3 that $\left(c_n^\Phi(s)\right)$ is a decreasing sequence of integers, with $c_0^\Phi(s) = \dim X_s$, as well as $c_n^\Phi(s) = 0$ for all large n. With this, consider non-negative integers $d_n^\Phi(s) := c_{n-1}^\Phi(s) - c_n^\Phi(s)$, with any $n \in \mathbb{N}$. As a consequence of (3.10), $d_n^\Phi(0)$ simply equals the number of blocks J_n in the real Jordan normal form of A, whereas $\frac{1}{2}d_n^\Phi(s)$ equals, for every $s \in \mathbb{R}^+$, the number of blocks $\left[\begin{array}{c|c} J_n & -sI_n \\ \hline sI_n & J_n \end{array}\right]$.

Recall first that $h(X_0) = Y_0$, by Proposition 2.3, and that $h\left(C^{\epsilon(n)}(\Phi, X)\right) = C^{\epsilon(n)}(\Psi, Y)$ for every $n \in \mathbb{N}_0$, by Lemma 3.6. It follows that $c_n^\Phi(0) = c_n^\Psi(0)$ for all $n \in \mathbb{N}_0$, and hence also $d_n^\Phi(0) = d_n^\Psi(0)$ for all $n \in \mathbb{N}$. Thus, A, B (and in fact αB for any $\alpha \in \mathbb{R}^+$) contain the same number (possibly, zero) of blocks J_n in their respective real Jordan normal forms, for each $n \in \mathbb{N}$. Since this clearly proves (iv) in case $\sigma(\Phi) \cap \imath\mathbb{R} \subset \{0\}$, henceforth assume that $\sigma(\Phi) \cap \imath\mathbb{R} \setminus \{0\} \neq \varnothing$.

Pick any $\lambda \in \sigma(\Phi) \cap \imath\mathbb{R} \setminus \{0\}$, and recall that $X_{\lambda\mathbb{Q}}^\Phi \subset \mathrm{Bnd}\,\Phi$ as well as $h(\mathrm{Bnd}\,\Phi) = \mathrm{Bnd}\,\Psi$. Thus $h(X_{\lambda\mathbb{Q}}^\Phi) = Y_{h_\mathbb{Q}(\lambda\mathbb{Q})}^\Psi$, by Lemma 4.4. As in the proof of Lemma 4.3, for convenience let $\lambda\mathbb{Q} \cap \sigma(\Phi) \setminus \{0\} = \{\pm\imath a_1, \ldots, \pm\imath a_m\}$ and $h_\mathbb{Q}(\lambda\mathbb{Q}) \cap \sigma(\Psi) \setminus \{0\} = \{\pm\imath b_1, \ldots, \pm\imath b_m\}$, with $m \in \mathbb{N}$ and real numbers $a_1 > \cdots > a_m > 0$ and $b_1 > \cdots > b_m > 0$; again, $a_0 := b_0 := 0$. As seen in that proof, $a_k = \alpha b_k$ for every $k = 0, 1, \ldots, m$, with $\alpha = T_{h_\mathbb{Q}(\lambda\mathbb{Q})}^\Psi / T_{\lambda\mathbb{Q}}^\Phi \in \mathbb{R}^+$, but also, with the sets $K_\ell \subset \mathbb{N}_0$ defined there,

$$h\left(X_{a_{\ell+1}} \oplus \bigoplus\nolimits_{k\in K_\ell} X_{ka_{\ell+1}}\right) = Y_{b_{\ell+1}} \oplus \bigoplus\nolimits_{k\in K_\ell} Y_{kb_{\ell+1}} \quad \forall \ell = 0, 1, \ldots, m-1. \tag{5.1}$$

Now, assume that, for some $0 \leq \ell < m$,

$$d_n^\Phi(a_k) = d_n^\Psi(b_k) \quad \forall n \in \mathbb{N}, k = 0, 1, \ldots, \ell; \tag{5.2}$$

as seen earlier, (5.2) holds for $\ell = 0$. With (5.1) and Lemma 3.6, for any $n \in \mathbb{N}_0$,

$$\begin{aligned} c_n^{\Phi}(a_{\ell+1}) + \sum\nolimits_{k\in K_\ell} c_n^{\Phi}(ka_{\ell+1}) &= \dim\left(\left(X_{a_{\ell+1}} \oplus \bigoplus\nolimits_{k\in K_\ell} X_{ka_{\ell+1}}\right) \cap C^{\epsilon(n)}(\Phi, X)\right) \\ &= \dim\left(\left(Y_{b_{\ell+1}} \oplus \bigoplus\nolimits_{k\in K_\ell} Y_{kb_{\ell+1}}\right) \cap C^{\epsilon(n)}(\Psi, Y)\right) = c_n^{\Psi}(b_{\ell+1}) + \sum\nolimits_{k\in K_\ell} c_n^{\Psi}(kb_{\ell+1}). \end{aligned}$$

Together with (5.2), this implies that $d_n^{\Phi}(a_{\ell+1}) = d_n^{\Psi}(b_{\ell+1})$ for every $n \in \mathbb{N}$, i.e., (5.2) holds with $\ell+1$ instead of ℓ, and by induction (on ℓ) in fact for $\ell = m$ as well. Thus, A, αB contain the same number of blocks $\left[\begin{array}{c|c} J_n & -a_k I_n \\ \hline a_k I_n & J_n \end{array}\right]$ in their respective real Jordan normal forms, for each $n \in \mathbb{N}$ and $k = 1, \ldots, m$. The same argument can be applied to every rational class $\lambda\mathbb{Q}$ with $\lambda \in \sigma(\Phi) \cap \imath\mathbb{R} \setminus \{0\}$. By Lemma 4.7, the resulting value of α is independent of λ. Thus $A^{\Phi_C} = A$ and $\alpha A^{\Psi_C} = \alpha B$ are similar, as claimed.

Showing that (iv)⇒(iii)⇒(ii) requires straightforward, mostly routine arguments. Since details of the latter can be found in many textbooks, e.g., [3,4,12,20,30], only a brief outline is included here for completeness. To prove that (iv)⇒(iii), note first that $\|x\|_S^{\Phi} := \int_0^{+\infty} \|\Phi_t P_S^{\Phi} x\| \,\mathrm{d}t$ and its counterpart $\|\cdot\|_S^{\Psi}$ on Y define norms on X_S^{Φ} and Y_S^{Ψ} respectively, for which $\|\Phi.x\|_S^{\Phi}$ and $\|\Psi.y\|_S^{\Psi}$ are strictly decreasing to 0 as $t \to +\infty$ whenever $x \neq 0$, $y \neq 0$. Consequently, given $x \in X_S^{\Phi} \setminus \{0\}$, there exists a unique $t_x \in \mathbb{R}$ with $\|\Phi_{t_x} x\|_S = 1$. Also, by assumption, there exists a linear isomorphism $H_S : X_S^{\Phi} \to Y_S^{\Psi}$. It is readily confirmed that $h_S : X_S^{\Phi} \to Y_S^{\Psi}$, given by

$$h_S(x) = \begin{cases} \dfrac{\Psi_{-\alpha t_x} H_S \Phi_{t_x} x}{\|H_S \Phi_{t_x} x\|_S^{\Phi}} & \text{if } x \in X_S^{\Phi} \setminus \{0\}, \\ 0 & \text{if } x = 0, \end{cases}$$

is a homeomorphism, and

$$h_S(\Phi_t P_S^{\Phi} x) = \Psi_{\alpha t} h_S(P_S^{\Phi} x) \quad \forall (t, x) \in \mathbb{R} \times X. \tag{5.3}$$

A completely analogous argument, utilizing $\|x\|_U^{\Phi} := \int_{-\infty}^0 \|\Phi_t P_U^{\Phi} x\| \,\mathrm{d}t$, its counterpart $\|\cdot\|_U^{\Psi}$ on Y, and a linear isomorphism $H_U : X_U^{\Phi} \to Y_U^{\Psi}$, yields a homeomorphism $h_U : X_U^{\Phi} \to Y_U^{\Psi}$ for which (5.3) holds with U instead of S. With this, clearly $\Phi_S \times \Phi_U$, $\Psi_S \times \Psi_U$ are C^0-flow equivalent via the homeomorphism $h_S \times h_U$ and with $\tau_x = \alpha\, \mathrm{id}_{\mathbb{R}}$ for all $x \in X_S^{\Phi} \times X_U^{\Phi}$. Since $HA^{\Phi_C} = \alpha A^{\Psi_C} H$ by assumption, $H(\Phi_C)_t = (\Psi_C)_{\alpha t} H$ for all $t \in \mathbb{R}$, that is, Φ_C, Ψ_C are linearly flow equivalent.

To prove that (iii)⇒(ii), assume that $\Phi_S \times \Phi_U$, $\Psi_S \times \Psi_U$ are C^0-flow equivalent and $H_C(\Phi_C)_t = (\Psi_C)_{\alpha t} H_C$ for all $t \in \mathbb{R}$, with some linear isomorphism $H_C : X_C^{\Phi} \to Y_C^{\Psi}$ and $\alpha \in \mathbb{R}^+$. By the implication (i)⇒(iv) already proved, $\dim X_S^{\Phi} = \dim Y_S^{\Psi}$, $\dim X_U^{\Phi} = \dim Y_U^{\Psi}$, and the argument used above to prove that (iv)⇒(iii) yields a homeomorphism $h_S : X_S^{\Phi} \to Y_S^{\Psi}$ satisfying (5.3), as well as its counterpart $h_U : X_U^{\Phi} \to Y_U^{\Psi}$. Combining these ingredients,

$$h(x) := h_S(P_S^{\Phi} x) + H_C P_C^{\Phi} x + h_U(P_U^{\Phi} x) \quad \forall x \in X,$$

defines a homeomorphism $h : X \to Y$ with $h(\Phi_t x) = \Psi_{\alpha t} h(x)$ for all $(t, x) \in \mathbb{R} \times X$. Thus Φ, Ψ are C^0-flow equivalent. The implication (ii)⇒(i) is trivial. □

The proof of Theorem 1.2 given below relies on two simple observations, both of which are straightforward linear algebra exercises [37]; recall that X, Y are finite-dimensional linear spaces over $\mathbb{R}$.

Proposition 5.1 *Let $A, \widetilde{A} : X \to Y$ be linear, and assume that $Z \neq X$ is a subspace of X with $Z \supset \ker A + \ker \widetilde{A}$. Then the following are equivalent:*

(i) *$Ax, \widetilde{A}x$ are linearly dependent for each $x \in X \setminus Z$;*
(ii) *$\widetilde{A} = \alpha A$ for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proposition 5.2 *Let $A : X \to X$ be linear. Then the following are equivalent:*

(i) *A is nilpotent, i.e., $A^n = 0$ for some $n \in \mathbb{N}$;*
(ii) *$A, \alpha A$ are similar for every $\alpha \in \mathbb{R} \setminus \{0\}$;*
(iii) *$A, \alpha A$ are similar for some $\alpha > 1$.*

Remark 5.3 While the non-trivial implication (i)⇒(ii) in Proposition 5.1 may fail if $Z \not\supset \ker A + \ker \widetilde{A}$, even when $\dim X = 1$, finite-dimensionality of X (or Y) is irrelevant for the result. By contrast, although (i)⇒(ii)⇒(iii) remains valid in Proposition 5.2 when $\dim X = \infty$, every other implication may fail in this case. Provided that $\mathbb{R}$ is replaced with $\mathbb{C}$ in (ii), Propositions 5.1 and 5.2 also hold when X, Y are linear spaces over $\mathbb{C}$.

Proof of Theorem 1.2 Clearly (iv)⇒(iii)⇒(ii)⇒(i), so only the implication (i)⇒(iv) requires proof. To prepare for the argument, assume Φ is (h, τ)-related to Ψ with a C^1-diffeomorphism $h : X \to Y$. For convenience, denote the linear isomorphism $D_0 h$ by H, the generators A^Φ, A^Ψ by A, B, and the projections $P_\bullet^\Phi, P_\bullet^\Psi$ by $P_\bullet, Q_\bullet$, respectively. As seen earlier, $h(X_S^\Phi) = Y_S^\Psi$ and hence $HX_S^\Phi = Y_S^\Psi$, and similarly for X_U^Φ. It is possible, however, that $HX_C^\Phi \neq Y_C^\Psi$, and this in turn necessitates usage of one additional pair of invariant subspaces as follows: Recall that $X_C^\Phi \supset \mathrm{Bnd}\,\Phi \supset \ker A$ and $Y_C^\Psi \supset \mathrm{Bnd}\,\Psi \supset \ker B$. By Proposition 2.3, $h(\mathrm{Bnd}\,\Phi) = \mathrm{Bnd}\,\Psi$, and hence $H\,\mathrm{Bnd}\,\Phi = \mathrm{Bnd}\,\Psi$, but also $A\,\mathrm{Bnd}\,\Phi \subset \mathrm{Bnd}\,\Phi$ and $B\,\mathrm{Bnd}\,\Psi \subset \mathrm{Bnd}\,\Psi$, due to invariance. With this, let $X_{HB} = X_S^\Phi \oplus \mathrm{Bnd}\,\Phi \oplus X_U^\Phi$ and $Y_{HB} = Y_S^\Psi \oplus \mathrm{Bnd}\,\Psi \oplus Y_U^\Psi$. Plainly, $HX_{HB} = Y_{HB}$, and crucially,

$$Q_\bullet Hx = HP_\bullet x \quad \forall x \in X_{HB},\ \bullet = S, C, U\,.$$

By Theorem 1.1, there is nothing to prove if $X_C^\Phi = X$, or equivalently if $X_{HB} \setminus X_C^\Phi = \varnothing$. Thus, henceforth assume that $X_{HB} \setminus X_C^\Phi \neq \varnothing$; notice that this in particular includes the possibility of $X_C^\Phi = \{0\}$, i.e., the case of a hyperbolic flow Φ.

With the notations introduced above, pick any $x \in X_{HB} \setminus X_C^\Phi$ and $t \in \mathbb{R}^+$. Note that if τ was differentiable, then differentiating the identity $h(e^{tA}x) = e^{\tau_x(t)B}h(x)$ at $(0, 0)$ would immediately yield $HA = \tau_0'(0)BH$; cf. [34, p. 233]. The following argument mimics this process of differentiation for arbitrary τ. First observe that, for every $\varepsilon > 0$,

$$h(e^{tA}\varepsilon x)/\varepsilon = e^{\tau_{\varepsilon x}(t)B}h(\varepsilon x)/\varepsilon. \tag{5.4}$$

Suppose that $\lim_{\varepsilon \downarrow 0} \tau_{\varepsilon x}(t) = +\infty$. If so, $\lim_{n\to\infty} \tau_{\varepsilon_n x}(t) = +\infty$ for every strictly decreasing sequence (ε_n) with $\lim_{n\to\infty} \varepsilon_n = 0$. In this case, applying Q_S to (5.4) yields

$$He^{tA}P_S x = Q_S He^{tA}x = \lim_{n\to\infty} e^{\tau_{\varepsilon_n x}(t)B} Q_S h(\varepsilon_n x)/\varepsilon_n = 0,$$

and hence $P_S x = 0$, whereas applying Q_U yields

$$0 = \lim_{n\to\infty} e^{-\tau_{\varepsilon_n x}(t)B} Q_U h(e^{tA}\varepsilon_n x)/\varepsilon_n = Q_U Hx = HP_U x,$$

and hence $P_U x = 0$. Taken together, $x \in \ker(P_S + P_U) = X_C^\Phi$, contradicting the fact that $x \in X_{HB} \setminus X_C^\Phi$. Consequently, $\rho_0(t, x) := \liminf_{\varepsilon\downarrow 0} \tau_{\varepsilon x}(t) < +\infty$ and

$$He^{tA}x = e^{\rho_0(t,x)B}Hx. \tag{5.5}$$

Since $\rho_0(t,x) = 0$ would imply $x \in \mathrm{Per}\,\Phi \subset X_C^\Phi$, clearly $\rho_0(t,x) \in \mathbb{R}^+$. Also, notice that if $\limsup_{t\downarrow 0} \rho_0(t,x)$ was positive, possibly $+\infty$, then $P_S x = 0$ and $P_U x = 0$ would follow from applying Q_S and Q_U respectively to (5.5), again contradicting the fact that $x \in X_{HB} \setminus X_C^\Phi$. Thus $\lim_{t\downarrow 0} \rho_0(t,x) = 0$.

Next, deduce from (5.5) that

$$HAx = \lim_{t\downarrow 0} H \frac{e^{tA} - \mathrm{id}_X}{t} x = \lim_{t\downarrow 0} \frac{\rho_0(t,x)}{t} \cdot \frac{e^{\rho_0(t,x)B} - \mathrm{id}_Y}{\rho_0(t,x)} Hx = \lim_{t\downarrow 0} \frac{\rho_0(t,x)}{t} BHx, \tag{5.6}$$

and so $\rho_{0,0}(x) := \lim_{t\downarrow 0} \rho_0(t,x)/t$ exists because $BHx \neq 0$. Clearly, $\rho_{0,0}(x) \geq 0$. In summary, for every $x \in X_{HB} \setminus X_C^\Phi$ there exists $\rho_{0,0}(x) \geq 0$ such that $HAx = \rho_{0,0}(x)BHx$; in particular, BHx, HA are linearly dependent for each $x \in X_{HB} \setminus X_C^\Phi$. Notice that $\ker BH = H^{-1}\ker B \subset H^{-1}\mathrm{Bnd}\,\Psi = \mathrm{Bnd}\,\Phi$, as well as $\ker HA = \ker A \subset \mathrm{Bnd}\,\Phi$, and hence $\ker BH + \ker HA \subset \mathrm{Bnd}\,\Phi \neq X_{HB}$. Proposition 5.1, applied to $BH, HA : X_{HB} \to Y_{HB}$ and $Z = \mathrm{Bnd}\,\Phi = X_{HB} \cap X_C^\Phi$, guarantees the existence of $\alpha \in \mathbb{R} \setminus \{0\}$ such that

$$HAx = \alpha BHx \quad \forall x \in X_{HB}, \tag{5.7}$$

and from (5.6) it is clear that in fact $\alpha \in \mathbb{R}^+$. Thus the proof is complete in case $X_{HB} = X$, or equivalently whenever $\mathrm{Bnd}\,\Phi = X_C^\Phi$. (This, for instance, includes the case of a hyperbolic flow Φ.)

It remains to consider the case of $\mathrm{Bnd}\,\Phi$ being a proper subspace of X_C^Φ, where necessarily $\mathrm{Bnd}\,\Phi \neq \{0\}$. Deduce from Theorem 1.1 that there exists a linear isomorphism $K : X \to Y$, with $KX_\bullet^\Phi = Y_\bullet^\Psi$ for each $\bullet = S, C, U$, and a $\beta \in \mathbb{R}^+$ such that

$$KAx = \beta BKx \quad \forall x \in X_C^\Phi. \tag{5.8}$$

Notice that (5.8) implies $K\,\mathrm{Bnd}\,\Phi = \mathrm{Bnd}\,\Psi$. Combine (5.7) and (5.8) to obtain

$$\alpha H^{-1}BHx = \beta K^{-1}BKx \quad \forall x \in \mathrm{Bnd}\,\Phi. \tag{5.9}$$

For convenience, denote the generators of $\Psi_{\mathrm{Bnd}\Psi}$ and $\Psi_{Y_C^\Psi}$ by B_B and B_C respectively. Since $H\,\mathrm{Bnd}\,\Phi = \mathrm{Bnd}\,\Psi = K\,\mathrm{Bnd}\,\Phi$, (5.9) simply asserts that αB_B, βB_B are similar. It is now helpful to distinguish two cases: On the one hand, if B_B is not nilpotent, then $\alpha = \beta$ by Proposition 5.2. In this case, $L : X \to Y$ with

$$L = HP_S + KP_C + HP_U \tag{5.10}$$

is a linear isomorphism, and $LAx = \alpha BLx$ for all $x \in X$. On the other hand, if B_B is nilpotent then so is B_C, and Proposition 5.2 shows that αB_C, βB_C are similar. Consequently, there exists a linear isomorphism $\widetilde{K} : X \to Y$, with $\widetilde{K}X_\bullet^\Phi = Y_\bullet^\Phi$ for each $\bullet = S, C, U$, such that $\widetilde{K}Ax = \alpha B\widetilde{K}x$ for all $x \in X_C^\Phi$. The same argument as in the non-nilpotent case then applies, with $\widetilde{K}$ in place of K in (5.10). In either case, therefore, $LA = \alpha BL$, that is, $A^\Phi = A$ and $\alpha A^\Psi = \alpha B$ are similar, and the proof is complete. □

With the main results established, the remainder of this section provides a brief discussion relating them to the existing literature.

In the case of hyperbolic flows, Theorem 1.1 is classical [3,4,12,20,30]. What makes the result more challenging in general, then, is the presence of a non-trivial central space. On this matter, two key references are [24,26]. In [24], the equivalence (ii)⇔(iv) of Theorem 1.1 is proved utilizing a version of flow equivalence (termed **homeomorphy**, also allowing for negative α in (iv), that is, for time-reversal). To put this in perspective, notice that insisting on *flow* (rather than mere *orbit*) equivalence greatly simplifies the arguments in the present article as well. For instance, Proposition 2.4(i) simply reads $T^{\psi}_{h(x)} = \alpha T^{\varphi}_x$ in this case, and Lemma 4.7 (the proof of which required considerable effort) trivially holds. Consequently, to decide whether two bounded real linear flows are C^0-flow equivalent, all that is needed is an elementary analysis of periodic points, as developed in Sect. 4. In particular, one may bypass the topological considerations of [24, §3–4] which the authors found unduly hard to grasp. To deal with non-semisimple eigenvalues on $\imath\mathbb{R}$, [24, §5] introduces a proximality relation $\Re_\varphi$: Specifically, $x\Re_\varphi\widetilde{x}$ if, given any neighbourhoods $U, \widetilde{U}$ of $x, \widetilde{x} \in X$ respectively, there exists a $v \in X$ such that $\varphi(\mathbb{R}, v) \cap U \neq \varnothing$ and $\varphi(\mathbb{R}, v) \cap \widetilde{U} \neq \varnothing$. Plainly, $\Re_\varphi$ is reflexive and symmetric, but not, in general, transitive, and if φ is (h, τ)-related to ψ, then $x\Re_\varphi\widetilde{x}$ is equivalent to $h(x)\Re_\psi h(\widetilde{x})$. Moreover, if $x \in C_0(\varphi, X)$, then $x\Re_\varphi 0$, and for irreducible linear flows the converse is true also. While the usage of $\Re_\varphi$ in [24] thus resembles the usage of C_0 (and C) in the present article, recall from Sect. 2 that these non-uniform cores may be ill-behaved under products—and so may be $\Re_\varphi$. In fact, as per Example 2.7 with $u, \widetilde{u}$ as in (2.9), it is readily seen that $u\Re_\varphi 0$ and $\widetilde{u}\Re_\varphi 0$, yet $(u, \widetilde{u}) \not\Re_{\varphi\times\varphi} (0, 0)$. Good behaviour of $\Re_\varphi$ under products, which even for linear flows may or may not occur in general, appears to have been taken for granted throughout [24] without proper justification. For comparison, recall from Sect. 2 that using *uniform* cores allows one to avoid this difficulty altogether; see also [18,36].

The focus in [26] is on C^0-orbit equivalence for linear flows, real or complex, for which (i)⇔(iv) of Theorem 1.1 and, in essence, a version of Theorem 6.1 below are established. In the process, the following terminology is employed (cf. also [9, sec. II.4]): For every $x \in X$, consider the φ-invariant closed sets

$$D^-_\varphi(x) = \bigcap\nolimits_{t,\varepsilon\in\mathbb{R}^+} \overline{\varphi\big(]-\infty, -t], B_\varepsilon(x)\big)}, \quad D^+_\varphi(x) = \bigcap\nolimits_{t,\varepsilon\in\mathbb{R}^+} \overline{\varphi\big([t, +\infty[, B_\varepsilon(x)\big)},$$

where $B_\varepsilon(x)$ denotes the open ε-ball centered at x. With this, $D_\varphi(x) := D^-_\varphi(x) \cap D^+_\varphi(x)$ and $S_\varphi := \{x \in X : D^-_\varphi(x) \neq \varnothing, D^+_\varphi(x) \neq \varnothing\}$ are called the φ-**prolongation** of x and the φ-**separatrix**, respectively. Note that, in the parlance of Sect. 2, simply $D_\varphi(x) = C_{x,x}(\varphi, X)$ and $S_\varphi = C(\varphi, X)$. A crucial lemma [26, Lem. 7] asserts that these sets are well-behaved under products, in that, for instance, $S_{\varphi\times\psi} = S_\varphi \times S_\psi$. As demonstrated by Example 2.7, this is incorrect in general. Another crucial lemma [26, Lem. 8] asserts that prolongations and separatrices are well-behaved under orbit equivalence. Although this assertion is correct (and a special case of Lemma 2.5), its proof in [26] assumes $\tau : \mathbb{R} \times X \to \mathbb{R}$ in (1.1) to be continuous. The reader will have no difficulty constructing examples of C^0-orbit equivalent flows on $X = \mathbb{R}^2$ for which τ is not even measurable, let alone continuous. Sometimes τ can be replaced by a continuous modification, but simple examples show that this may not always be the case. Obviously, by Theorem 1.1, a continuous modification of τ always exists between *linear* flows, but surely this should be a consequence, rather than an assumption, of any topological classification theorem—as it is in the present article, where no regularity whatsoever is assumed for τ beyond the requirement that τ_x be strictly increasing for each $x \in X$. One observation regarding a counterpart of Lemma 4.7 is worth mentioning also: [26, Prop. 3] implicitly assumes that no more than two different rational classes have to be considered simultaneously. In the notation of the proof of Lemma 4.7, this amounts to assuming that $X_{1,2} = X^\Phi_{\lambda_1\mathbb{Q}} \oplus X^\Phi_{\lambda_2\mathbb{Q}}$. As the reader may want to check, this drastically simplifies the proof of that lemma, since Step II and much of Step IV become obsolete. In general, however, such an assumption is unfounded, as it is quite possible for three or more rational classes to be rationally dependent, and hence for $X_{1,2}$ to be strictly larger than $X^\Phi_{\lambda_1\mathbb{Q}} \oplus X^\Phi_{\lambda_2\mathbb{Q}}$.

As far as the smooth classification of linear flows is concerned, most textbooks mention the special case (ii)⇔(iv) of Theorem 1.2 which, of course, can be established immediately by differentiating $h(e^{tA}x) = e^{\alpha t B}h(x)$ w.r.t. x and t; see, e.g., [3,12,30,32]. However, if one only assumes C^1-*orbit* equivalence, where τ may depend on x in a potentially very rough way, differentiation clearly is not available, and a finer analysis is needed. A substantial literature exists of further classification results for linear flows (considering, e.g., Lipschitz [22] and Hölder [29] equivalence) as well as non-autonomous [14] and control systems [7,27,35], and also for non-linear flows derived from them [8,23].

Finally, it is worth pointing out that a similar classification problem presents itself in discrete time, i.e., for linear operators $A : X \to X$, $B : Y \to Y$ which are C^ℓ-**equivalent** if $h(Ax) = Bh(x)$ for all $x \in X$. While for $\ell \geq 1$ this problem is easier than its continuous-time analogue, for $\ell = 0$ it is significantly more difficult and, to some extent, still unresolved; see, e.g., [10,11,15,19,25] and the references therein for the long history of the problem and its many ramifications.

6 Equivalence of Complex Linear Flows

So far, the classification of finite-dimensional linear flows developed in this article has focussed entirely on *real* flows. Such focus is warranted by the fact that the main result, Theorem 1.1, is a truly real theorem, whereas Theorem 1.2 carries over verbatim to *complex* flows. The goal of this concluding section is to make these two assertions precise, via Theorems 6.1 and 6.2 below.

Throughout, let X be a finite-dimensional normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$; to avoid notational conflicts with previous sections, the field of scalars is indicated explicitly wherever appropriate. Further, let $X_{\mathbb{R}}$ be the **realification** of X, i.e., the linear space $X_{\mathbb{R}}$ equals X as a set, but with the field of scalars being $\mathbb{R}$, and define $\iota_X : X \to X_{\mathbb{R}}$ as $\iota_X(x) = x$. Thus, if $\mathbb{K} = \mathbb{C}$, then ι_X is a homeomorphism as well as an $\mathbb{R}$-linear bijection, and $\dim X_{\mathbb{R}} = 2 \dim X$. (Trivially, if $\mathbb{K} = \mathbb{R}$ then $X_{\mathbb{R}}$ equals X as a linear space, and $\iota_X = \mathrm{id}_X$.) Every map $h : X \to Y$ induces a map $h_{\mathbb{R}} = \iota_Y \circ h \circ \iota_X^{-1} : X_{\mathbb{R}} \to Y_{\mathbb{R}}$ which is continuous (one-to-one, onto) if and only if h is. If h is C^ℓ or linear then so is $h_{\mathbb{R}}$, but the converse is not true in general when $\mathbb{K} = \mathbb{C}$. In particular, an $\mathbb{R}$-linear map $h : X \to Y$ is $\mathbb{C}$-linear precisely if $h_{\mathbb{R}} J_X = J_Y h_{\mathbb{R}}$ where $J_X : X_{\mathbb{R}} \to X_{\mathbb{R}}$ is the unique linear operator with $J_X(\,\cdot\,) = \iota_X\big(\imath\, \iota_X^{-1}(\,\cdot\,)\big)$. Given any (smooth) flow φ on X, its realification $\varphi_{\mathbb{R}}$ on $X_{\mathbb{R}}$ is defined via $(\varphi_{\mathbb{R}})_t = (\varphi_t)_{\mathbb{R}}$ for all $t \in \mathbb{R}$. Clearly, if φ, ψ are C^ℓ-orbit (or -flow) equivalent then so are $\varphi_{\mathbb{R}}, \psi_{\mathbb{R}}$, and for $\ell = 0$ the converse also holds. For a $\mathbb{K}$-linear flow Φ on X, it is readily confirmed that all fundamental dynamical objects associated with Φ are well-behaved under realification in that, for instance, $A^{\Phi_{\mathbb{R}}} = A^{\Phi}_{\mathbb{R}}$ and also $X^{\Phi_{\mathbb{R}}}_{\mathbb{R}\bullet} = \iota_X(X^{\Phi}_{\bullet})$ for $\bullet = S, C, U$. With this, the topological classification theorem for $\mathbb{K}$-linear flows, a generalization and immediate consequence of Theorem 1.1, presents itself as a truly *real* result in that topological equivalence is determined completely by the associated realifications. (The reader familiar with [26] will notice how usage of realifications avoids the somewhat cumbersome notion of *c-analog*.)

Theorem 6.1 *Let Φ, Ψ be $\mathbb{K}$-linear flows on X, Y, respectively. Then each of the following five statements implies the other four:*

(i) Φ, Ψ *are C^0-orbit equivalent;*
(ii) Φ, Ψ *are C^0-flow equivalent;*
(iii) $\Phi_{\mathbb{R}}, \Psi_{\mathbb{R}}$ *are C^0-orbit equivalent;*
(iv) $\Phi_{\mathbb{R}}, \Psi_{\mathbb{R}}$ *are C^0-flow equivalent;*

(v) *dim* X_S^Φ = *dim* Y_S^Ψ, *dim* X_U^Φ = *dim* Y_U^Ψ, *and* $A_{\mathbb{R}}^{\Phi_C}, \alpha A_{\mathbb{R}}^{\Psi_C}$ *are* $\mathbb{R}$*-similar for some* $\alpha \in \mathbb{R}^+$.

Proof For $\mathbb{K} = \mathbb{R}$, this is part of Theorem 1.1, so assume $\mathbb{K} = \mathbb{C}$. Since $h_{\mathbb{R}} : X_{\mathbb{R}} \to Y_{\mathbb{R}}$ is a homeomorphism if and only if $h : X \to Y$ is, clearly (i)$\Leftrightarrow$(iii) and (ii)$\Leftrightarrow$(iv). By Theorem 1.1, (iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v). □

By contrast, smooth equivalence of $\mathbb{C}$-linear flows is *not* determined by the associated realifications. To appreciate this basic difference, consider the $\mathbb{C}$-linear flows Φ, Ψ generated by $[\,\imath\,]$, $[-\imath]$, respectively: While $[\,\imath\,]_{\mathbb{R}}$, $[-\imath]_{\mathbb{R}}$ are $\mathbb{R}$-similar, and hence $\Phi_{\mathbb{R}}$, $\Psi_{\mathbb{R}}$ are C^1- (in fact, linearly) flow equivalent, $[\,\imath\,]$, $\alpha[-\imath]$ are not $\mathbb{C}$-similar for any $\alpha \in \mathbb{R}^+$, and correspondingly Φ, Ψ are not C^1-orbit equivalent—though, of course, they are C^0-flow equivalent by Theorem 6.1. The following generalization of Theorem 1.2 shows that, just as in this simple example, smooth equivalence always is determined by the $\mathbb{K}$-similarity of generators (and not by the $\mathbb{R}$-similarity of realified generators).

Theorem 6.2 *Let* Φ, Ψ *be* $\mathbb{K}$*-linear flows. Then each of the following four statements implies the other three:*

(i) Φ, Ψ *are* C^1*-orbit equivalent;*
(ii) Φ, Ψ *are* C^1*-flow equivalent;*
(iii) Φ, Ψ *are* $\mathbb{K}$*-linearly flow equivalent;*
(iv) $A^\Phi, \alpha A^\Psi$ *are* $\mathbb{K}$*-similar for some* $\alpha \in \mathbb{R}^+$.

Apart from a few simple but crucial modifications, the proof of Theorem 6.2 closely follows the arguments in previous sections and only is outlined here, with most details left to the interested reader. A noteworthy stepping stone is the following extension of Theorem 4.1; note that the increased smoothness is irrelevant when $\mathbb{K} = \mathbb{R}$, but is essential (for the "only if" part) when $\mathbb{K} = \mathbb{C}$, as demonstrated by the simple example considered earlier.

Lemma 6.3 *Two bounded* $\mathbb{K}$*-linear flows* Φ, Ψ *are* C^1*-orbit equivalent if and only if* $A^\Phi, \alpha A^\Psi$ *are* $\mathbb{K}$*-similar for some* $\alpha \in \mathbb{R}^+$.

Proof Only the case of $\mathbb{K} = \mathbb{C}$ needs to be considered. Note that the definition of $X_{\omega\mathbb{Q}}^\Phi$ makes sense in this case, in fact, $X_{\omega\mathbb{Q}}^\Phi = \bigoplus_{s\in\mathbb{R}:\imath s\in\omega\mathbb{Q}} \ker(A^\Phi - \imath s\, \mathrm{id}_X)$, and Proposition 4.2 carries over verbatim. A crucial step, then, is to show that Lemma 4.3, with similarity in (iii) understood to mean $\mathbb{C}$-similarity, also remains valid provided that $h : X \to Y$ is a C^1-diffeomorphism. For assertions (i) and (ii), this is obvious, even when h is only a homeomorphism. Differentiability of h, however, in addition yields $H\,\mathrm{Per}_T\,\Phi = \mathrm{Per}_{\alpha T}\,\Psi$ for every $T \in \mathbb{R}^+$, where $H = D_0 h$ for convenience. To establish (iii), analogously to the proof of Lemma 4.3, denote A^Φ, A^Ψ by A, B respectively, and let $\sigma(\Phi) \setminus \{0\} = \{\imath a_1, \ldots, \imath a_m\}$ with the appropriate $m \in \mathbb{N}_0$ as well as real numbers a_j such that $|a_1| \geq \ldots \geq |a_m| > 0$, and $a_j > a_{j+1}$ in case $|a_j| = |a_{j+1}|$. Similarly, $\sigma(\Psi) \setminus \{0\} = \{\imath b_1, \ldots, \imath b_n\}$ with $n \in \mathbb{N}_0$ as well as $|b_1| \geq \ldots \geq |b_n| > 0$, and $b_j > b_{j+1}$ whenever $|b_j| = |b_{j+1}|$. For convenience, $a_0 = b_0 = 0$, and $X_s = \ker(A - \imath s\, \mathrm{id}_X)$, $Y_s = \ker(B - \imath s\, \mathrm{id}_Y)$ for every $s \in \mathbb{R}$. Since A, B are diagonalisable, it suffices to prove that $m = n$, and moreover that

$$a_k = \alpha b_k \quad \text{and} \quad H X_{a_k} = Y_{b_k} \quad \forall k = 0, 1, \ldots, m. \tag{6.1}$$

To this end, notice that $\mathrm{Per}_{2\pi/|s|}\Phi = \bigoplus_{k\in\mathbb{Z}} X_{ks}$ and $\mathrm{Per}_{2\pi/|s|}\Psi = \bigoplus_{k\in\mathbb{Z}} Y_{ks}$ for every $s \in \mathbb{R} \setminus \{0\}$. Clearly, if $mn = 0$ then $m = n = 0$, and (6.1) holds. Henceforth, let $m, n \geq 1$, and assume that, for some integer $0 \leq \ell < \min\{m, n\}$,

$$a_k = \alpha b_k \quad \text{and} \quad H X_{a_k} = Y_{b_k} \quad \forall k = 0, 1, \ldots, \ell; \tag{6.2}$$

since $HX_0 = Y_0$, this is clearly correct for $\ell = 0$. Letting

$$K_\ell = \bigl\{k \in \mathbb{Z} \setminus \{-1, 1\} : k|a_{\ell+1}| \in \{a_0, a_1, \ldots, a_\ell\}\bigr\},$$

note that K_ℓ is finite, and $0 \in K_\ell$. Deduce from

$$\mathrm{Per}_{2\pi/|a_{\ell+1}|}\Phi = \bigoplus\nolimits_{k\in\mathbb{Z}} X_{k|a_{\ell+1}|} = X_{-|a_{\ell+1}|} \oplus X_{|a_{\ell+1}|} \oplus \bigoplus\nolimits_{k\in K_\ell} X_{k|a_{\ell+1}|}\,,$$

together with (6.2) and

$$\begin{aligned} HX_{-|a_{\ell+1}|} \oplus H_{|a_{\ell+1}|} \oplus \bigoplus\nolimits_{k\in K_\ell} HX_{k|a_{\ell+1}|} &= H\,\mathrm{Per}_{2\pi/|a_{\ell+1}|}\Phi = \mathrm{Per}_{2\pi\alpha/|a_{\ell+1}|}\Psi \\ &= \bigoplus\nolimits_{k\in\mathbb{Z}\setminus K_\ell} Y_{k|a_{\ell+1}|/\alpha} \oplus \bigoplus\nolimits_{k\in K_\ell} Y_{k|a_{\ell+1}|/\alpha}, \end{aligned} \tag{6.3}$$

that $\dim(X_{-|a_{\ell+1}|} \oplus X_{|a_{\ell+1}|}) = \sum_{k\in\mathbb{Z}\setminus K_\ell} \dim Y_{k|a_{\ell+1}|/\alpha} > 0$. Hence $\imath k|a_{\ell+1}|/\alpha \in \sigma(\Psi)$ for some $k \in \mathbb{Z} \setminus K_\ell$, and so in fact $|a_{\ell+1}| \le \alpha|b_{\ell+1}|$, but also $\dim(Y_{-|a_{\ell+1}|/\alpha} \oplus Y_{|a_{\ell+1}|/\alpha}) \le \dim(X_{-|a_{\ell+1}|} \oplus X_{|a_{\ell+1}|})$ because $\{-1, 1\} \subset \mathbb{Z} \setminus K_\ell$. Reversing the roles of Φ and Ψ yields that $|a_{\ell+1}| = \alpha|b_{\ell+1}|$ and $\dim(X_{-|a_{\ell+1}|} \oplus X_{|a_{\ell+1}|}) = \dim(Y_{-|b_{\ell+1}|} \oplus Y_{|b_{\ell+1}|})$. Consequently, (6.3) becomes

$$HX_{-|a_{\ell+1}|} \oplus HX_{|a_{\ell+1}|} \oplus \bigoplus\nolimits_{k\in K_\ell} HX_{k|a_{\ell+1}|} = Y_{-|b_{\ell+1}|} \oplus Y_{|b_{\ell+1}|} \oplus \bigoplus\nolimits_{k\in K_\ell} Y_{k|b_{\ell+1}|}, \tag{6.4}$$

and the goal now is to show that (6.2) holds with $\ell + 1$ instead of ℓ. To this end, begin by assuming that $X_{|a_{\ell+1}|} \ne \{0\}$, and pick any $x \in X_{|a_{\ell+1}|} \setminus \{0\}$. Then $\varepsilon x \in \mathrm{Per}_{2\pi/|a_{\ell+1}|}\Phi$ and $h(\varepsilon x) \in \mathrm{Per}_{2\pi/|b_{\ell+1}|}\Psi$ for every $\varepsilon > 0$, as well as

$$h(e^{tA}\varepsilon x)/\varepsilon = h(e^{\imath t|a_{\ell+1}|}\varepsilon x)/\varepsilon = e^{\tau_{\varepsilon x}(t)B}h(\varepsilon x)/\varepsilon. \tag{6.5}$$

Note that $0 \le \tau_{\varepsilon x}(t) \le 2\pi/|b_{\ell+1}|$ for every $0 \le t \le 2\pi/|a_{\ell+1}|$, and $\tau_{\varepsilon x}(\,\cdot\,)$ is increasing. By the Helly selection theorem, there exists a strictly decreasing sequence (ε_n) with $\lim_{n\to\infty}\varepsilon_n = 0$, along with an increasing function ρ with $\rho(0) = 0$, $\rho(2\pi/|a_{\ell+1}|) = 2\pi/|b_{\ell+1}|$ such that $\lim_{n\to\infty}\tau_{\varepsilon_n x}(t) = \rho(t)$ for almost all (in fact, all but countably many) $0 \le t \le 2\pi/|a_{\ell+1}|$. With this, (6.5) yields

$$He^{\imath t|a_{\ell+1}|}x = e^{\rho(t)B}Hx \quad \text{for almost all } 0 \le t \le 2\pi/|a_{\ell+1}|.$$

Note that $0 < \rho(t) < 2\pi/|b_{\ell+1}|$ for all $0 < t < 2\pi/|a_{\ell+1}|$. By monotonicity, $\rho_0 := \lim_{t\downarrow 0}\rho(t)$ exists, with $0 \le \rho_0 < 2\pi/|b_{\ell+1}|$. If $\rho_0 > 0$ then $Hx \in \mathrm{Per}_{\rho_0}\Psi$, and hence $\rho_0|b_{\ell+1}| \in 2\pi\mathbb{N}$, which is impossible. Thus $\rho_0 = 0$, and

$$\imath|a_{\ell+1}|Hx = \lim\nolimits_{t\downarrow 0} H\frac{e^{\imath t|a_{\ell+1}|} - 1}{t}x = \lim\nolimits_{t\downarrow 0}\frac{\rho(t)}{t}\cdot\frac{e^{\rho(t)B} - \mathrm{id}_Y}{\rho(t)}Hx = \lim\nolimits_{t\downarrow 0}\frac{\rho(t)}{t}BHx,$$

showing that $\rho_{0,0} := \lim_{t\downarrow 0}\rho(t)/t$ exists, with $\imath|a_{\ell+1}|Hx = \rho_{0,0}BHx$. Clearly $\rho_{0,0} \ge 0$, in fact, $\rho_{0,0} > 0$ since $Hx \ne 0$, and hence $Hx \in Y_{|a_{\ell+1}|/\rho_{0,0}}$. In other words, if $x \in X_{|a_{\ell+1}|}$ then $Hx \in Y_b$ for some $b \in \mathbb{R}^+$. Completely analogous reasoning yields $Hx \in Y_{-b}$ for some $b \in \mathbb{R}^+$ whenever $x \in X_{-|a_{\ell+1}|}$.

Recall that the goal is to establish (6.2) with $\ell + 1$ instead of ℓ. To this end, assume first that $|a_{\ell+1}| = |a_\ell|$, and hence $a_{\ell+1} = -a_\ell < 0$, but also $b_{\ell+1} = -b_\ell < 0$. In this case $X_{a_{\ell+1}} = X_{-|a_{\ell+1}|} \ne \{0\}$, and utilizing the preceding considerations, together with (6.2) and (6.4), it follows that $HX_{a_{\ell+1}} \subset Y_{-|b_{\ell+1}|} = Y_{b_{\ell+1}}$. Reversing the roles of Φ and Ψ yields $HX_{a_{\ell+1}} = Y_{b_{\ell+1}}$. Since $a_{\ell+1} = \alpha b_{\ell+1}$ in this case, (6.2) holds with $\ell + 1$ instead of ℓ.

It remains to consider the case of $|a_{\ell+1}| < |a_\ell|$. Here it is convenient to distinguish two possibilities: On the one hand, if $\ell = m - 1$ or $|a_{\ell+2}| < |a_{\ell+1}|$ then exactly one of the spaces

$X_{\pm|a_{\ell+1}|}$ is different from $\{0\}$. As before, it is readily seen that $a_{\ell+1}, b_{\ell+1}$ have the same sign, hence $a_{\ell+1} = \alpha b_{\ell+1}$, and $HX_{a_{\ell+1}} = Y_{b_{\ell+1}}$, so again (6.2) holds with $\ell + 1$ instead of ℓ. On the other hand, if $|a_{\ell+2}| = |a_{\ell+1}|$ then $a_{\ell+2} = -a_{\ell+1} < 0$, and the argument immediately following (6.3) shows that $|a_{\ell+2}| = \alpha|b_{\ell+2}|$ also. Thus $a_{\ell+1} = \alpha b_{\ell+1} > 0$ and $a_{\ell+2} = \alpha b_{\ell+2} < 0$, and analogous reasoning as before results in $HX_{a_{\ell+1}} = Y_{b_{\ell+1}}$, $HX_{a_{\ell+2}} = Y_{b_{\ell+2}}$. Again, (6.2) holds with $\ell + 1$ (in fact, $\ell + 2$) instead of ℓ. Induction now proves (6.1), and since $X = \bigoplus_{\ell=0}^{m} X_{a_\ell}$, $Y = \bigoplus_{\ell=0}^{n} Y_{b_\ell}$, clearly $m = n$. As indicated earlier, this establishes Lemma 4.3(iii) in the case of $\mathbb{K} = \mathbb{C}$ and under the assumption that h is a C^1-diffeomorphism.

With Lemma 4.3 thus extended to complex linear flows, the remainder of the proof proceeds exactly as in Sect. 4, since Lemmas 4.4 and 4.7 carry over without any modifications, and so does the proof of Theorem 4.1. (In fact, with the notation used in that proof, the linear isomorphism H_j can be taken to be the restriction of D_0h to $X^{\Phi}_{\lambda_j\mathbb{Q}}$. Thus, instead of being defined abstractly by $A^{\Phi}, \alpha A^{\Psi}$ both being diagonalisable and having the same eigenvalues with matching geometric multiplicities, H now simply equals D_0h.) $\square$

Outline of Proof of Theorem 6.2 Again, one only needs to consider the case of $\mathbb{K} = \mathbb{C}$ and establish (i)$\Rightarrow$(iv), as in the proof of Theorem 1.2. The crucial step is to extend Lemma 6.3 from *bounded* to *central* $\mathbb{K}$-linear flows, i.e., to show that (i) implies $\mathbb{C}$-similarity of $A^{\Phi_C}, \alpha A^{\Psi_C}$ for some $\alpha \in \mathbb{R}^+$. To prove the latter along the lines of the proof of Theorem 1.1, with $X_s = \ker(A^{\Phi_C} - \imath s\, \mathrm{id}_{X_C^{\Phi}})$, $Y_s = \ker(A^{\Psi_C} - \imath s\, \mathrm{id}_{Y_C^{\Psi}})$ for every $s \in \mathbb{R}$, it is necessary to first adjust the auxiliary results of Sect. 3, notably Lemmas 3.2 and 3.7, for complex linear flows. With the details of these routine adjustments left to the reader, the non-negative integer $d_n^{\Phi}(s)$ now equals, for each $n \in \mathbb{N}$ and $s \in \mathbb{R}$, the number of blocks $\imath s I_n + J_n$ in the (complex) Jordan normal form of A^{Φ}. Utilizing the proof of Lemma 6.3, deduce that $m = n$, as well as $a_k = \alpha b_k$ for $k = 0, 1 \ldots, m$ and an appropriate $\alpha \in \mathbb{R}^+$, and that moreover $d_n^{\Phi}(a_k) = d_n^{\Psi}(b_k)$ for all n, k. Again, the differentiability of h, h^{-1} is essential here, unlike in the proof of Theorem 1.1. Thus, $A^{\Phi_C}, \alpha A^{\Psi_C}$ indeed are $\mathbb{C}$-similar, which in turn proves that (i)$\Rightarrow$(iv) in case $X_C^{\Phi} = X$. Apart from the fact that this latter extension of Lemma 6.3, rather than Theorem 1.1, has to be used to establish (5.8), the remaining argument now is identical to the one proving Theorem 1.2 in Sect. 5. $\square$

To finally illustrate the difference between real and complex linear flows in dimensions 1 and 2, recall that on $X = \mathbb{R}$ there are exactly three (C^0- or C^1-) equivalence classes of $\mathbb{R}$-linear flows, represented by $\Phi(t, x) = e^{ta}x$ with $a \in \{-1, 0, 1\}$. By contrast, on $X = \mathbb{C}$ there are four C^0-equivalence classes of $\mathbb{C}$-linear flows, represented by $\Phi(t, x) = e^{tc}x$ with $c \in \{-1, 0, 1, \imath\}$, but infinitely many C^1-equivalence classes, corresponding to $c \in \{\omega \in \mathbb{C} : |\omega| = 1\} \cup \{0\}$. Similarly, on $X = \mathbb{R}^2$ there are exactly eight C^0-equivalence classes of $\mathbb{R}$-linear flows, listed in (1.2), whereas for $\mathbb{C}$-linear flows on $X = \mathbb{C}^2$, all C^0-equivalence classes are given by all the matrices in (1.2) except for the left-most, together with

$$\pm\begin{bmatrix} 1 & 0 \\ 0 & \imath \end{bmatrix}, \begin{bmatrix} \imath & 1 \\ 0 & \imath \end{bmatrix}, \begin{bmatrix} \imath & 0 \\ 0 & \imath a \end{bmatrix} \quad (0 \le a \le 1),$$

and all C^1-equivalence classes are given by

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & \omega \end{bmatrix} \quad (c, \omega \in \mathbb{C}, |c| = 1, |\omega| \le 1).$$

The reader may want to compare the latter to the seven singleton classes and five infinite families that make up all C^1-equivalence classes of $\mathbb{R}$-linear flows on $X = \mathbb{R}^2$, as listed in the Introduction; cf. also [28, Ex. 1].

Acknowledgements The first author was partially supported by an NSERC Discovery Grant. The authors gratefully acknowledge helpful comments and suggestions made by C. Kawan, T. Oertel-Jäger, G. Peschke, C. Pötzsche, and V. Troitsky.

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